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Ageostrophic instability of ocean currents

by

R.W. Griffiths, Peter D. Killworth and Melvin E. Stern

APPENDIX B

The inner solution and matching

We set

$$\epsilon Y = y - y_c \tag{B1}$$

and write

$$\begin{aligned}\phi &= \Phi_0 + \epsilon \Phi_1 + \epsilon^2 \Phi_2 + \epsilon^3 \log\epsilon\, \Phi_3 + \epsilon^3 \Phi_4 + \cdots \\ \hat{u} &= u_0 + \epsilon \log\epsilon\, u_1 + \epsilon u_2 + \cdots \\ i\hat{v} &= iv_0 + \epsilon i v_1 + \epsilon^2 \log\epsilon\, i v_2 + \epsilon^2 i v_3 + \cdots\end{aligned} \tag{B2}$$

where the expansions of $\hat{u}, \hat{v}$ have taken the stretched y-coordinate into account. The equations of motion (2.15), (2.16) and (2,17) then take the (symbolic) form, for n=0,1,2

$$(\bar{u}_{yc} Y - c_1) u_n + (1 - \bar{u}_{yc}) i v_{n+1} + \Phi_{(n+2)Y} + a_n = 0 \tag{B3}$$

$$u_n + \Phi_{(n+2)YY} = 0 \tag{B4}$$

$$u_n - i v_{(n+1)Y} = 0 \tag{B5}$$

where $a_n(Y)$ involve lower-order terms in the expansion but are not here produced, and a suffix c denotes evaluation at the critical layer. After matching to the outer solution, the solutions for Φ_0, Φ_1 are trivially found to be

$$\Phi_0 = \bar{h}_c\,,\ iv_0 = 0,\ \Phi_1 = \phi_1^{\pm}(y_c) \tag{B6}$$

so that ϕ_1 is continuous at $y = y_c$. At next order we find

$$\Phi_2 = \alpha_2 + \beta_2 Y + \gamma_2 Y^2 \tag{B7}$$

where α_2, β_2, γ_2 are to be found. At this order the matching with the outer solution yields

$$\Phi_2 = \phi_{2c} + \frac{c_1}{\bar{u}_{yc}} Y - \frac{\bar{u}_{yc}}{2} Y^2 \tag{B8}$$

$$iv_1 = \bar{u}_{yc} Y - \frac{c_1}{\bar{u}_{yc}}\left(1 + \bar{u}_{yc}\right) \tag{B9}$$

$$u_0 = \bar{u}_{yc}, \tag{B10}$$

so that ϕ_2 is continuous at $y = y_c$.

At next order, again we have

$$\Phi_3 = \alpha_3 + \beta_3 Y + \gamma_3 Y^2. \tag{B11}$$

As is well known, the matching can only proceed together with the $O(\epsilon^3)$ terms. We therefore write the equation for Φ_4 as

$$\Phi_{4YYY}\left(\bar{u}_{yc} Y - c_1\right) = -\bar{u}_{yc}\bar{u}_{yyc} Y + \frac{c_1 \bar{u}_{yyc}}{\bar{u}_{yc}}\left(1 + \bar{u}_{yc}\right) \tag{B12}$$

or

$$\Phi_4 = \alpha_4 + \beta_4 Y + \gamma_4 Y^2 - \frac{\bar{u}_{yyc}}{6} Y^3 + \frac{c_1 \bar{u}_{yyc}}{2\bar{u}_{yc}^4}\left(\bar{u}_{yc} Y - c_1\right)^2\left\{\log\left(\bar{u}_{yc} Y - c_1\right) - 3/2\right\}. \tag{B13}$$

We write the outer $O(\epsilon^3)$ expansion as

$$\phi \sim \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \epsilon^3 \log_e \epsilon\, \phi_3 + \epsilon^3 \phi_4$$

$$= \phi_{oc} - \epsilon^2 \frac{\bar{u}_{yc}}{2} - \epsilon^3 \frac{\bar{u}_{yyc}}{6} Y^3$$

$$+ \epsilon\left\{ \phi_{1c} + \epsilon \frac{c_1}{\bar{u}_{yc}} Y + \frac{c_1 \bar{u}_{yyc}}{2\bar{u}_{yc}^2} \epsilon^2 Y^2 (\log\epsilon + \log Y) + \frac{1}{2}\phi^{\pm}_{1yyc} \epsilon^2 Y^2 \right\}$$

$$+ \epsilon^2 \left\{ \phi_{2c} - \frac{c_1^2 \bar{u}_{yyc}}{\bar{u}_{yc}^3} \epsilon Y (\log\epsilon + \log Y) + \phi^{\pm}_{2yc} \epsilon Y \right\} + \epsilon^3 \log\epsilon\, \phi^{\pm}_{3c} \quad \text{(B14)}$$

$$+ \epsilon^3 \left\{ \frac{c_1^3 \bar{u}_{yyc}}{2\bar{u}_{yc}^4} (\log\epsilon + \log Y) + \phi^{\pm}_{4c} \right\}$$

after use of the expansions of the outer solutions and truncations to $O(\varepsilon^3)$, where $\frac{1}{2}\phi^{\pm}_{1yyc}$, etc., are the coefficients of the well-behaved part of $\frac{1}{2}\phi_{1yy}$ above and below the critical layer (i.e. the coefficients of η^2 in (4.18), (4.19)).

The inner $O(\varepsilon^3)$ expansion is

$$\phi \sim \Phi_0 + \epsilon \Phi_1 + \epsilon^2 \Phi_2 + \epsilon^3 \log\epsilon\, \Phi_3 + \epsilon^3 \Phi_4 \quad \text{(B15)}$$

which, after substitution, becomes in outer variables

$$\phi = \phi_{oc} + \epsilon \phi_{1c} + \epsilon^2 \phi_{2c} + \frac{\epsilon c_1 \eta}{\bar{u}_{yc}} - \frac{\bar{u}_{yc}}{2} \eta^2 + \epsilon^3 \log\epsilon . \alpha_3 + \epsilon^2 \log\epsilon . \beta_3 \eta$$

$$+ \epsilon \log\epsilon . \gamma_3 \eta^2 + \epsilon^3 \alpha_4 + \epsilon^2 \beta_4 \eta + \epsilon \gamma_4 \eta^2 - \frac{\bar{u}_{yyc}}{6} \eta^3 \quad \text{(B16)}$$

$$+ \frac{\epsilon c_1 \bar{u}_{yyc}}{2\bar{u}_{yc}^4} (\bar{u}_{yc}\eta - \epsilon c_1)^2 \left\{ \log\left(\frac{\bar{u}_{yc}\eta}{\epsilon} - c_1 \right) - \frac{3}{2} \right\}.$$

The argument of the logarithm takes values with a negative imaginary part (as $I_m(c_1) > 0$ for an unstable mode), with the signs of the real part depending on $\mathrm{sgn}(\bar{u}_{yc})$. Hence $\log(-\delta)$ is to be interpreted as $\log\delta - i\pi\,\mathrm{sgn}(\bar{u}_{yc})$ in section 4. Expanding (B16) to $O(\varepsilon^3)$, and rewriting in inner variables gives

$$\phi = \phi_{0c} + \epsilon\phi_{1c} + \epsilon^2\left(\phi_{2c} + \frac{c_1}{\bar{u}_{yc}} Y - \frac{\bar{u}_{yc}}{2} Y^2\right)$$
$$+ \epsilon^3 \log\epsilon \left(\alpha_3 + \beta_3 Y + \gamma_3 Y^2\right)$$
$$+ \epsilon^3\left\{\alpha_4 + \beta_4 Y + \gamma_4 Y^2 - \frac{\bar{u}_{yyc}}{6} Y^3 + \frac{c_1\bar{u}_{yyc}}{2\bar{u}_{yc}^2} Y^2\left(\log Y + \log\bar{u}_{yc} - 3/2\right)\right. \qquad \text{(B17)}$$
$$+ \frac{c_1^2\bar{u}_{yyc}}{2\bar{u}_{yc}^3} Y - \frac{\bar{u}_{yyc} c_1^2 Y}{\bar{u}_{yc}^3}\left(\log Y + \log\bar{u}_{yc} - 3/2\right)$$
$$\left. - \frac{3c_1^3\bar{u}_{yyc}}{4\bar{u}_{yc}^4} + \frac{c_1^3\bar{u}_{yyc}}{2\bar{u}_{yc}^4}\left(\log Y + \log\bar{u}_{yc} - 3/2\right)\right\}.$$

Matching (B17) with (B14) yields three equations at $O(\varepsilon^3\log\varepsilon)$, which give α_3, β_3 and γ_3. At $O(\varepsilon^3)$, there are seven equations, for the coefficients of $Y^3, Y^2\log Y, Y^2, Y\log Y, Y, \log Y$ and 1. Those for Y^3 and the terms proportional to $\log Y$ merely give identities; those for Y and 1 give β_4 and α_4. However, that for Y^2 gives

$$\frac{1}{2}\phi^{\pm}_{1yyc} = \gamma_4 + \frac{c_1\bar{u}_{yyc}}{2\bar{u}_{yc}^2}\left(\log\bar{u}_{yc} - 3/2\right) \qquad \text{(B18)}$$

so that the well-behaved part of ϕ_{1yy} is continuous at $y = y_c$, as required for the matching in section 4.