Ageostrophic instability of ocean currents
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APPENDIX B

The inner solution and matching

We set

$$\epsilon Y = y - y_c$$  \hspace{1cm} (B1)

and write

$$\phi = \phi_0 + \epsilon \phi_1 + \epsilon^2 \phi_2 + \epsilon^3 \log \epsilon \phi_3 + \epsilon^4 \phi_4 + \ldots$$

$$\hat{u} = u_0 + \epsilon \log \epsilon u_1 + \epsilon u_2 + \ldots$$  \hspace{1cm} (B2)

$$\hat{v} = \epsilon v_0 + \epsilon \epsilon v_1 + \epsilon^2 \log \epsilon v_2 + \epsilon^3 v_3 + \ldots$$

where the expansions of $\hat{u}, \hat{v}$ have taken the stretched $y$-coordinate into account. The equations of motion (2.15), (2.16) and (2.17) then take the (symbolic) form, for $n=0,1,2$

$$\left( \frac{\partial}{\partial Y} \right)_c u_n + (1 - \hat{u}_c) i v_{n+1} + \hat{\phi}_{(n+1)} + a_n = 0$$  \hspace{1cm} (B3)

$$u_n + \hat{\phi}_{(n+1)} \gamma = 0$$  \hspace{1cm} (B4)

$$u_n - i v_{(n+1)} \gamma = 0$$  \hspace{1cm} (B5)

where $a_n(Y)$ involve lower-order terms in the expansion but are not here produced, and a suffix c denotes evaluation at
the critical layer. After matching to the outer solution, the solutions for \( \Phi_0, \Phi_1 \) are trivially found to be

\[
\Phi_0 = \tilde{\Phi}_0, \quad i \nu_0 = 0, \quad \Phi_1 = \Phi_1^* (y_c) \tag{B6}
\]

so that \( \Phi_1 \) is continuous at \( y = y_c \). At next order we find

\[
\Phi_2 = \kappa + \beta_1 Y + \gamma_2 Y^2 \tag{B7}
\]

where \( \alpha, \beta, \gamma \) are to be found. At this order the matching with the outer solution yields

\[
\Phi_2 = \phi_2 + \frac{c}{u_{yc}} Y - \frac{u_{yc}}{2} \tag{B8}
\]

\[
i \nu_1 = \frac{u_{yc}}{2} Y - \frac{c}{u_{yc}} \left( 1 + \frac{u_{yc}}{2} \right) \tag{B9}
\]

\[
u_0 = \frac{u_{yc}}{2} \tag{B10}
\]

so that \( \Phi_2 \) is continuous at \( y = y_c \).

At next order, again we have

\[
\Phi_3 = \alpha_3 + \beta_3 Y + \gamma_3 Y^2 \tag{B11}
\]

As is well known, the matching can only proceed together with the \( O(\varepsilon^3) \) terms. We therefore write the equation for \( \Phi_4 \) as

\[
\Phi_4 = \frac{\varepsilon}{u_{yc}} \left( \frac{\varepsilon}{u_{yc}} - \varepsilon \right) \tag{B12}
\]

\[
\Phi_4 = \frac{\varepsilon}{u_{yc}} \left( \frac{\varepsilon}{u_{yc}} - \varepsilon \right) \tag{B13}
\]

We write the outer \( O(\varepsilon^3) \) expansion as

\[
\Phi \sim \Phi_0 + \varepsilon \Phi_1 + \varepsilon^2 \Phi_2 + \varepsilon^3 \log \varepsilon \Phi_3 + \varepsilon^3 \Phi_4
\]
\[\phi = \phi_0 + \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \varepsilon^3 \log \varepsilon \phi_3 + \varepsilon^3 \phi_4 \tag{B15}\]

which, after substitution, becomes in outer variables

\[\phi = \phi_0 + \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \varepsilon^3 \left( \frac{c_i}{\tilde{u}_{yc}} \right)^2 \left( \log \left( \frac{\tilde{u}_{yc} \gamma - c_i}{\varepsilon} \right) - \frac{3 \gamma}{2} \right) \tag{B16}\]

The argument of the logarithm takes values with a negative imaginary part (as \(I_m(c_i) > 0\) for an unstable mode), with the signs of the real part depending on \(\text{sgn}(\tilde{u}_{yc})\). Hence \(\log(-\delta)\) is to be interpreted as \(\log \delta - i \pi \text{sgn}(\tilde{u}_{yc})\) in section 4. Expanding (B16) to \(O(\varepsilon^3)\), and rewriting in inner variables gives...
\[ \phi = \phi_0 + \epsilon \phi_{1c} + \epsilon^2 \left( \phi_{1c} + \frac{c_t}{w_y} Y - \bar{u}_{yc} Y^2 \right) \]
\[ + \epsilon^3 \log \epsilon \left( \alpha_3 + \beta_3 Y + \gamma_3 Y^2 \right) \]
\[ + \epsilon^3 \left\{ \kappa_4 + \beta_4 Y + \epsilon \frac{\bar{u}_{yy}}{b} - \frac{\bar{u}_{yy}}{a} \epsilon^3 + \frac{c_t \bar{u}_{yc}}{2 \bar{u}_{yc}} Y^2 \left( \log Y + \log \bar{u}_{yc} - \frac{3}{2} \right) \right\} \quad \text{(B17)} \]
\[ + \epsilon^3 \frac{\bar{u}_{yy}}{b} \text{ Y} - \frac{\bar{u}_{yy}}{a} \epsilon \left( \log Y + \log \bar{u}_{yc} - \frac{3}{2} \right) \]
\[ - \frac{3}{2} \epsilon^3 \frac{\bar{u}_{yy}}{b} + \epsilon^3 \left( \log Y + \log \bar{u}_{yc} - \frac{3}{2} \right) \right\} \right\}. \]

Matching (B17) with (B14) yields three equations at \( O(\epsilon^3 \log \epsilon) \), which give \( \alpha_3, \beta_3 \) and \( \gamma_3 \). At \( O(\epsilon^3) \), there are seven equations, for the coefficients of \( Y^3, Y^2 \log Y, Y \log Y, Y, \log Y \) and 1. Those for \( Y^3 \) and the terms proportional to \( \log Y \) merely give identities; those for \( Y \) and 1 give \( \beta_4 \) and \( \alpha_4 \). However, that for \( Y^2 \) gives
\[ \frac{1}{n} \phi_{1yy} = \beta_4 + \frac{c_t \bar{u}_{yy}}{2 \bar{u}_{yc}} \left( \log \bar{u}_{yc} - \frac{3}{2} \right) \quad \text{(B18)} \]
so that the well-behaved part of \( \phi_{1yy} \) is continuous at \( y = y_c \), as required for the matching in section 4.