Renormalized expansions in the theory of turbulence
with the use of Lagrangian position function.

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Note that the author would like to have all requests for copies of this appendix referred to him, so that he can send them the most up-to-date information.
Appendix.

Here are given algebraic details to derive (2.35) \(\sim\) (2.43) and (2.48) \(\sim\) (2.52).

(A-1)

To derive (2.36), it is convenient to write (2.16) in another form as;

\[
\frac{\partial}{\partial s} v(s) = L_i(x, t|s) = \int d^3x'' \left[ \frac{1}{\rho} \frac{\partial}{\partial x_i} p(x'', s) + \nabla^2 u_i(x'', s) \right] \psi(x'', s; x, t),
\]

where

\[
- \frac{1}{\rho} \frac{\partial}{\partial x_i} p(x, s) = \lambda \Im (\nabla^2 X) \left\{ u_m(x, s) u_n(x, s) \right\},
\]

\[
\Im (\nabla^2 X) = \left[ \frac{\partial}{\partial x_n} \frac{\partial}{\partial x_n} + \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_m} \right].
\]

Equation (A 1) can be verified by noting the physical meaning of the MTD \(\frac{\partial}{\partial s} v_i(x, t|s)\); which is equivalent to the usual Lagrangian time derivative. Equation (A 1) can, of course, be verified also directly from (2.16). For, by applying Gauss' theorem, we can transform the following volume integral;

\[
\int d^3x'' \left\{ \frac{1}{2} \left( \frac{\partial}{\partial x_m} + \frac{\partial}{\partial x_n} \right) (u_m(x'', s) u_n(x'', s)) \right\} \psi(x'', s; x, t)
\]

\[
+ u_i(x'', s) u_m(x'', s) \frac{\partial}{\partial x_m} \psi(x'', s; x, t)
\]

\[
= \int d^3x'' \frac{\partial}{\partial x_m} \left\{ u_i(x'', s) u_m(x'', s) \psi(x'', s; x, t) \right\},
\]

(A-2)

\[
(A-1)
\]
into a surface integral over the boundary surface $S$ (say). It is clear that this surface integral is zero as far as the position vector $y(x, t|s)$ at time $s$ (see (1.2)) of the fluid element whose space-time trajectory passes through $(x, t)$ does not lie on $S$. By putting the integral of (A.2) equal to zero and using (2.4), we can easily obtain (A.1) from (2.16).

Hence we can write $B_{ij}$ in (2.19) as

$$B_{ij}(x, t; x', t') = B_{ij}^u(x, t; x', t') + B_{ij}^\nu(x, t; x', t'), \quad (A\, 3\, a)$$

where

$$B_{ij}^u(x, t; x', t') = \nu \left\langle \int d^3x'' \left[ \nabla_{x''}^2 u_i(x'', t) \right] \psi(x'', t; x, t') \cdot u_j(x', t') \right\rangle, \quad (A\, 3\, b)$$

and

$$B_{ij}^\nu(x, t; x', t')$$

$$= \frac{\lambda}{2} \left\langle \int d^3x'' \left\{ \nabla_{x''} \left[ \nabla_{x''} \right] \left[ u_m(x'', t) u_n(x'', t) \right] \right\} \psi(x'', t; x, t') \cdot u_j(x', t') \right\rangle. \quad (A\, 3\, c)$$

Similarly, by applying Gauss' theorem, we can simplify a few terms in (2.17a); if we write $C_{ij} (\equiv \hat{C}_{ij})$ as

$$C_{ij}(x, t; x', t') = <\nu\text{-terms}> + <\nabla\text{-terms}> + <\hat{C}_{ij}^\nu(x, t; x', t')>, \quad (A\, 4\, a)$$

where $\hat{C}_{ij}^\nu$ represents the 2nd, 3rd and last terms of (2.17a) (i.e. those which do not contain $\nu$ nor $\nabla$), $<\nu\text{-terms}>$ represents the 1st and 4th terms, and $<\nabla\text{-terms}>$ represents the 5th and 6th terms of (2.17a), then we can write $<\hat{C}_{ij}^\nu>$ as

(A-2)
\[
\langle \hat{C}_{ij}^{\pi}(x,t;x',t') \rangle = \lambda \int d^3x'' \{ S_{ijmn}(x'',t) \hat{C}^E_{nj}(x'',t;x',t') \} \times \psi(x'',t;x',t') > . \tag{A 4b}
\]

[A-III]

From (2.1) and (2.2), we obtain the expansions of \( u_i \) and \( \psi \) in powers of \( \lambda \) as:
\[
\begin{align*}
    u_i(x,t) &= u_i^0(x,t) - \frac{\lambda}{2} \int_s^t ds' \int d^3x' \hat{G}^{E0}_{ij}(x,t;x',s) \\
    &\times P_{jmn}(x',s)[u_m^0(x',s)u_n^0(x',s)] + ..., \tag{A 5}
\end{align*}
\]
\[
\psi(x,t;x',t') = \psi^0(x,t;x',t') - \lambda \int_s^t ds \{ u_j^0(x,s) \frac{\partial}{\partial x_j} \psi^0(x,s,x',t') \} \\
+ \ldots . \tag{A 6}
\]

The expansions of \( \hat{G}^E \), \( \psi \) and \( \hat{G} \) are also obtained similarly from (2.1), (2.2), (2.12) \& (2.15) in terms of \( \overline{u}_i^0(\Xi u_i^0(x,t)) \), \( \hat{G}^{E0} \) and \( \psi^0 \).

Here, from (2.24), \( \psi^0 \) is known to be equal to a \( \delta \)-function, and from (2.25) \( \hat{G}^{E0} = G^0 \). Hence we can obtain expansions of \( u, \psi, \hat{G}^E \), \( \hat{G} \) and \( \hat{C} \) in terms of \( \overline{u}_i^0 \) and \( G^0 \).

By substituting these expansions into \( A, \overline{B} \) and \( \overline{C} \) (see (2.18), (2.32) \& (2.34), (A 3) and (A 4)), we obtain expansions of \( A, \overline{B} \) and \( \overline{C} \) in terms of \( \overline{u}_i^0 \) and \( G^0 \).

Now let us first consider about \( B_{ij} \) in (A.3). Because the distribution over the ensemble of the initial velocity field \( u_i(x,t_0) \) is assumed to be Gaussian with zero mean, \( B_{ij}^{\pi} \) (see (A 3c)) yields zero in \( O(\lambda) \). In \( O(\lambda^2) \), it yields
\[
(u_m \rightarrow u_m^1) + (u_n \rightarrow u_n^1) + (\psi \rightarrow \psi^1) + (u_j \rightarrow u_j^1), \tag{A 7}
\]

(A-3)
where e.g. \((\psi \to \psi^1)\) means that \(\psi\) in (A.3c) is to be replaced by
the first order term \(\psi^1\) of (A.6) and the other \(u\)-terms in it are
to be replaced by the zeroth order term \(u^0\) of (A.5), i.e.

\[
(\psi \to \psi^1) \equiv \frac{\lambda}{2} \left< \int d^3x'' \{ [Si_{mn}(\nabla_{x''}) [u^0_m(x'',t)u^0_n(x'',t)] \\
\times [ -\int ds \frac{\partial}{\partial x''_a} \psi^0(x'',s;x,t')] \} \cdot u^0_j(x',t') \right>
\]

\[
= B_{ij}^{\Pi}(x,t;x',t'). \tag{A.8}
\]

By using (2.24) and integrating (A.8) by parts, we obtain

\[
B_{ij}^{\Pi}(x,t;x',t') = \frac{\lambda}{2} \left< \int d^3x \frac{\partial}{\partial x'_a} \{ [Si_{mn}(\nabla_x) [u^0_m(x,t)u^0_n(x,t)] \\
\times u^0_a(x,s) \} \cdot u^0_j(x',t') \right> . \tag{A.9}
\]

The remaining three terms of (A.7) give after substitution of (2.24)

\[
\frac{\lambda}{2} \{ [Si_{mn}(\nabla_x)(u^1_m(x,t)u^0_n(x,t)+u^0_m(x,t)u^1_n(x,t))] \cdot u^0_j(x',t') \\
+ [Si_{mn}(\nabla_x)(u^0_m(x,t)u^0_n(x,t))] \cdot u^1_j(x',t') \} \equiv B_{ij}^{\Pi}(x,t;x',t'), \tag{A.10}
\]

where \(u^1\) denotes the first order term of (A.5).

The fourth order moment \(<u^0_m u^0_n u^0_u>\) in (A.9) and (A.10) can
be expressed in terms of \(<u^0_m u^0_n> = U^E_0\), i.e. by virtue of (2.26)
in terms of \(U^0\). By transforming (A.9) and (A.10) into the wave-
vector space defined similariy to (2.27), we can calculate the
contributions of these terms to \(B_{ij}(k,t,t') \equiv P_{ib}(k)B_{pj}(k,t,t')\).
By noting (2.31), we obtain

(A-4)
\[ P_{ib}(k) B_{bj}^\Pi (k,t,t') = -\frac{\lambda^2}{2} \int_t^\Delta ds \sum_{p,r} P_{ib}(k) k_{a} S_{bmn}(p) \]
\[ \times [Q_{ma}^0(-r,t,s) Q_{nj}^0(k,t,t') + Q_{mj}^0(k,t,t') Q_{na}^0(-r,t,s)] \]
\[ = -\lambda^2 [ \int_t^\Delta ds \sum_{p,r} P_{ib}(k) k_{a} S_{bmn}(p) Q_{ma}^0(-r,t,s) ] Q_{nj}^0(k,t,t') \]
\[ = \lambda^2 I_{ij}(k,t,t'), \quad \text{(A.11)} \]

where
\[ S_{bmn}(p) = 2p_b p_m p_n / p^2, \]
and we have used \( S_{bmn}(p) = S_{bmn}(p) \). While, because of the
presence of the factor \( \delta/\delta x_i \) in \( S_{imn}(\nabla) \)(see (A.1c) and (2.5)),
it can be shown that \( B_{ij}^\Pi (k,t,t') \) contain the factor \( k_i \) and
consequently
\[ P_{ib}(k) B_{bj}^\Pi (k,t,t') = 0, \quad \text{(A.12)} \]
for \( P_{ib}(k) k_b = 0 \). Thus \( B_{ij}^\Pi (k) \) does not contribute to \( B_{ij} (k) \)
\[ \equiv P_{ib}(k) B_{bj}(k). \] Moreover we readily see that
\[ P_{ib}(k) B_{bj}^\gamma (k,t,t') = -\nu [k^2 Q_{ij}^0(k,t,t') + O(\lambda)]. \quad \text{(A.13)} \]

From (A.3), (A.7) \( \cap \) (A.13), we have
\[ \overline{B}_{ij}(k,t,t') = -\nu X_{ij}(k,t,t') + \lambda^2 I_{ij}(k,t,t'), \quad \text{(A.14a)} \]
with
\[ X_{ij}(k,t,t') = k^2 Q_{ij}^0(k,t,t') + O(\lambda), \quad \text{(A.14b)} \]
\[ I_{ij}(k,t,t') = I_{ij}^0(k,t,t') + O(\lambda), \quad \text{(A.14c)} \]

(A-5)
where $I_{ij}^{0}$ is given by (A.11).

Next let us consider about $C_{ij}$ in (A.4). By substituting the primitive expansions of $u, \psi$ and $\hat{\mathbf{E}}$ into (A.4b), we obtain expansions of $C_{ij}^{\pi}(\equiv \bar{C}_{ij}^{\pi})$. In $O(\lambda)$, it gives zero. In $O(\lambda^2)$, it yields

$$
(u_{m}^{0} - u_{m}^{\perp}) + (\hat{\mathbf{E}}^{\perp} - \hat{\mathbf{E}}^{\parallel}) + (\psi - \psi^{\perp}) \quad \text{(A.15)}
$$

The meanings of these terms should be understood analogously to (A.7). The third term $(\psi - \psi^{\perp})$ yields after substitution of (2.24), (cf. (A.8) and (A.9)),

$$(\psi - \psi^{\perp}) = \lambda^2 \int_{t'}^{t} ds \left[ i \partial_{x} \left( S_{imn} (q_{x}) [u_{m}^{0}(x,t) q_{nj}(x,t;x',t')] \right) u_{a}^{0}(x,s) \right] 
\equiv C_{ij}^{\pi}(x,t;x',t') \quad \text{(A.16)}
$$

By using $<u_{m}^{0} u_{a}^{0}> = U_{0}$ and (2.31), and transforming (A.16) into the wavevector space defined similarly to (2.28), we obtain

$$
P_{ib}^{ci} (k) C_{ij}^{\pi}(k,t,t') P_{cj}^{ci} (k) = -\lambda^2 \left[ \int_{t'}^{t} ds \int_{t'}^{t} \int_{t'}^{t} P_{ib}^{ci} (k) S_{bmn} (p) Q_{ma}^{ci} (-r,t,s) \right] 
\times P_{nj}^{ci} (k,t,t') \equiv \lambda^2 J_{ij}^{0}(k,t,t') \quad \text{(A.17)}
$$

The first two terms of (A.15) does not contribute to $\bar{C}_{ij}^{\pi}(k)$ $\equiv P_{ib}^{ci} (k) C_{bc}^{ci} (k) P_{cj}^{ci} (k)$ by the same reason as $\bar{B}_{ij}^{\pi}$ does not to $\bar{B}_{ij}^{\pi}(k)$. It is not difficult to see that $(\psi$-terms) gives zero in $O(\lambda^2)$ and to calculate $(\nu$-terms) in $O(\lambda^0)$. Thus we obtain

$$
\bar{C}_{ij}^{\pi}(k,t,t') \equiv P_{ib}^{ci} (k) C_{bc}^{ci} (k,t,t') P_{cj}^{ci} (k) = -\nu Y_{ij}^{\pi}(k,t,t') + \lambda^2 J_{ij}^{0}(k) \quad \text{(A.18a)}
$$

(A-6)
with
\[ Y_{ij}(k,t,t') = k^2 F_{ij}(k,t,t') + O(\lambda), \]  \hspace{1cm} (A.18b)
\[ J_{ij}(k,t,t') = J_{ij}^0(k,t,t') + O(\lambda), \]  \hspace{1cm} (A.18c)

where \( J_{ij}^0 \) is given by (A.17).

The expansion of \( A \) (see (2.18)) can be obtained in a similar manner to that of \( \bar{B} \) and \( \bar{C} \).

As is clear from the above calculations, we can obtain the expansions of \( A, \bar{B} \) and \( \bar{C} \) in terms of \( Q^0 \) and \( F^0 \). In the same way as for \( A, \bar{B} \) and \( \bar{C} \), we can expand also \( Q \) and \( F \) in terms of \( Q^0 \) and \( F^0 \). (These \( Q, F, Q^0 \) and \( F^0 \) have respectively only two, not four, time arguments.) By reverting these expansions of \( Q \) and \( F \), we obtain expansions of \( Q^0 \) and \( F^0 \) in functional powers of \( Q \) and \( F \). By the substitutions of these expansions of \( Q^0 \) and \( F^0 \) into \( A, \bar{B} \) and \( \bar{C} \) obtained above, we can express \( A, \bar{B} \) and \( \bar{C} \) in functional powers of \( Q \) and \( F \).

It is worthwhile to note that in the lowest order in \( \lambda \),
\[ Q^0 = Q \] \hspace{1cm} and \hspace{1cm} \[ F^0 = F. \]

In order to obtain the lowest order terms of \( X, I, Y \) and \( J \) in (A.14) and (A.18), we have only to replace \( Q^0 \) and \( F^0 \) in (A.14b,c), (A.11), (A.18b,c) and (A.17) by \( Q \) and \( F \) respectively. Thus we can obtain (2.36), (2.37) with (2.39), (2.40), (2.42), (2.43). Similarly, we can verify (2.35) with (2.38), (2.41). The structure of \( H_{ij} \) in (2.41) is the same as that which appears in DIA.
Finally, (2.48b), (2.48c) with (2.50), (2.52) can be obtained from (2.36) and (2.37) by using (2.47) and by noting

\[ P_{ib}(k)k_aS_{bmn}(p)P_{ma}(r)P_{ni}(k) \]

\[ = 2k_aP_{b}P_{m}P_{bn}(k)P_{ma}(r)/p^2 = 2k_aP_{m}P_{ma}(r)P_{b}P_{n}P_{bn}(k)/p^2 \]

\[ = 2k^2(1-y^2)(1-z^2), \quad \text{(A.19)} \]

where we have used \( P_{m}P_{ma}(r)=k_{m}P_{ma}(r) \), because \( k=p+r \) and \( r_{ma}(r)=0 \).

As for the derivation of (2.48a) with (2.49), (2.51), the reader may consult Leslie's book (1973).