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APPENDIX (to remain unpublished) to:

"2nd order effects in free convection"

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3. The First-Order Boundary Layer

Equation (2.11b) means that the pressure is constant across the boundary layer and takes its value at the outer edge. Hence $\partial p_0/\partial s = 0$. Equation (2.11d) is satisfied directly by the introduction of a stream function, ψ_0, defined by

$$u_0 = \frac{1}{r}\frac{\partial \psi_0}{\partial n}, \quad v_0 = -\frac{1}{r}\frac{\partial \psi_0}{\partial s}.$$

We then have

$$\begin{aligned}
&\psi_{0n}\left(\psi_{0ns} - \frac{1}{r}\frac{dr}{ds}\psi_{0n}\right) - \psi_{0s}\psi_{0nn} = r^2\Phi_{0}\sin\theta + r\,\psi_{0nnn}, \\
&\psi_{0n}\Phi_{0s} - \psi_{0s}\Phi_{0n} = \frac{r}{\sigma}\Phi_{0nn}, \\
&\psi_{0s}(s,0) = \psi_{0n}(s,0) = 0, \quad \Phi_0(s,0) = 1, \\
&\lim_{n\to\infty}\psi_{0n}(s,n) = \lim_{n\to\infty}\Phi_0(s,n) = 0,
\end{aligned} \tag{3.1}$$

where subscripts s,n denote differentiation.

Solutions are best obtained by dividing the range of integration in the s-direction into three parts. Near the stagnation point ψ_0, Φ_0 are expanded in power series in s with coefficients as functions of n which satisfy certain ordinary differential equations. For $s \gg 1$ asymptotic series are obtained in inverse powers of s, while in the middle region equations (3.1) usually must be integrated numerically. For isothermal streaming flows Görtler (1957) found that a transformation of the variables gave improved series solutions and Saville and Churchill (1967) have provided a similar transformation for flows with a body force. It is this that we use here.

We first use Mangler's transformation to reduce equations (3.1) to a form similar to that for planar flows. We write

$$\bar{s} = \int_0^s r^2\, ds, \qquad \bar{n} = rn,$$

$$\bar{\psi}_{0\bar{n}} = \frac{1}{r}\psi_{0n}, \quad \bar{\psi}_{0\bar{s}} = \frac{1}{r^2}\psi_{0s} - \frac{dr}{ds}\frac{n}{r^3}\psi_{0n}.$$

Then (3.1) become on dropping the bars

$$\left.\begin{aligned} &\psi_{0n}\psi_{0ns} - \psi_{0s}\psi_{0nn} = \sin\theta/r^2\, T_0 + \psi_{0nnn}, \\ &T_{0nn} + \sigma(\psi_{0s}T_{0n} - \psi_{0n}T_{0s}) = 0, \end{aligned}\right\} \quad (3.2)$$

Using the Saville-Churchill transformation

$$\xi = \int_0^s \left(\frac{\sin\theta}{r^2}\right)^{1/3} ds, \qquad \eta = \left(\frac{4\xi}{3}\right)^{-1/4}\left(\frac{\sin\theta}{r^2}\right)^{1/3} n,$$

$$\psi_0 = \left(\frac{4\xi}{3}\right)^{3/4} F_0(\xi,\eta), \qquad T_0 = T_0(\xi,\eta),$$

equations (3.2) become

$$\left.\begin{aligned} &\left(\frac{4\xi}{3}\right)(F_{0\eta}F_{0\xi\eta} - F_{0\xi}F_{0\eta\eta}) - F_0F_{0\eta\eta} + \frac{4}{3}K(\xi)F_{0\eta}^2 = T_0 + F_{0\eta\eta\eta}, \\ &\left(\frac{4\xi}{3}\right)(F_{0\eta}T_{0\xi} - F_{0\xi}T_{0\eta}) - F_0T_{0\eta} = \frac{1}{\sigma}T_{0\eta\eta}, \end{aligned}\right\} \quad (3.3)$$

where $F_0 = F_{0\eta} = 0$ and $T_0 = 1$ at $\eta = 0$ and $F_{0\eta} \to 0$, $T_0 \to 0$ as $\eta \to \infty$.
$K(\xi)$ corresponds to Gortler's "principal function" and is defined by

$$K(\xi) = \frac{1}{2} + \frac{\xi}{3}\frac{r^2}{\sin\theta}\frac{d}{d\xi}\left(\frac{\sin\theta}{r^2}\right).$$

The shape of the body enters these equations only through the coefficient $K(\xi)$ and for particular body shapes for which $K(\xi)$ is constant we may obtain similarity solutions. Here $K(\xi)$ is not constant and we seek solutions for $\xi \ll 1$ by expanding in powers of ξ. Thus

$$K(\xi) = \sum_{j=0}^{\infty} K_j \left(\frac{\xi}{k_0}\right)^{\alpha j},$$

where for round-nosed bodies $k_0 = \frac{3}{8}, \alpha = \frac{3}{4}$ and in particular for a paraboloid $K_1 = \frac{-3}{28}$, $K_2 = \frac{141}{980}$. ~~and~~ We expand F_0, T_0 in powers of $\xi^{3/4}$, i.e.

$$F_0 = F_{00} + K_1 F_{01} (\xi/k_0)^{3/4} + (K_2 F_{02} + K_1^2 F_{011})(\xi/k_0)^{3/2} + \ldots,$$

$$T_0 = T_{00} + K_1 T_{01} (\xi/k_0)^{3/4} + (K_2 T_{02} + K_1^2 T_{011})(\xi/k_0)^{3/2} + \ldots,$$

then equating coefficients of $\xi^{\alpha j}$ in (3.3) gives

$$\left.\begin{aligned} &F_{00}''' + F_{00}F_{00}'' - \tfrac{1}{2}F_{00}'^2 + T_{00} = 0, \\ &T_{00}'' + \sigma F_{00} T_{00}' = 0, \\ \text{where}\quad &F_{00}(0) = F_{00}'(0) = F_{00}'(\infty) = T_{00}(\infty) = 0,\ T_{00}(0) = 1; \end{aligned}\right\} \quad (3.4)$$

$$\left.\begin{aligned} &F_{01}''' + F_{00}F_{01}'' - 2F_{00}'F_{01}' + 2F_{00}''F_{01} + T_{01} = 4/3\, F_{00}'^2, \\ &T_{01}'' + \sigma(F_{00}T_{01}' - F_{00}'T_{01} + 2F_{01}T_{00}') = 0, \\ \text{where}\quad &F_{01}(0) = F_{01}'(0) = F_{01}'(\infty) = T_{01}(0) = T_{01}(\infty) = 0; \end{aligned}\right\} \quad (3.5)$$

$$\left.\begin{aligned} &F_{02}''' + F_{00}F_{02}'' - 3F_{00}'F_{02}' + 3F_{00}''F_{02} + T_{02} = 4/3\, F_{00}'^2, \\ &T_{02}'' + \sigma(F_{00}T_{02}' - 2F_{00}'T_{02} + 3F_{02}T_{00}') = 0, \\ \text{where}\quad &F_{02}(0) = F_{02}'(0) = F_{02}'(\infty) = T_{02}(0) = T_{02}(\infty) = 0; \end{aligned}\right\} \quad (3.6)$$

$$\left.\begin{aligned} &F_{011}''' + F_{00}F_{011}'' - 3F_{00}'F_{011}' + 3F_{00}''F_{011} + T_{011} = \tfrac{3}{2}F_{01}'^2 - 2F_{01}F_{01}'' + \tfrac{8}{3}F_{00}'F_{01}', \\ &T_{011}'' + \sigma(F_{00}T_{011}' - 2F_{00}'T_{011} + 3F_{011}T_{00}') = \sigma(T_{01}F_{01}' - 3F_{01}T_{01}'), \\ \text{where}\quad &F_{011}(0) = F_{011}'(0) = F_{011}'(\infty) = T_{011}(0) = T_{011}(\infty) = 0. \end{aligned}\right\} \quad (3.7)$$

For $\xi \gg 1$ we find that for a paraboloid

$$K(\xi) \sim \frac{3}{10} + \bar{K}_1 \xi^{-1} + \bar{K}_2 \xi^{-6/5} + O(\xi^{-9/5})$$

where $\bar{K}_1 = 0.052998$, $\bar{K}_2 = -\left(\frac{3}{10}\right)^{6/5}\frac{3}{20}$. This suggests that we look for asymptotic solutions of (3.3) in the form

$$F_0(\xi,\eta) = \bar F_{00} + \bar K_1 \bar F_{01}\xi^{-1} + \bar K_2 \bar F_{02}\xi^{-6/5} + \sum \beta_{\lambda_i}\bar F_{0\lambda_i}\xi^{-\lambda_i} + o(\xi^{-9/5}),$$

and similarly for T_0, where λ_i are the possible eigenvalues of (3.3) and β_{λ_i} are scaling factors chosen so that $\bar F_{0\lambda_i}''(0) = 1$. Equating coefficients of powers of ξ, we obtain

$$\left.\begin{aligned} &\bar F_{00}''' + \bar T_{00} + \bar F_{00}\bar F_{00}'' - \tfrac{2}{5}\bar F_{00}'^2 = 0,\\ &\bar T_{00}'' + \sigma\bar F_{00}\bar T_{00}' = 0,\\ &\text{where } \bar F_{00}(0) = \bar F_{00}'(0) = \bar F_{00}'(\infty) = \bar T_{00}(\infty) = 0,\ \bar T_{00}(0) = 1; \end{aligned}\right\} \quad (3.8)$$

$$\left.\begin{aligned} &\bar F_{01}''' + \bar T_{01} + \tfrac{4}{3}(\bar F_{00}'\bar F_{01}' - \bar F_{00}''\bar F_{01}) + \bar F_{00}\bar F_{01}'' + \bar F_{01}\bar F_{00}'' - \tfrac{4}{5}\bar F_{00}\bar F_{01}' = \tfrac{4}{3}\bar F_{00}'^2,\\ &\bar T_{01}'' + \tfrac{4\sigma}{3}(\bar F_{00}'\bar T_{01} - \bar F_{01}\bar T_{00}') + \sigma(\bar F_{00}\bar T_{01}' + \bar F_{01}\bar T_{00}') = 0,\\ &\text{where } \bar F_{01}(0) = \bar F_{01}'(0) = \bar F_{01}'(\infty) = \bar T_{01}(0) = \bar T_{01}(\infty) = 0; \end{aligned}\right\} \quad (3.9)$$

$$\left.\begin{aligned} &\bar F_{02}''' + \bar T_{02} + \tfrac{8}{5}(\bar F_{00}'\bar F_{02}' - \bar F_{02}\bar F_{00}'') + \bar F_{00}\bar F_{02}'' + \bar F_{02}\bar F_{00}'' - \tfrac{4}{5}\bar F_{00}\bar F_{02}' = \tfrac{4}{3}\bar F_{00}'^2,\\ &\bar T_{02}'' + \tfrac{8\sigma}{5}(\bar F_{00}'\bar T_{00} - \bar F_{02}\bar T_{00}') + \sigma(\bar F_{00}\bar T_{02}' + \bar F_{02}\bar T_{00}') = 0,\\ &\text{where } \bar F_{02}(0) = \bar F_{02}'(0) = \bar F_{02}'(\infty) = \bar T_{02}(0) = \bar T_{02}(\infty) = 0; \end{aligned}\right\} \quad (3.10)$$

$$\left.\begin{aligned} &\bar F_{0\lambda_i}''' + \bar T_{0\lambda_i} + \tfrac{4\lambda_i}{3}(\bar F_{00}'\bar F_{0\lambda_i}' - \bar F_{00}''\bar F_{0\lambda_i}) + \bar F_{00}\bar F_{0\lambda_i}'' + \bar F_{00}''\bar F_{0\lambda_i} - \tfrac{4}{5}\bar F_{00}'\bar F_{0\lambda_i}' = 0,\\ &\bar T_{0\lambda_i}'' + \sigma\tfrac{4\lambda_i}{3}(\bar F_{00}'\bar T_{0\lambda_i} + \bar F_{0\lambda_i}\bar T_{00}) + \sigma(\bar F_{00}\bar T_{0\lambda_i}' + \bar F_{0\lambda_i}\bar T_{00}') = 0,\\ &\text{where } \bar F_{0\lambda_i}(0) = \bar F_{0\lambda_i}'(0) = \bar F_{0\lambda_i}'(\infty) = \bar T_{0\lambda_i}(0) = \bar T_{0\lambda_i}(\infty) = 0;\ \bar F_{0\lambda_i}''(0) = 1. \end{aligned}\right\} \quad (3.11)$$

β_{λ_1} is found by comparing the velocity and temperature profiles with those obtained in the following section for some value of ξ where both solutions are valid. For $\sigma = 1$ we obtain $\beta_{\lambda_1} = -0.052 \pm 0.001$ by comparison of profiles at $\xi = 10$ and numerical integration of (3.11) gives $\lambda_1 = 0.962914$ for $\sigma = 1$.

3.1 Solution in the region where neither series holds

In order to calculate the outer flow we need solutions of the boundary layer equations at all points along the body. Van Dyke (1964) treating the flow past a parabolic cylinder achieved this by manipulation of series in s while Clark and Watson (1971) were able to use similarity solutions. Neither approach seems likely to succeed here and we must turn our attention to numerical methods. Terrill (1960), Merkin (1969), and Switzer (1969) have developed step-by-step finite difference integration procedures to continue the series solution for $\xi \ll 1$ into the region of validity of the series for $\xi \gg 1$. Large but sparse matrices must be inverted and, because the equations are non-linear, iteration is necessary at each step; the step length along the body is very small at first but is systematically increased later.

A much faster method due to Merk (1959) is applicable whenever $\xi K'(\xi)$ is small, as it is here. This is equivalent to assuming that $K(\xi)$ is almost constant, i.e. the departure from similarity solutions is small. In fact, the first term in the expansion (3.13) below is a local similarity solution and the subsequent terms are corrections to it. We first change independent variables from (ξ, η) to (K, η) and equations (3.3) become

$$\left.\begin{aligned} F_{0\eta\eta\eta} + T_{0\eta} + F_0 F_{0\eta\eta} - \tfrac{4}{3}K(\xi) F_{0\eta}^2 &= \frac{4\xi}{3} K'(\xi)\left(F_{0\eta}F_{0K\eta} - F_{0K}F_{0\eta\eta}\right), \\ T_{0\eta\eta} + \sigma F_0 T_{0\eta} &= \sigma \frac{4\xi}{3} K'(\xi)\left(F_{0\eta}T_{0K} - F_{0K}T_{0\eta}\right), \end{aligned}\right\} \quad (3.6)$$

where $F_0(0) = F_{0\eta}(0) = F_{0\eta}(\infty) = T_0(\infty) = 0,\ T_0(0) = 1.$

F_0 may then be expanded as

$$F_0 = F_{00}(K,\eta) + \frac{4\xi}{3} K'(\xi) F_{01}(K,\eta) + \left(\frac{4\xi}{3}\right)^2 \left[K''(\xi) F_{021}(K,\eta) + K'^2(\xi) F_{022}(K,\eta)\right] + \cdots, \quad (3.7)$$

and similarly for T_0. Substituting into (3.12) we get

$$F_{00}''' + F_{00}F_{00}'' + T_{00} - \tfrac{4}{3}K\,F_{00}'^{2} = 0,$$
$$T_{00}'' + \sigma F_{00}T_{00}' = 0, \qquad (3.14)$$
where $F_{00}(0) = F_{00}'(0) = F_{00}'(\infty) = T_{00}(\infty) = 0,\ T_{00}(0) = 1;$

$$F_{01}''' + T_{01} + F_{00}F_{01}'' + \tfrac{7}{3}F_{00}''F_{01} - \left(\tfrac{4}{3}+\tfrac{8}{3}K\right)F_{00}'F_{01}' = F_{00}'\frac{\partial F_{00}'}{\partial K} - F_{00}''\frac{\partial F_{00}}{\partial K},$$
$$T_{01}'' + \sigma\left(F_{00}T_{01}' + \tfrac{7}{3}F_{01}T_{00}' - \tfrac{4}{3}F_{00}'T_{01}\right) = \sigma\left(F_{00}'\frac{\partial T_{00}}{\partial K} - T_{00}'\frac{\partial F_{00}}{\partial K}\right), \qquad (3.15)$$
where $F_{01}(0) = F_{01}'(0) = F_{01}'(\infty) = T_{01}(0) = T_{01}(\infty) = 0;$

$$\left(\frac{\partial F_{00}}{\partial K}\right)''' + \left(\frac{\partial T_0}{\partial K}\right) + F_{00}\left(\frac{\partial F_{00}}{\partial K}\right)'' + F_{00}'\left(\frac{\partial F_{00}}{\partial K}\right) - \frac{8K}{3}F_{00}''\left(\frac{\partial F_{00}}{\partial K}\right)' = \frac{4}{3}F_{00}'^{2},$$
$$\left(\frac{\partial T_0}{\partial K}\right)'' + \sigma\left(F_{00}\left(\frac{\partial T_{00}}{\partial K}\right)' + T_{00}'\left(\frac{\partial F_{00}}{\partial K}\right)\right) = 0, \qquad (3.16)$$
where $\left(\frac{\partial F_{00}}{\partial K}\right)(0) = \left(\frac{\partial F_{00}}{\partial K}\right)'(0) = \left(\frac{\partial F_{00}}{\partial K}\right)'(\infty) = \left(\frac{\partial T_{00}}{\partial K}\right)(0) = \left(\frac{\partial T_{00}}{\partial K}\right)(\infty) = 0,$

Equations (3.14) correspond to those given by Merk (his equations (20), (25)) but his equations (21), (26) do not correspond exactly to our (3.15) because he has omitted a factor σ on the right of (26) and some terms on the left of (25), (26). In calculating $2\xi\Lambda'\,\partial(f',f)/\partial(\Lambda,\eta)$ where $f = f_0 + 2\xi\Lambda' f_1 + \dots$ we get $2\xi\Lambda'\{\partial(f_0',f_0)/\partial(\Lambda,\eta) + 2(f_1'f_0' - f_1f_0'') + o(\Lambda')\}$ and these additional terms in f mean that (25), (26) should read

$$f_1''' + f_0f_1'' + 3f_0''f_1 - 2(\Lambda+1)f_0'f_1' = \partial(f_0',f_0)/\partial(\Lambda,\eta),$$
$$\theta_1 + \sigma(f_0\theta_1' + 2f_1\theta_0' - \theta_1f_0') = \sigma\,\partial(\theta_0,f_0)/\partial(\Lambda,\eta).$$

It is now a straightforward matter to obtain the solution for any ξ by solving a system of ordinary differential equations in which K is a known parameter. In fact, initial guesses for the "shooting" method used here are furnished by the solution at the previous value of ξ and convergence is rapid.* The step-by-step integration is lengthy and yields only values of F_0' and T_0 for certain values of (ξ,η), the grid points.

In order to calculate $F_0(\infty)$ it is necessary to integrate F_0' numerically and $F_0''(0)$ and $T_0'(0)$ must be approximated by finite difference formulae. Values of $F_0(\infty), F_0''(0), T_0'(0)$ obtained by the two methods are shown in Table 2. Good agreement is reached for $0.1 \leq \xi \leq 10.0$ and outside this range the series solutions are to be used.

5. The Second Order Boundary Layer

The pressure gradient in (2.12a) is obtained from (2.12b) as

$$\frac{\partial p_1}{\partial s} = -\frac{\partial}{\partial s}\left\{\kappa\int_n^\infty u_0^2\,dn\right\} + \frac{\partial}{\partial s}\left\{\cos\theta\int_n^\infty T_0\,dn\right\}. \qquad (5.1)$$

We define the second order stream function ψ_1 to satisfy (2.12d) identically, i.e.

$$\left.\begin{aligned} \psi_{1n} &= r u_1 + n\sin\theta\, u_0 - \bar{x}\, r u_0 T_0, \\ \psi_{1s} &= -r v_1 - v_0(r n \kappa + n\sin\theta - \bar{x}\, r T_0). \end{aligned}\right\} \qquad (5.2)$$

Using (2.11), (5.2) and the definition of ψ_0, (2.12a) and (2.12c) become

$$\begin{aligned} &\left(\psi_{1n}\frac{\partial}{\partial s} - \psi_{1s}\frac{\partial}{\partial n}\right)(\psi_{0n}/r) + \left(\psi_{0n}\partial/\partial s - \psi_{0s}\partial/\partial n\right)(\psi_{1n}/r) - \psi_{1nnn} - r\sin\theta\, T_1 \\ &= -r\frac{\partial p_1}{\partial s} + \kappa r\left\{n\frac{\partial^2 u_0}{\partial n^2} + n\sin\theta\, T_0 + \frac{\partial u_0}{\partial n} - u_0 v_0\right\} + \sin\theta\left\{n\sin\theta\, T_0 - \frac{\partial u_0}{\partial n}\right\} + \\ &+ r\left(u_0\frac{\partial}{\partial s} + v_0\frac{\partial}{\partial n}\right)\left(\frac{n\sin\theta}{r}u_0\right) + \bar{x}\left\{\mu_1 T_0\frac{\partial^2 u_0}{\partial n^2} + (2+\mu_1)\frac{\partial T_0}{\partial n}\frac{\partial u_0}{\partial n} + \right. \\ &\left. + \left(1-\frac{1}{\sigma}\right)u_0\frac{\partial^2 T_0}{\partial n^2} - \sin\theta\, T_0^2 + \beta_1 T_0^2\sin\theta\right\}, \end{aligned} \qquad (5.3)$$

$$\begin{aligned} &\frac{1}{r}\left(\psi_{1n}\frac{\partial T_0}{\partial s} - \psi_{1s}\frac{\partial T_0}{\partial n}\right) + \frac{1}{r}\left(\psi_{0n}\frac{\partial T_1}{\partial s} - \psi_{0s}\frac{\partial T_1}{\partial n}\right) - \frac{1}{\sigma}\frac{\partial^2 T_1}{\partial n^2} \\ &= \frac{1}{\sigma}\left\{\kappa + \frac{\sin\theta}{r}\right\}\left\{n\frac{\partial^2 T_0}{\partial n^2} + \frac{\partial T_0}{\partial n}\right\} + \bar{x}\left\{\frac{k_1}{\sigma}\left\{T_0\frac{\partial^2 T_0}{\partial n^2} + \left(\frac{\partial T_0}{\partial n}\right)^2\right\}\right\}. \end{aligned} \qquad (5.4)$$

These equations are linear in ψ_1, T_1 and may be subdivided into a number of simpler problems. Of the terms on the right of (5.3), (5.4) those in κ represent the effect of longitudinal curvature (l) and those in $\sin\theta/r$ arise from transverse curvature effects (t). Buoyancy effects are present in both terms and also couple the equations through the term $r\sin\theta\, T_1$ on the left of (5.3). The second term in (5.1) is due to a heat flux (h) arising from the component of the body force along the surface and terms in $\bar{x}$ are due to the variation in the physical properties of the fluid with temperature. We distinguish those due to variation of density (dv) (a correction to the Boussinesq approximation) viscosity (vv), thermometric conductivity (tv) and coefficient of expansion (cv). The boundary conditions are $\psi_1 = \psi_{1n} = T_1 = 0$ at $n = 0$ and the matching conditions give $u_1 \to U_0$, $T_1 \to 0$ as $n \to \infty$. This means that we may distinguish the contribution of the displacement flow (d) as a specified tangential velocity at infinity. We thus define

$$\psi_1 = \psi_1^{(e)} + \psi_1^{(t)} + \psi_1^{(h)} + \bar{x}\left(\psi_1^{(dv)} + \mu_1 \psi_1^{(vv)} + k_1 \psi_1^{(tv)} + \beta_1 \psi_1^{(cv)}\right) + \psi_1^{(d)} \qquad (5.5)$$

and similarly for T_1.

We now define the operators Θ, Φ by

$$\left.\begin{aligned} \Theta(F,G) &\equiv \frac{\partial^3 F}{\partial n^3} - \left(\frac{\partial F}{\partial n}\frac{\partial u_0}{\partial s} - \frac{\partial F}{\partial s}\frac{\partial u_0}{\partial n}\right) - r\left(u_0 \frac{\partial}{\partial s} + v_0 \frac{\partial}{\partial n}\right)\left(\frac{1}{r}\partial F/\partial n\right) + r\sin\theta\, G\,; \\ \Phi(F,G) &\equiv \frac{1}{\sigma}\frac{\partial^2 G}{\partial n^2} - \frac{1}{r}\left(\frac{\partial T_0}{\partial s}\frac{\partial F}{\partial n} - \frac{\partial T_0}{\partial n}\frac{\partial F}{\partial s}\right) - \left(u_0 \frac{\partial G}{\partial s} + v_0 \frac{\partial G}{\partial n}\right). \end{aligned}\right\} \quad (5.6)$$

Then

$$\begin{aligned}
\Theta(\psi_1^{(e)}, T_1^{(e)}) &= -r\frac{\partial}{\partial s}\left\{K\int_n^\infty u_0^2\,dn\right\} - rK\left\{\frac{\partial u_0}{\partial n} - u_0 v_0 + n\frac{\partial^2 u_0}{\partial n^2} + n\sin\theta\, T_0\right\}, \\
\Phi(\psi_1^{(e)}, T_1^{(e)}) &= -K/\sigma\ \partial/\partial n\left\{n\,\partial T_0/\partial n\right\}, \\
\Theta(\psi_1^{(t)}, T_1^{(t)}) &= -r\left(u_0\,\partial/\partial s + v_0\,\partial/\partial n\right)\left(\frac{\sin\theta}{r} n u_0\right) + \sin\theta\left\{\frac{\partial u_0}{\partial n} - n\sin\theta\, T_0\right\}, \\
\Phi(\psi_1^{(\theta)}, T_1^{(\theta)}) &= -\frac{1}{\sigma}\frac{\sin\theta}{r}\frac{\partial}{\partial n}\left\{n\frac{\partial T_0}{\partial n}\right\}, \\
\Theta(\psi_1^{(h)}, T_1^{(h)}) &= r\frac{\partial}{\partial s}\left\{\int_n^\infty \cos\theta\, T_0\,dn\right\}, \\
\Theta(\psi_1^{(dv)}, T_1^{(dv)}) &= -\left\{2r\frac{\partial T_0}{\partial n}\frac{\partial u_0}{\partial n} + \left(1-\frac{1}{\sigma}\right) r u_0 \frac{\partial^2 T_0}{\partial n^2} - r\sin\theta\, T_0^2\right\}, \qquad (5.7) \\
\Theta(\psi_1^{(vv)}, T_1^{(vv)}) &= -r\left\{T_0\frac{\partial^2 u_0}{\partial n^2} + \frac{\partial T_0}{\partial n}\frac{\partial u_0}{\partial n}\right\}, \\
\Phi(\psi_1^{(tv)}, T_1^{(tv)}) &= -\frac{1}{\sigma}\left\{T_0\frac{\partial^2 T_0}{\partial n^2} + \left(\frac{\partial T_0}{\partial n}\right)^2\right\}, \\
\Theta(\psi_1^{(cv)}, T_1^{(cv)}) &= -r\sin\theta\, T_0^2, \\
\Phi(\psi_1^{(h)}, T_1^{(h)}) &= \Phi(\psi_1^{(dv)}, T_1^{(dv)}) = \Phi(\psi_1^{(vv)}, T_1^{(vv)}) = 0, \\
\Theta(\psi_1^{(tv)}, T_1^{(tv)}) &= \Phi(\psi_1^{(cv)}, T_1^{(cv)}) = 0; \\
\Theta(\psi_1^{(d)}, T_1^{(d)}) &= \Phi(\psi_1^{(d)}, T_1^{(d)}) = 0.
\end{aligned}$$

The boundary conditions for all components are

$$\psi_1 = \psi_{1\eta} = T_1 = 0 \quad \text{at } \eta = 0; \quad \psi_{1\eta} \to 0,\ T_1 \to 0 \quad \text{as } \eta \to \infty$$

except for $\psi_1^{(d)}$ which satisfies $\psi_{1\eta}^{(d)} \to Hu_1$ as $\eta \to \infty$

After applying the Saville-Churchill transformation equations (5.6),(5.7) become

$$\text{\textcircled{H}}(F_1, T_1) \equiv F_1''' + T_1 - \frac{4\xi}{3}\left\{(F_{0\xi}F_1' - F_0''F_{1\xi}) + (F_0'F_{1\xi}' - F_1''F_{0\xi})\right\} + F_0''F_1 + F_0F_1'' - \frac{8}{3}k(\xi)F_1$$

$$\bar{\Phi}(F_1 T_1) \equiv \frac{1}{\sigma}T_1'' - \frac{4\xi}{3}\left\{(T_{0\xi}F_1' - T_0'F_{1\xi}) + (F_0'T_{1\xi} - F_{0\xi}T_1')\right\} + T_0'F_1 + F_0T_1', \qquad (5$$

$$\text{\textcircled{H}}(F_1^{(l)}, T_1^{(l)}) = L_1(\xi)\left\{F_0'' + \eta F_0''' + \eta T_0 + F_0F_0' + \frac{4\xi}{3}F_0'F_{0\xi}\right\} + L_2(\xi)\int_\eta^\infty F_0'^2\,d\eta,$$

$$\bar{\Phi}(F_1^{(l)}, T_1^{(l)}) = L_1(\xi)\frac{1}{\sigma}\left\{T_0' + \eta T_0''\right\},$$

$$\text{\textcircled{H}}(F_1^{(t)}, T_1^{(t)}) = T_1(\xi)\left\{\frac{4\xi}{3}\eta(F_0'F_{0\xi}' - F_0''F_{0\xi}) - \frac{4\xi}{3}F_0'F_{0\xi} - \eta F_0F_0'' - F_0F_0' - F_0'' + \eta T_0\right\} +$$

$$+ T_2(\xi)\eta F_0'^2,$$

$$\bar{\Phi}(F_1^{(t)}, T_1^{(t)}) = \frac{1}{\sigma}T_1(\xi)\left\{T_0' + \eta T_0''\right\},$$

$$\text{\textcircled{H}}(F_1^{(h)}, T_1^{(h)}) = H_1(\xi)\eta T_0 + H_2(\xi)\int_\eta^\infty T_0\,d\eta + H_3(\xi)\int_\eta^\infty T_{0\xi}\,d\eta,$$

$$\text{\textcircled{H}}(F_1^{(dw)}, T_1^{(dw)}) = \left(\frac{1}{\sigma} - 1\right)F_0'T_0'' - 2T_0'F_0'' - T_0^2, \qquad (5.9)$$

$$\text{\textcircled{H}}(F_1^{(vw)}, T_1^{(vw)}) = -\left\{T_0F_0''' + T_0'F_0''\right\},$$

$$\bar{\Phi}(F_1^{(tw)}, T_1^{(tw)}) = -\frac{1}{\sigma}\left\{T_0T_0'' + T_0'^2\right\},$$

$$\text{\textcircled{H}}(F_1^{(cw)}, T_1^{(cw)}) = -T_0^2,$$

$$\bar{\Phi}(F_1^{(h)}, T_1^{(h)}) = \bar{\Phi}(F_1^{(dw)}, T_1^{(dw)}) = \bar{\Phi}(F_1^{(vw)}, T_1^{(vw)}) = 0,$$

$$\text{\textcircled{H}}(F_1^{(tw)}, T_1^{(tw)}) = \bar{\Phi}(F_1^{(cw)}, T_1^{(cw)}) = 0,$$

$$\text{\textcircled{H}}(F_1^{(d)}, T_1^{(d)}) = \bar{\Phi}(F_1^{(d)}, T_1^{(d)}) = 0,$$

and the boundary conditions for all components are $F_1 = F_1' = T_1 = 0$ at $\eta = 0$ and $F_1' \to 0, T_1 \to 0$ as $\eta \to \infty$ except for $F_1^{(d)}$ which satisfies $F_1^{(d)\prime} \to D(\xi)$ as $\eta \to \infty$. The shape of the body enters these equations only through the "secondary" functions

$$L_1(\xi) = -\frac{K}{rZ}\left(\frac{4\xi}{3}\right)^{1/4}, \quad L_2(\xi) = -\frac{d}{d\xi}\left(\frac{KZ}{r}\left(\frac{4\xi}{3}\right)^{5/4}\right)\frac{1}{Z^2},$$

$$T_1(\xi) = -\left(\frac{4\xi}{3}\right)^{1/4} Z^2, \quad T_2(\xi) = T_1(\xi)\left(1 + \frac{4\xi Z'}{Z}\right),$$

$$H_1(\xi) = -\left\{\frac{dr}{ds} + r^3\left(Z' - \frac{Z}{4\xi}\right)\right\}\left(\frac{4\xi}{3}\right)^{1/4}\frac{\cos\theta}{r^4 Z^4},$$

$$H_2(\xi) = \frac{d}{d\xi}\left\{\left(\frac{4\xi}{3}\right)^{1/4}\frac{\cos\theta}{rZ^3}\right\}\frac{1}{Z^2}, \quad H_3(\xi) = \left(\frac{4\xi}{3}\right)^{1/4}\frac{\cos\theta}{rZ^3},$$

$$D(\xi) = \left(\frac{4\xi}{3}\right)^{-1/2} r^{1/3}(1+r^2)^{1/6} U_1(\xi),$$

where $$Z = \left(\frac{s\sin\theta}{r}\right)^{1/3}$$

which are of fundamental importance in determining the nature of the secondary flow. The introduction of $L_2(\xi)$ as well as $L_1(\xi)$ and so on is a reflection of the incompleteness of our sub-division (5.5) of equations (5.3) and (5.4), but any further sub-division would be unnecessarily cumbersome. For a paraboloid these functions are

$$L_1(\xi) = -\left(\frac{4\xi}{3}\right)^{1/4} r^{-2/3}(1+r^2)^{-4/3}, \quad L_2(\xi) = \frac{4\xi}{3}L_1' + \frac{8L_1}{3}k(\xi),$$

$$T_1(\xi) = -\left(\frac{4\xi}{3}\right)^{1/4} r^{-2/3}(1+r^2)^{-1/3}, \quad T_2(\xi) = T_1(\xi)\left(4k(\xi) - 1\right),$$

$$H_1(\xi) = -\left(\frac{4\xi}{3}\right)^{1/4}\left(r^{-8/3}(1+r^2)^{-1/3} + (k(\xi) - 3/4)/\xi\right),$$

$$H_2(\xi) = \left(\frac{4\xi}{3}\right)^{-3/4}\left(8k(\xi)/3 - 1\right), \quad H_3(\xi) = \left(4\xi/3\right)^{1/4}.$$

Expanding for $\xi \ll 1$ we get $L_1 \sim L_2 \sim T_1 \sim 2T_2 \sim -\frac{7}{3}H_1 \sim \frac{7}{4}H_2 \sim -2^{-1/4}$, $H_3\int_\eta^\infty T_{0\xi}\,d\eta \sim \int_\eta^\infty T_{01}\,d\eta$, $D \sim 0.6331$ (for $\sigma = 1$) and the first terms in the expansions of F_1, T_1 about $\xi = 0$ are obtained by substituting these values into equations (5.9). For $\xi \gg 1$ we find that

$$L_1(\xi) \sim -(2/5)\left(4\xi/3\right)^{-3/4}, \quad L_2(\xi) \sim 1/5(2/5)\left(4\xi/3\right)^{-3/4},$$

$$T_1(\xi) \sim -(2/5)^{1/4}\left(10\xi/3\right)^{-3/20}, \quad T_2(\xi) \sim -1/5(2/5)^{1/4}\left(10\xi/3\right)^{-3/20},$$

$$H_1(\xi) \sim 1/5\,(4\xi/3)^{-3/4}, \quad H_2(\xi) \sim -1/5\,(4\xi/3)^{-3/4},$$

$$H_3(\xi)\int_\eta^\infty T_{0\xi}\,d\eta \sim -8/15\,(4\xi/3)^{-7/4}\int_\eta^\infty \bar{T}_{01}\,d\eta,$$

$$D(\xi) \sim \bar{F}_{00}(\infty)\left(\frac{4\xi}{3}\right)^{-3/4}\left\{1+\frac{3}{20}\log\frac{10\xi}{3}-\frac{1}{4}(\gamma+2\psi(5/4)+\pi)\right\},$$

the latter being obtained by use of equation (4.7). This suggests that we expand F_1 in inverse powers of ξ as

$$F_1^{(e)} = (2/5)(4\xi/3)^{-3/4}\{\bar{F}_{10}^{(e)}+\ldots\},$$

$$F_1^{(t)} = (2/5)^{1/4}(10\xi/3)^{-3/10}\{\bar{F}_{10}^{(t)}+\ldots\},$$

$$F_1^{(h)} = 1/5\,(4\xi/3)^{-3/4}\{\bar{F}_{10}^{(h)}+\ldots\},$$

$$F_1^{(v)} = \bar{F}_{10}^{(v)} + O(\xi^{-3/4}),$$

$$F_1^{(d)} = \bar{F}_{00}(\infty)\left\{(4\xi/3)^{-3/4}\log\xi\,\bar{F}_{10}^{(d)} + a_1(4\xi/3)^{-3/4}\bar{F}_{11}^{(d)}+\ldots\right\},$$

and similarly for T_1, where (v) denotes (dv), (vv), (tv), (cv) in turn and

$$a_1 = \tfrac{1}{4}\left(4-\pi-\gamma-2\psi(5/4)-3/5\log(10/3)\right).$$

It is readily shown that the first eigensolutions of (5.9) for $\sigma = 1$ satisfy equations identical to (3.11) and therefore the first eigenvalue for $\sigma = 1$ is 0.962914. Now let

$$\Theta_0(\bar{F}_{10},\bar{T}_{10}) \equiv \bar{F}_{10}''' + \bar{T}_{10} - 4/5\,\bar{F}_{00}'\bar{F}_{10}' + \bar{F}_{00}''\bar{F}_{10} + \bar{F}_{00}\bar{F}_{10}'',$$

$$\Phi_0(\bar{F}_{10},\bar{T}_{10}) \equiv \tfrac{1}{\sigma}\bar{T}_{10}'' + \bar{T}_{00}'\bar{F}_{00} + \bar{F}_{00}\bar{T}_{10}',$$

$$\Theta_1(\bar{F}_{10},\bar{T}_{10}) \equiv \bar{F}_{10}''' + \bar{T}_{10} + 1/5\,\bar{F}_{10}'\bar{F}_{00}' + \bar{F}_{00}\bar{F}_{10}'',$$

$$\Phi_1(\bar{F}_{10},\bar{T}_{10}) \equiv \tfrac{1}{\sigma}\bar{T}_{10}'' + \bar{F}_{00}'\bar{T}_{10} + \bar{F}_{00}\bar{T}_{10}',$$

$$\Theta_2(\bar{F}_{10},\bar{T}_{10}) \equiv \bar{F}_{10}''' + \bar{T}_{10} - 3/5\,\bar{F}_{00}'\bar{F}_{10}' + 4/5\,\bar{F}_{00}''\bar{F}_{10} + \bar{F}_{00}\bar{F}_{10}'',$$

$$\Phi_2(\bar{F}_{10},\bar{T}_{10}) \equiv \tfrac{1}{\sigma}\bar{T}_{10}'' + 4/5\,\bar{T}_{00}'\bar{F}_{10} + \bar{F}_{00}\bar{T}_{10}' + 1/5\,\bar{F}_{00}'\bar{T}_{10},$$

then the first approximations to the components of F_1, T_1 for $\xi \gg 1$ satisfy

$$\Theta_1(\bar{F}_{10}^{(e)},\bar{T}_{10}^{(e)}) = -\{\bar{F}_{00}'' + \eta\bar{F}_{00}''' + \eta\bar{T}_{00} + \bar{F}_{00}\bar{F}_{00}'\} + 1/5\int_\eta^\infty \bar{F}_{00}'^2\,d\eta,$$

$$\Phi_1(\bar{F}_{10}^{(e)},\bar{T}_{10}^{(e)}) = -\tfrac{1}{\sigma}\{\bar{T}_{00}' + \eta\bar{T}_{00}''\},$$

$$\Theta_2(\bar{F}_{10}^{(t)},\bar{T}_{10}^{(t)}) = -\{\eta\bar{T}_{00} - \eta\bar{F}_{00}\bar{F}_{00}'' - \bar{F}_{00}\bar{F}_{00}' - \bar{F}_{00}''\} - 2\eta/25\,\bar{F}_{00}'^2,$$

$$\Phi_2(\bar{F}_{10}^{(t)},\bar{T}_{10}^{(t)}) = -\tfrac{1}{\sigma}\{\bar{T}_{00}' + \eta\bar{T}_{00}''\},$$

$$\text{⊕}_1(\bar F_{10}^{(h)}, \bar T_{10}^{(h)}) = -\int_1^\infty \bar T_{00}\, d\eta + 1\, \bar T_{00},$$
$$\text{⊕}_0(\bar F_{10}^{(dv)}, \bar T_{10}^{(dv)}) = (\tfrac{1}{\sigma}-1)\bar F_{00}'\bar T_{00}'' - 2\bar T_{00}'\bar F_{00}'' + \bar T_{00}^2,$$
$$\text{⊕}_0(\bar F_{10}^{(vv)}, \bar T_{10}^{(vv)}) = -\tfrac{1}{\sigma}(\bar T_{00}\bar F_{00}''' + \bar T_{00}'\bar F_{00}''),$$
$$\Phi_0(\bar F_{10}^{(tv)}, \bar T_{10}^{(tv)}) = -\tfrac{1}{\sigma}(\bar T_{00}\bar T_{00}'' + \bar T_{00}'^2),$$
$$\text{⊕}_0(\bar F_{10}^{(cv)}, \bar T_{10}^{(cv)}) = -\bar T_{00}^2,$$
$$\Phi_1(\bar F_{10}^{(h)}, \bar T_{10}^{(h)}) = \Phi_0(\bar F_{10}^{(dv)}, \bar T_{10}^{(dv)}) = \Phi(\bar F_{10}^{(vv)}, \bar T_{10}^{(vv)}) = 0,$$
$$\text{⊕}_0(\bar F_1^{(tv)}, \bar T_1^{(tv)}) = \Phi_0(\bar F_1^{(cv)}, \bar T_1^{(cv)}) = 0,$$
$$\text{⊕}_1(\bar F_{10}^{(d)}, \bar T_{10}^{(dv)}) = \Phi_1(F_{10}^{(d)}, T_{10}^{(d)}) = 0,$$
$$\text{⊕}_1(\bar F_{11}^{(d)}, \bar T_{11}^{(d)}) = \frac{4}{3a_1}(\bar F_{00}'\bar F_{10}^{(d)\prime} - \bar F_{00}''\bar F_{10}^{(d)}),$$
$$\Phi_1(\bar F_{11}^{(d)}, \bar T_{11}^{(d)}) = \frac{4}{3a_1}(\bar F_{00}'\bar T_{10}^{(d)} - \bar T_{00}'\bar F_{10}^{(d)}).$$

The boundary conditions for all components are $\bar F_{10} = \bar F_{10}' = \bar T_{10} = 0$ at $\eta = 0$, $\bar F_{10}' \to 0, \bar T_{10} \to 0$ as $\eta \to \infty$ except $\bar F_{10}^{(d)\prime} \to 1$, $\bar F_{11}^{(d)} \to 1$ as $\eta \to \infty$. The first eigen-solution is almost comparable to the first terms in the (l) and (h) components and the second term in the expansion is important in the (t) component, so that we use this first approximation only for very large ξ e.g. $\xi > 100$.

As in section 3 we adopt Merk's method in the region where neither series holds and accept the first term as being sufficiently accurate as far as $\xi = 100$ (the error is $O(10^{-4})$). Whilst the individual terms on the right-hand-sides of (5.9) vary slowly with ξ, the secondary functions, L_1, L_2, and so on do not and it is necessary to remove them as factors. Writing $F_1^{(e)} = L_1 F_1^{(e)}$ etc. we are left with coefficients of the form L_2/L_1 and $\frac{4\xi}{3} L_1'/L_1$ which do vary slowly, at least for a paraboloid. If we now define

$$\text{⊕}(F_1, T_1, L_1) = F_{10}''' + T_{10} + (F_{10}F_{00}'' + F_{00}F_{10}'') - 8K/3\,(F_{00}'F_{10}) - \frac{4\xi}{3}\frac{L_1'}{L_1}(F_{00}'F_{10}' - F_{00}''F_{10})$$

$$\Phi(F_1, T_1, L_1) \equiv \frac{1}{\sigma} T_{10}'' + (F_{10} T_{00}' + F_{00} T_{10}') - \frac{4J}{3} \frac{L_1'}{L_1} \left(F_{00}' T_{10} - T_{00}' F_{10}\right),$$

and take $\Theta(F_1, T_1, 0)$ to mean that the term in L_1'/L_1 is omitted, we have

$$\Theta(F_1^{(\ell)}, T_1^{(\ell)}, L_1) = F_{00}'' + \eta F_{00}''' + \eta T_0 + F_{00} F_{00}' + L_2/L_1 \int_\eta^\infty F_{00}'^2 \, d\eta,$$

$$\Phi(F_1^{(\ell)}, T_1^{(\ell)}, L_1) = \frac{1}{\sigma}(T_{00}' + \eta T_{00}''),$$

$$\Theta(F_1^{(t)}, T_1^{(t)}, T_1) = \eta T_{00} - \eta F_{00} F_{00}'' - F_{00} F_{00}' - F_{00}'' + T_2/T_1 \, \eta F_{00}'^2;$$

$$\Phi(F_1^{(t)}, T_1^{(t)}, T_1) = \frac{1}{\sigma}(T_{00}' + \eta T_{00}''),$$

$$\Theta(F_1^{(h)}, T_1^{(h)}, H_1) = \eta T_{00} + H_2/H_1 \int_\eta^\infty T_{00} \, d\eta,$$

and the remaining equations and the boundary conditions are as in (5.9) with the appropaiate new left-hand-sides ($\Theta(F_1^{(dw)}, T_1^{(dw)}, 0)$, $\Theta(F_1^{(d)}, T_1^{(d)}, D_1)$ etc.).