On some problems of similarity flow of fluid with a free surface

By Z.N. DOBROVOL'SKAYA Volume 36, 1969, 805.

On the derivation of Equation (4.19)

Using the Sokhotsky-Plemelj formula for the limit value of the Cauchy-type integral of expression (3.18), we obtain the limit value $\zeta'(u+i0)$ of function $\zeta'(w)$

$$\zeta'(u+i0) = -icu^{-\frac{1}{2}+\alpha}(u-1)^{-\alpha}\exp\left[-\int_{-\infty}^{0}\frac{f(u_1)}{u_1-u}du_1\right]e^{-i\pi f(u)} . \qquad \text{(i)}$$

Substituting the expression (i) into the kinematic condition (3.15) and taking into account function

$$-icu^{-\frac{1}{2}+\alpha}(u-1)^{-\alpha}\exp\left[-\int_{-\infty}^{0}\frac{f(u_1)}{u_1-u}du_1\right]$$

to be real for negative u , we obtain

$$\mathrm{Re}\left\{ie^{-i\pi f(u)}\int_0^u e^{i\pi f(u)}\left[-iu^{-\frac{1}{2}+\alpha}(u-1)^{-\alpha}\exp\left(-\int_{-\infty}^{0}\frac{f(u_1)}{u_1-u}du_1\right)+\right.\right.$$
$$\left.\left.+i\frac{c_0^2}{c^2}u^{-1-\alpha}(u-1)^{-1+\alpha}\exp\left(\int_{-\infty}^{0}\frac{f(u_1)}{u_1-u}du_1\right)\right]du\right\}=0 \quad (-\infty<u\leq 0) \qquad \text{(ii)}$$

or

$$\sin\pi f(u)\int_0^u\left[G(u)\cos\pi f(u)-\frac{c_0^2}{c^2}H(u)\sin\pi f(u)\right]du -$$
$$-\cos\pi f(u)\int_0^u\left[G(u)\sin\pi f(u)+\frac{c_0^2}{c^2}H(u)\cos\pi f(u)\right]du=0, \qquad \text{(iii)}$$

where G(u) and H(u) are the following real functions:

$$G(u)=-iu^{-\frac{1}{2}+\alpha}(u-1)^{-\alpha}\exp\left[-\int_{-\infty}^{0}\frac{f(u_1)}{u_1-u}du_1\right],$$
$$H(u)=u^{-1-\alpha}(u-1)^{-1+\alpha}\exp\left[\int_{-\infty}^{0}\frac{f(u_1)}{u_1-u}du_1\right]. \qquad (-\infty<u\leq 0) \quad \text{(iv)}$$

Let us consider equation (iii) and introduce the notations

$$M(u) = \int_0^u \left[G(u) \sin \pi f(u) + \frac{c_0^2}{c^2} H(u) \cos \pi f(u) \right] du, \qquad \text{(v)}$$

$$N(u) = \int_0^u \left[G(u) \cos \pi f(u) - \frac{c_0^2}{c^2} H(u) \sin \pi f(u) \right] du. \qquad \text{(vi)}$$

Then equation (iii) can be written in the form

$$N(u) \sin \pi f(u) - M(u) \cos \pi f(u) = 0 . \qquad \text{(vii)}$$

The differentiation of (vii) with respect to u gives

$$\pi f'(u) \left(N \cos \pi f + M \sin \pi f \right) + \sin \pi f \left(G \cos \pi f - \frac{c_0^2}{c^2} H \sin \pi f \right) -$$
$$- \cos \pi f \left(G \sin \pi f + \frac{c_0^2}{c^2} H \cos \pi f \right) =$$
$$= \pi f'(u) \left(N \cos \pi f + M \sin \pi f \right) - \frac{c_0^2}{c^2} H(u) = 0 \qquad \text{(viii)}$$

and

$$f'(u) = \frac{1}{\pi} \frac{c_0^2}{c^2} \frac{H(u)}{N(u) \cos \pi f(u) + M(u) \sin \pi f(u)} . \qquad \text{(ix)}$$

Let us analyse the denominator of the right-hand side of (ix). From (vii) we obtain

$$\frac{M(u)}{N(u)} = \frac{\sin \pi f(u)}{\cos \pi f(u)} . \qquad \text{(x)}$$

Thus, we can write

$$M(u) = R(u) \sin \pi f(u), \quad N(u) = R(u) \cos \pi f(u), \qquad \text{(xi)}$$

where $R(u)$ is an unknown function. So, the denominator of (ix) can be written in the form

$$N \cos \pi f + M \sin \pi f \equiv R(u). \qquad \text{(xii)}$$

The differentiation of (xii), taking into account (v), (vi), and (xi), gives

$$R'(u) = N'\cos\pi f - \pi f' N \sin\pi f + M'\sin\pi f + \pi f' M \cos\pi f =$$

$$= G \cos^2\pi f - \frac{c_o^2}{c^2} H \sin\pi f \cos\pi f + G \sin^2\pi f +$$

$$+ \frac{c_o^2}{c^2} H \sin\pi f \cos\pi f - \pi f' N \sin\pi f + \pi f' M \cos\pi f =$$

$$= G + \pi f' (M \cos\pi f - N \sin\pi f) =$$

$$= G + \pi f' (R \sin\pi f \cos\pi f - R \sin\pi f \cos\pi f) = G. \quad \text{(xiii)}$$

Hence,

$$R(u) = \int_0^u G(u)\, du \qquad \text{(xiv)}$$

($M(0) = N(0) = R(0) = 0$, see (xii) and (v)-(vi)), and we obtain from (ix), finally, the following equation:

$$f'(u) = \frac{1}{\pi} \frac{c_o^2}{c^2} \frac{H(u)}{\int_0^u G(u)\, du}. \qquad \text{(xv)}$$

The integration of (xv) with respect to u, taking into account that $f(-\infty) = 0$, gives

$$f(u) = -\frac{1}{\pi} \frac{c_o^2}{c^2} \int_{-\infty}^{u} \frac{u^{-1-\alpha}(u-1)^{-1+\alpha} \exp\left[\int_{-\infty}^{0} \frac{f(u_1)}{u_1 - u} du_1\right]}{\int_0^u i u^{-\frac{1}{2}+\alpha}(u-1)^{-\alpha} \exp\left[-\int_{-\infty}^{0} \frac{f(u_1)}{u_1 - u} du_1\right] du}\, du ,$$

quod erat demonstrandum.

B. On the determination of functional factor c_o^2/c^2.

After the solution of equation (3.19) has been found, function $\varsigma(w)$ can be determined from the expression

$$\varsigma(w) = \varsigma_B - ic \int_0^w w^{-\frac{1}{2}+\alpha} (w-1)^{-\alpha} \exp\left[-\int_{-\infty}^{0} \frac{f(u)}{u - w} du\right] dw, \qquad \text{(xvi)}$$

and the complex velocity at any point of the flow region is determined by the formula

$$V'(\zeta) = \bar{\zeta}_B - c_o^2 \left[\int_0^w w^{-\frac{3}{2}} (w-1)^{-1} \frac{1}{\zeta'(w)} \, dw \right]_{w = w(\zeta)} . \qquad \text{(xvii)}$$

Let us derive the equations for the determination of three real parameters c , c_o , and η_B . As

$$\zeta(w) = \zeta_B + \int_0^w \zeta'(w) \, dw , \qquad \text{(xviii)}$$

condition (3.21) gives

$$-i = \zeta_B + \int_0^1 \zeta'(u) \, du . \qquad \text{(xix)}$$

Equation (xix) , taking into account

$$\zeta'(u) = e^{-i\left(\frac{1}{2}\pi + \alpha_o\right)} |\zeta'(u)| \quad \text{for} \quad 0 \leq u \leq 1 , \qquad \text{(xx)}$$

can be written in the form

$$-i = \zeta_B - i e^{-i\alpha_o} \int_0^1 |\zeta'(u)| \, du , \qquad \text{(xxi)}$$

which in turn gives only one condition

$$-1 = \eta_B - \cos\alpha_o \int_0^1 |\zeta'(u)| \, du . \qquad \text{(xxii)}$$

After the expression for $|\zeta'(u)|$ ($0 \leq u \leq 1$) has been substituted into the above condition, we obtain the first equation for the determination of c , c_o , and η_B :

$$1 + \eta_B = c \cdot \cos\alpha_o \int_0^1 u^{-\frac{1}{2}+\alpha} (1-u)^{-\alpha} \exp\left[- \int_{-\infty}^0 \frac{f(u_1)}{u_1 - u} \, du_1 \right] du . \qquad \text{(xxiii)}$$

Condition (3.22) , after the limit value of $\zeta(w)$ on the real semi-axis $-\infty < u \leq 0$ has been substituted into it, is reduced to the form

$$\eta_B = ic \int_{-\infty}^{0} u^{-\frac{1}{2}+\alpha} (u-1)^{-\alpha} \exp\left[-\int_{-\infty}^{0} \frac{f(u_1)}{u_1 - u} du_1 \right] \sin \pi f(u)\, du \,, \qquad \text{(xxiv)}$$

which is the second equation for c , c_o , and η_B . To obtain the third equation, we use condition (3.23) at the wedge apex. Then (3.23) and (xvii) give

$$i = \bar{\xi}_B - c_o^2 \int_0^1 u^{-\frac{3}{2}} (u-1)^{-1} \frac{1}{\xi'(u)} du \,, \qquad \text{(xxv)}$$

from where only one condition is obtained :

$$1 + \eta_B = \frac{c_o^2}{c} \cos\alpha_o \int_0^1 u^{-1-\alpha} (1-u)^{-1+\alpha} \exp\left[\int_{-\infty}^{0} \frac{f(u_1)}{u_1 - u} du_1 \right] du \qquad \text{(xxvi)}$$

(taking into account (xx)).

Equations (xxiii), (xxiv), and (xxvi) give three conditions for the determination of c , c_o , and η_B . The value c_o^2/c^2 in equation (3.19) is obtained from (xxiii) and (xxvi) in the form (3.24).

B. Dobrynin