Supplementary Material

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1. Derivation of the adjoint model

This section provides the derivation of the adjoint model. Based on Eq. (A 16) in the appendix of the main paper, the perturbation of Lagrangian can be rewritten as

$$\begin{aligned}\delta L = \langle \delta\eta, \eta - \eta_M \rangle &+ \underbrace{\left\langle \mathcal{M}'(\delta\eta), \lambda_1 \right\rangle}_{\text{Term1}} + \underbrace{\left\langle \mathcal{M}'(\delta\Phi^s), \lambda_2 \right\rangle}_{\text{Term2}} + \underbrace{\left\langle \mathcal{M}'(\delta W), \gamma \right\rangle}_{\text{Term3}} \\ &+ \underbrace{\sum_{m=2}^{M} \left\langle \mathcal{M}'(\delta\Phi^{(m)}(x,0,t)), \alpha_1^{(m)} \right\rangle}_{\text{Term4}} + \underbrace{\sum_{m=2}^{M} \left\langle \mathcal{M}'(\delta\Phi_z^{(m)}(x,0,t)), \alpha_2^{(m)} \right\rangle}_{\text{Term5}} \\ &+ \left\langle -\delta\Phi^{(1)} + \delta\Phi^s, \alpha_1^{(1)} \right\rangle + \left\langle -\delta\Phi_z^{(1)}, \alpha_2^{(1)} \right\rangle . \end{aligned} \tag{S1}$$

With the state variables and adjoint variables satisfying the boundary conditions as shown in (2.29) and (2.31) in the main paper, each term can be further written as

$$\begin{aligned}\text{Term1} &= \langle \delta\eta_t + \nabla\delta\Phi^s \cdot \nabla\eta + \nabla\Phi^s \cdot \nabla\delta\eta - (1 + \nabla\eta\cdot\nabla\eta)\delta W - 2(\nabla\delta\eta\cdot\nabla\eta)W, \lambda_1 \rangle \\ &= \left\langle \delta\eta, -\lambda_{1,t} - \nabla\cdot(\lambda_1\nabla\Phi^s) + 2\nabla\cdot(\lambda_1 W\nabla\eta) \right\rangle \\ &\quad + \langle \delta\Phi^s, -\nabla\cdot(\lambda_1\nabla\eta) \rangle + \langle \delta W, -(1+\nabla\eta\cdot\nabla\eta)\lambda_1 \rangle , \end{aligned} \tag{S2}$$

$$\begin{aligned}\text{Term2} &= \left\langle \delta\Phi_t^s + \nabla\delta\Phi^s\cdot\nabla\Phi^s + g\delta\eta - (1+\nabla\eta\cdot\nabla\eta)W\delta W - (\nabla\delta\eta\cdot\nabla\eta)W^2, \lambda_2 \right\rangle \\ &= \left\langle \delta\eta, g\lambda_2 + \nabla\cdot(\lambda_2 W^2\nabla\eta) \right\rangle + \left\langle \delta\Phi^s, -\lambda_{2,t} - \nabla\cdot(\lambda_2\nabla\Phi^s) \right\rangle \\ &\quad + \langle \delta W, -(1+\nabla\eta\cdot\nabla\eta)W\lambda_2 \rangle , \end{aligned} \tag{S3}$$

$$\begin{aligned}\text{Term3} &= \Bigg\langle -\delta W + \delta\eta \sum_{m=1}^{M}\sum_{l=1}^{M-m} \frac{\eta^{l-1}}{(l-1)!}\frac{\partial^{l+1}}{\partial z^{l+1}}\Phi^{(m)}(x,0,t) \\ &\quad + \sum_{m=1}^{M}\sum_{l=0}^{M-m} \frac{\eta^l}{l!}\left(a^{(l+1)}\left[\delta\Phi^{(m)}(x,0,t)\right] + b^{(l+1)}\left[\delta\Phi_z^{(m)}(x,-h,t)\right]\right), \gamma \Bigg\rangle \\ &= \langle \delta W, -\gamma \rangle + \left\langle \delta\eta, \left[\sum_{m=1}^{M}\sum_{l=1}^{M-m} \frac{\eta^{l-1}}{(l-1)!}\frac{\partial^{l+1}}{\partial z^{l+1}}\Phi^{(m)}(x,0,t)\right]\gamma \right\rangle \end{aligned}$$

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$$+\sum_{m=1}^{M}\left\langle \delta\Phi^{(m)}(x,0,t), \sum_{l=0}^{M-m} a^{(l+1)}\left[\frac{\eta^l}{l!}\gamma\right]\right\rangle$$
$$+\sum_{m=1}^{M}\left\langle \delta\Phi_z^{(m)}(x,-h,t), \sum_{l=0}^{M-m} b^{(l+1)}\left[\frac{\eta^l}{l!}\gamma\right]\right\rangle, \tag{S4}$$

$$\begin{aligned}
\text{Term4} = &\sum_{m=2}^{M}\Bigg\langle -\delta\Phi^{(m)}(x,0,t) - \delta\eta\sum_{l=1}^{m-1}\frac{\eta^{l-1}}{(l-1)!}\frac{\partial^l}{\partial z^l}\Phi^{(m-l)}(x,0,t) \\
&-\sum_{l=1}^{m-1}\frac{\eta^l}{l!}\left(a^{(l)}\left[\delta\Phi^{(m-l)}(x,0,t)\right] + b^{(l)}\left[\delta\Phi_z^{(m-l)}(x,-h,t)\right]\right), \alpha_1^{(m)}\Bigg\rangle \\
=&\sum_{m=2}^{M}\left\langle \delta\Phi^{(m)}(x,0,t), -\alpha_1^{(m)}\right\rangle \\
&+\left\langle \delta\eta, -\sum_{m=2}^{M}\left[\sum_{l=1}^{m-1}\frac{\eta^{l-1}}{(l-1)!}\frac{\partial^l}{\partial z^l}\Phi^{(m-l)}(x,0,t)\right]\alpha_1^{(m)}\right\rangle \\
&+\sum_{m=2}^{M}\sum_{l=1}^{m-1}\left\langle \delta\Phi^{(m-l)}(x,0,t), -a^{(l)}\left[\frac{\eta^l}{l!}\alpha_1^{(m)}\right]\right\rangle \\
&+\sum_{m=2}^{M}\sum_{l=1}^{m-1}\left\langle \delta\Phi_z^{(m-l)}(x,-h,t), -b^{(l)}\left[\frac{\eta^l}{l!}\alpha_1^{(m)}\right]\right\rangle \\
=&\sum_{m=2}^{M}\left\langle \delta\Phi^{(m)}(x,0,t), -\alpha_1^{(m)}\right\rangle \\
&+\left\langle \delta\eta, -\sum_{m=2}^{M}\left[\sum_{l=1}^{m-1}\frac{\eta^{l-1}}{(l-1)!}\frac{\partial^l}{\partial z^l}\Phi^{(m-l)}(x,0,t)\right]\alpha_1^{(m)}\right\rangle \\
&+\sum_{k=1}^{M-1}\left\langle \delta\Phi^{(k)}(x,0,t), -\sum_{l=1}^{M-k} a^{(l)}\left[\frac{\eta^l}{l!}\alpha_1^{(l+k)}\right]\right\rangle \\
&+\sum_{k=1}^{M-1}\left\langle \delta\Phi_z^{(k)}(x,0,t), -\sum_{l=1}^{M-k} b^{(l)}\left[\frac{\eta^l}{l!}\alpha_1^{(l+k)}\right]\right\rangle,
\end{aligned} \tag{S5}$$

$$\begin{aligned}
\text{Term5} = &\sum_{m=2}^{M}\Bigg\langle -\delta\Phi_z^{(m)}(x,-h,t) + \sum_{l=1}^{m-1}\nabla\cdot\frac{\beta^{l-1}\delta\beta}{(l-1)!}\frac{\partial^{l-1}}{\partial z^{l-1}}\nabla\Phi^{(m-l)}(x,-h,t) \\
&+\sum_{l=1}^{m-1}\nabla\cdot\frac{\beta^l}{l!}\nabla\left(c^{(l-1)}\left[\delta\Phi^{(m-l)}(x,0,t)\right] + d^{(l-1)}\left[\delta\Phi_z^{(m-l)}(x,-h,t)\right]\right), \alpha_2^{(m)}\Bigg\rangle \\
=&\sum_{m=2}^{M}\left\langle \delta\Phi_z^{(m)}(x,-h,t), -\alpha_2^{(m)}\right\rangle \\
&+\left\langle \delta\beta, -\sum_{m=2}^{M}\sum_{l=1}^{m-1}\left(\frac{\beta^{l-1}}{(l-1)!}\nabla\alpha_2^{(m)}\right)\cdot\left(\nabla\frac{\partial^{l-1}}{\partial z^{l-1}}\Phi^{(m-l)}(x,-h,t)\right)\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{m=2}^{M} \sum_{l=1}^{m-1} \left\langle \delta\Phi^{(m-l)}(x,0,t), c^{(l-1)} \left[\nabla \cdot \left(\frac{\beta^l}{l!} \alpha_2^{(m)} \right) \right] \right\rangle \\
&+ \sum_{m=2}^{M} \sum_{l=1}^{m-1} \left\langle \delta\Phi_z^{(m-l)}(x,-h,t), d^{(l-1)} \left[\nabla \cdot \left(\frac{\beta^l}{l!} \alpha_2^{(m)} \right) \right] \right\rangle \\
= &\sum_{m=2}^{M} \left\langle \delta\Phi_z^{(m)}(x,-h,t), -\alpha_2^{(m)} \right\rangle \\
&+ \left\langle \delta\beta, -\sum_{m=2}^{M} \sum_{l=1}^{m-1} \left(\frac{\beta^{l-1}}{(l-1)!} \nabla\alpha_2^{(m)} \right) \cdot \left(\nabla \frac{\partial^{l-1}}{\partial z^{l-1}} \Phi^{(m-l)}(x,-h,t) \right) \right\rangle \\
&+ \sum_{k=1}^{M-1} \left\langle \delta\Phi^{(k)}(x,0,t), \sum_{l=1}^{M-k} c^{(l-1)} \left[\nabla \cdot \left(\frac{\beta^l}{l!} \alpha_2^{(l+k)} \right) \right] \right\rangle \\
&+ \sum_{k=1}^{M-1} \left\langle \delta\Phi_z^{(k)}(x,-h,t), \sum_{l=1}^{M-k} d^{(l-1)} \left[\nabla \cdot \left(\frac{\beta^l}{l!} \alpha_2^{(l+k)} \right) \right] \right\rangle . \qquad \text{(S6)}
\end{aligned}
$$

By collecting all the terms, we can write Eq. (S1) as

$$
\begin{aligned}
\delta L = &\left\langle \delta\beta, -\sum_{m=2}^{M} \sum_{l=1}^{m-1} \left(\frac{\beta^{l-1}}{(l-1)!} \nabla\alpha_2^{(m)} \right) \cdot \left(\nabla \frac{\partial^{l-1}}{\partial z^{l-1}} \Phi^{(m-l)}(x,-h,t) \right) \right\rangle \\
&+ \langle \delta\eta, \mathcal{M}^*(\eta) \rangle + \langle \delta\Phi^s, \mathcal{M}^*(\Phi^s) \rangle + \langle \delta W, \mathcal{M}^*(W) \rangle \\
&+ \sum_{m=1}^{M} \left\langle \delta\Phi^{(m)}(x,0,t), \mathcal{M}^*(\Phi^{(m)}(x,0,t)) \right\rangle \\
&+ \sum_{m=1}^{M} \left\langle \delta\Phi_z^{(m)}(x,-h,t), \mathcal{M}^*(\Phi_z^{(m)}(x,-h,t)) \right\rangle , \qquad \text{(S7)}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{M}^*(\eta) = &(\eta - \eta_M) - \lambda_{1,t} - \nabla \cdot (\lambda_1 \nabla\Phi^s) + 2\nabla \cdot (\lambda_1 W \nabla\eta) \\
&+ g\lambda_2 + \nabla \cdot (\lambda_2 W^2 \nabla\eta) + \left[\sum_{m=1}^{M} \sum_{l=1}^{M-m} \frac{\eta^{l-1}}{(l-1)!} \frac{\partial^{l+1}}{\partial z^{l+1}} \Phi^{(m)}(x,0,t) \right] \gamma \\
&- \sum_{m=2}^{M} \left[\sum_{l=1}^{m-1} \frac{\eta^{l-1}}{(l-1)!} \frac{\partial^l}{\partial z^l} \Phi^{(m-l)}(x,0,t) \right] \alpha_1^{(m)}, \qquad \text{(S8)}
\end{aligned}
$$

$$
\mathcal{M}^*(\Phi^s) = -\nabla \cdot (\lambda_1 \nabla\eta) - \lambda_{2,t} - \nabla \cdot (\lambda_2 \nabla\Phi^s) + \alpha_1^{(1)}, \qquad \text{(S9)}
$$

$$
\mathcal{M}^*(W) = -(1 + \nabla\eta \cdot \nabla\eta)\lambda_1 - (1 + \nabla\eta \cdot \nabla\eta) W \lambda_2 - \gamma, \qquad \text{(S10)}
$$

$$\mathcal{M}^*(\Phi^{(m)}(x,0,t)) = -\alpha_1^{(m)} + \sum_{l=0}^{M-m} a^{(l+1)}\left[\frac{\eta^l}{l!}\gamma\right] + \begin{cases} -\sum_{l=1}^{M-m} a^{(l)}\left[\frac{\eta^l}{l!}\alpha_1^{(l+m)}\right] & m = 1,2,..M-1 \\ 0 & m = M \end{cases} + \begin{cases} \sum_{l=1}^{M-m} c^{(l-1)}\left[\nabla\cdot\left(\frac{\beta^l}{l!}\alpha_2^{(l+m)}\right)\right] & m = 1,2,..M-1 \\ 0 & m = M \end{cases}, \tag{S11}$$

$$\mathcal{M}^*(\Phi_z^{(m)}(x,-h,t)) = -\alpha_2^{(m)} + \sum_{l=0}^{M-m} b^{(l+1)}\left[\frac{\eta^l}{l!}\gamma\right] + \begin{cases} -\sum_{l=1}^{M-m} b^{(l)}\left[\frac{\eta^l}{l!}\alpha_1^{(l+m)}\right] & m = 1,2,..M-1 \\ 0 & m = M \end{cases} + \begin{cases} \sum_{l=1}^{M-m} d^{(l-1)}\left[\nabla\cdot\left(\frac{\beta^l}{l!}\alpha_2^{(l+m)}\right)\right] & m = 1,2,..M-1 \\ 0 & m = M \end{cases}. \tag{S12}$$

By setting $\mathcal{M}^*(.) = 0$, we obtain the adjoint equations as shown in Eqs. (2.22)-(2.26) in the main paper and Eq. (S7) is reduced to

$$\delta L = \left\langle \delta\beta, -\sum_{m=2}^{M}\sum_{l=1}^{m-1}\left(\frac{\beta^{l-1}}{(l-1)!}\nabla\alpha_2^{(m)}\right)\cdot\left(\nabla\frac{\partial^{l-1}}{\partial z^{l-1}}\Phi^{(m-l)}(x,-h,t)\right)\right\rangle. \tag{S13}$$

2. Bathymetry detection without multiscale optimisation

Figure S1 presents the detected bathymetry after 400 iterations without the multiscale optimisation method, together with the comparison with the ground truth along $x = 0$ m and $y = 0$ m for the case Shoal-Mono in table 1 of the main paper. As shown in figure S1(a), the detected bathymetry deviates significantly from the ground truth. Figure S1(b) shows that the detection result underestimates the bump and cannot reflect the slope accurately in the y direction. In the x direction, the detection result exhibits high-wavenumber oscillations (figure S1c), a phenomenon consistent with the high-wavenumber modes in the gradients of the cost function shown in figure 5(a) of the main paper.

To further examine the optimisation performance, we present the variations of the cost function and detection error with the optimisation iteration in figure S2. The cost function decreases rapidly at the beginning of the optimisation, but the detection error saturates at a large value of approximately 0.88 after the second optimisation iteration. This phenomenon is associated with the high-wavenumber oscillations shown in figure S1. The gradient of cost function with respect to the bathymetry contains the high-wavenumber components, but for a global optimal solution, i.e., the ground truth, the high-wavenumber components in the optimisation procedure should be avoided or addressed later in the process, which is the idea of the multiscale optimisation method presented in § 2.5 in the main paper.

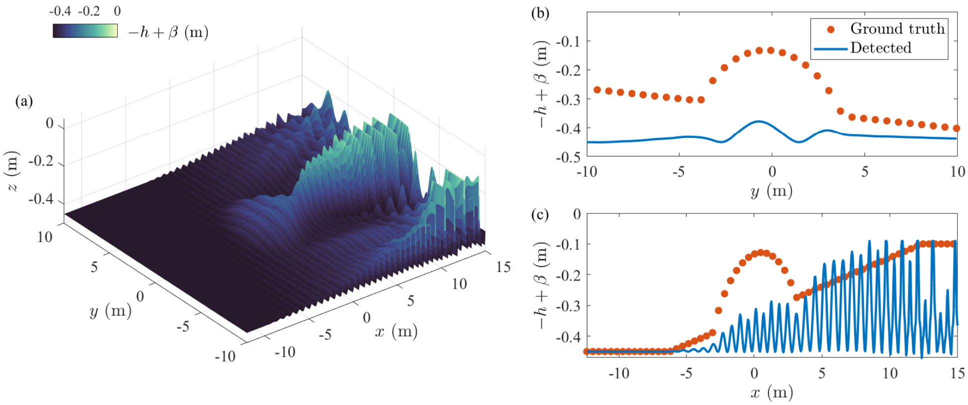

FIGURE S1. Detected bathymetry without the multiscale optimisation for case Shoal-Mono in the main paper. (*a*) Detected bathymetry after 400 iterations; (*b*) comparison between detected bathymetry and ground truth along $x = 0$ m; (*c*) same as (*b*) but for the line along $y = 0$ m.

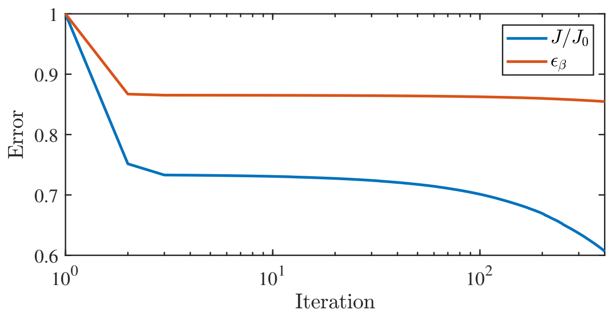

FIGURE S2. Variations of cost function and detection error with the number of optimisation iterations without the multiscale optimisation method for case Shoal-Mono in the main paper.