Dissipation characterisation in a turbulent boundary layer, based on SPIV and DNS data

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1 Definition of dissipation and its role in the RANS equations

1.1 The instantaneous equations

If we denote $\tilde{u}_i$ the instantaneous velocity components of a Newtonian incompressible fluid, ρ the fluid density, and μ the dynamic viscosity, then the flow is described for incompressible flow by the continuity equation and the Navier-Stokes equations given respectively by:

$$\begin{aligned} \frac{\partial \tilde{u}_i}{\partial x_i} &= 0 \\ \rho\frac{\partial \tilde{u}_i}{\partial t} + \rho\tilde{u}_j\frac{\partial \tilde{u}_i}{\partial x_j} &= -\frac{\partial \tilde{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\tilde{\tau}_{ij}\right) \end{aligned} \tag{1}$$

For an incompressible Newtonian fluid, $\tilde{\tau}_{ij} = 2\mu\tilde{s}_{ij}$ is the viscous stress tensor, $\nu = \mu/\rho$ is the kinematic viscosity and $\tilde{s}_{ij} = (1/2)(\partial\tilde{u}_i/\partial x_j + \partial\tilde{u}_j/\partial x_i)$ is the instantaneous strain-rate tensor, In this paper only flows of constant density will be of interest. We note for future reference that the deformation-rate tensor, $\partial\tilde{u}_i/\partial x_j$, can be decomposed into its symmetric and anti-symmetric parts as $\partial\tilde{u}_i/\partial x_j = \tilde{s}_{ij} + \tilde{\omega}_{ij}$ where $\tilde{\omega}_{ij} = (1/2)(\partial\tilde{u}_i/\partial x_j - \partial\tilde{u}_j/\partial x_i)$ is the rotation-rate tensor.

1.2 The averaged equations

In turbulence the flow velocity field is usually decomposed into mean and fluctuating parts using the Reynolds decomposition, say $\tilde{u}_i = U_i + u_i$, where $U_i = \langle\tilde{u}_i\rangle$ and $\langle u_i\rangle = 0$. Any instantaneous flow variable (e.g., $\tilde{p}, \tilde{\tau}_{ij}, \tilde{\varepsilon}$...) can be similarly decomposed. Based on such a decomposition, the Reynolds-averaged equations can be obtained:

$$
\begin{aligned}
\frac{\partial U_i}{\partial x_i} &= 0 \\
\rho\frac{\partial U_i}{\partial t} + \rho U_j\frac{\partial U_i}{\partial x_j} &= -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mathcal{T}_{ij} - \rho\langle u_i u_j\rangle\right)
\end{aligned} \tag{2}
$$

where $\mathcal{T}_{ij} = 2\mu S_{ij}$, $S_{ij} = (1/2)(\partial U_i/\partial x_j + \partial U_j/\partial x_i)$ and $\rho\langle u_i u_j\rangle$ are the well-known Reynolds stresses.

The classical next step in a statistical approach of turbulence is to derive transport equations for this Reynolds stresses:

$$
\begin{gathered}
\rho\frac{\partial\langle u_i u_j\rangle}{\partial t} + \rho U_l\frac{\partial\langle u_i u_j\rangle}{\partial x_l} = \\
-\rho\langle u_i u_l\rangle\frac{\partial U_j}{\partial x_l} - \rho\langle u_j u_l\rangle\frac{\partial U_i}{\partial x_l} \\
+\frac{\partial}{\partial x_l}\left(\langle u_i\tau_{jl}\rangle + \langle u_j\tau_{il}\rangle - \langle u_i p\rangle\delta_{jl} - \langle u_j p\rangle\delta_{il} - \rho\langle u_i u_j u_l\rangle\right) \\
+\left(\langle p\frac{\partial u_i}{\partial x_j}\rangle + \langle p\frac{\partial u_j}{\partial x_i}\rangle\right) \\
-\langle\tau_{il}\frac{\partial u_j}{\partial x_l}\rangle - \langle\tau_{jl}\frac{\partial u_i}{\partial x_l}\rangle
\end{gathered} \tag{3}
$$

The term on the second line is the production of the corresponding Reynolds stress, the third line corresponds to the diffusion by, respectively, the viscosity, the pressure fluctuations and the velocity fluctuations, the fourth line is the redistribution term and the last line corresponds to the dissipation.

The important point for this paper is to emphasize the form of the viscous diffusion (first two terms of the third line of equation (3) and the dissipation (last line of the same equation). Note that the notion of "dissipation" appears here for the first time as the equations have raised by one order and the transported terms have now the dimension of a kinetic energy. As equation (3) represents a set of 6 independent equations, the "dissipation" appears here as a tensor:

$$
\varepsilon_{ij} = \frac{1}{\rho}\langle\tau_{il}\frac{\partial u_j}{\partial x_l}\rangle + \frac{1}{\rho}\langle\tau_{jl}\frac{\partial u_i}{\partial x_l}\rangle \tag{4}
$$

which can also be written as:

$$
\varepsilon_{ij} = 2\nu\langle\frac{\partial u_i}{\partial x_l}\frac{\partial u_j}{\partial x_l}\rangle + \nu\langle\frac{\partial u_l}{\partial x_i}\frac{\partial u_j}{\partial x_l}\rangle + \nu\langle\frac{\partial u_l}{\partial x_j}\frac{\partial u_i}{\partial x_l}\rangle \tag{5}
$$

From equation (3), by contraction of the two indices i and j (or by taking the trace of the set of equations), and dividing by 2, the equation for the turbulence kinetic energy k is directly obtained as:

$$
\rho\frac{Dk}{Dt} = \rho\langle u_i u_j\rangle\frac{\partial U_j}{\partial x_j} + \frac{\partial}{\partial x_j}\left[-\langle p u_j\rangle_{-}\frac{\rho}{2}\langle u_i u_i u_j\rangle + \langle u_i\tau_{ij}\rangle\right] \; - \; \rho\varepsilon \tag{6}
$$

where the first term on the right hand side is the production of TKE, the second one is the diffusion term by, respectively, the pressure fluctuations, the velocity fluctuations and the viscosity. $\varepsilon \equiv (1/\rho)\langle\tau_{ij}\partial u_i/\partial x_j\rangle = \frac{1}{2}\varepsilon_{ii}$ is the true turbulence dissipation, $k \equiv \langle u_i u_i\rangle/2$ is the average kinetic energy per unit mass associated with turbulence, and the left-hand-side has been compacted using the averaged material derivative defined to be:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U_j \frac{\partial}{\partial x_j}. \tag{7}$$

Note that since ensemble averages are implied, the above equations (2), (3) and (6) are valid in both statistically stationary and non-stationary flows; i.e., there has been no assumption of time-averaging! Note also for future reference the form of the viscous diffusion term, $\langle u_i \tau_{ij} \rangle$.

1.3 The dissipation, ε

For reasons of simplicity and to conform to usual practice, in the following we will replace $\langle \epsilon \rangle$ by just ϵ. It is relatively easy to show that $\langle \varepsilon \rangle$ is indeed the real dissipation (at least in a Newtonian flow), since it is always positive *and occurs with the opposite sign in the Reynolds-averaged entropy equations* (Note that the latter is not true for the pseudo-dissipation $\mathcal{D}$ defined below.).

It is easy to show from the definitions that for incompressible Newtonian fluids:

$$\varepsilon = \langle \frac{1}{\rho} \tau_{ij} \frac{\partial u_i}{\partial x_j} \rangle = 2\nu \langle s_{ij} s_{ij} \rangle, \tag{8}$$

since only the symmetrical part of the velocity deformation survives the double-contraction of the indices. Expanding the strain-rate tensor in equation (8) yields the form most useful for experimental evaluation:

$$\varepsilon = \nu \langle \left[\frac{\partial u_i}{\partial x_j} \right]^2 \rangle + \nu \langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \rangle \tag{9}$$

If the flow is *statistically homogeneous*, then homogeniety alone implies that the upper and/or lower indices can be interchanged [George and Hussein, 1991]; e.g., permutting the lower yields:

$$\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \rangle = \langle \frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_j} \rangle \tag{10}$$

If the flow is also incompressible, continuity together with homogeneity implies that cross-derivative terms are identically zero ALWAYS since $\partial u_i / \partial x_i = 0$ instantaneously. But there are very few turbulent flows which are homogeneous. So strictly speaking, this is almost never true, and is at best an approximation. When it is true the flow is said to be "locally homogeneous" [George and Hussein, 1991].

1.4 The 'pseudo-dissipation'

Another quantity will also be investigated in the present contribution; namely the pseudo dissipation, which is defined as a tensor when looking at the Reynolds stress equations:

$$\mathcal{D}_{ij} = 2\nu \langle \frac{\partial u_i}{\partial x_l} \frac{\partial u_j}{\partial x_l} \rangle \tag{11}$$

or as a scalar when looking at the turbulence energy:

$$\mathcal{D} = \nu \langle \left[\frac{\partial u_i}{\partial x_j} \right]^2 \rangle. \tag{12}$$

Expanding the double contraction of the velocity gradient tensor yields:

$$\mathcal{D} = \nu\langle\left[\frac{\partial u_i}{\partial x_j}\right]^2\rangle = \nu\langle[s_{ij} + \omega_{ij}]^2\rangle = \nu[\langle s_{ij}s_{ij}\rangle + \langle\omega_{ij}\omega_{ij}\rangle] \tag{13}$$

where ω_{ij} is the fluctuating rotation-rate tensor and the product $s_{ij}\omega_{ij} = 0$ since s_{ij} is symmetric and ω_{ij} antisymmetric. Note that the mean square fluctuating rotation-rate tensor, $\langle\omega_{ij}\omega_{ij}\rangle$, is NOT in general equal to the mean square fluctuating strain-rate, $\langle s_{ij}s_{ij}\rangle$. So in general ε and $\mathcal{D}$ are different, even when the flow is turbulent.

A distinction should consequently be made between the true dissipation, ε, and the pseudo-dissipation, $\mathcal{D}$, which is defined from the mean square deformation rate. Even though $\mathcal{D}$ has only squared terms, it is NOT the dissipation, since it is ε which appears in the kinetic energy equation (6) and with opposite sign in the entropy transport equation (not detailed here).

From equations (5) and (11) the dissipation tensor ε_{ij} can be rewritten as:

$$\varepsilon_{ij} = \mathcal{D}_{ij} + \nu\langle\frac{\partial u_l}{\partial x_i}\frac{\partial u_j}{\partial x_l}\rangle + \nu\langle\frac{\partial u_l}{\partial x_j}\frac{\partial u_i}{\partial x_l}\rangle \tag{14}$$

and the dissipation ε as:

$$\varepsilon = \mathcal{D} + \nu\langle\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\rangle \tag{15}$$

There is a particular circumstance in which ε and $\mathcal{D}$ are equal, namely *incompressible homogeneous turbulence*. Then the second term on the right hand side of equation (9) is equal to zero and the two terms are equal. In a homogeneous turbulence, also $\langle\omega_{ij}\omega_{ij}\rangle = \langle s_{ij}s_{ij}\rangle$ [George and Hussein, 1991]. But only in homogeneous turbulence!

To use this simplification, even if the flow is not globally homogeneous, it is often assumed that turbulence is *locally homogeneous*, which was *defined* by [George and Hussein, 1991] to be the situation where the enstrophy and mean square strain-rate are *approximately* equal. This is usually justified by the same arguments used for the assumption of *local isotropy*; namely that the small scales which dominate velocity derivatives exist in an environment that is relatively independent of the large scale motions.

Also, turbulence modelers usually prefer to use a form of the energy equations in which $\mathcal{D}$ (and not ε) occurs explicitly:

$$\rho\frac{Dk}{Dt} = \rho\langle u_i u_l\rangle\frac{\partial U_j}{\partial x_l} + \frac{\partial}{\partial x_j}\left[-\langle pu_j\rangle - \rho\frac{1}{2}\langle u_i u_i u_j\rangle + \mu\frac{\partial k}{\partial x_j}\right] \; - \; \rho\mathcal{D} \tag{16}$$

Note that both ε and the viscous diffusion term $\langle u_i\tau_{ij}\rangle$ on the right-hand-side of equation (6) have been replaced by $\mathcal{D}$ and $\partial k/\partial x_j$.

Equation (16) is exact and strictly equivalent mathematically to equation (6). But with "dissipation" and "diffusion" terms that are not physically easy to interpret and which are not obtained directly by contraction of equation 3.

References

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2 Supplementary figures

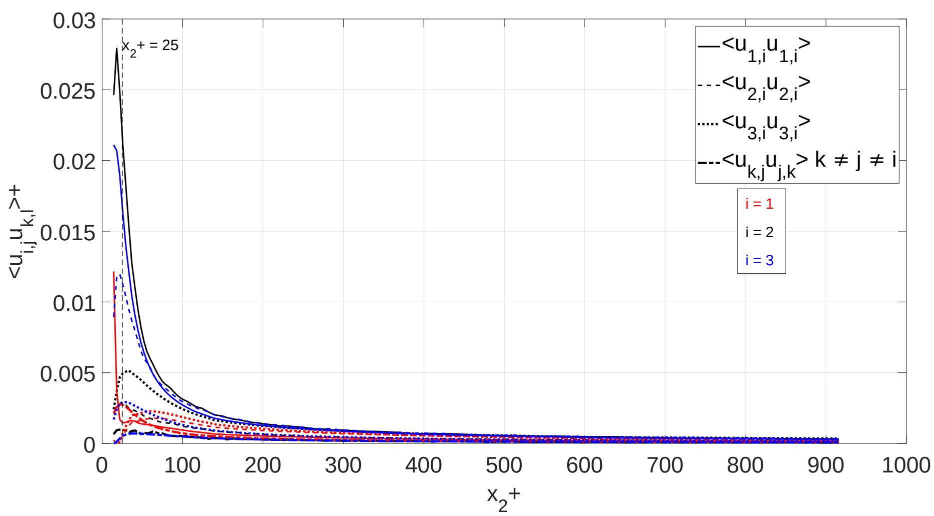

Figure 1: Linear-linear plot of all dissipation derivatives moments deduced from the PIV data plotted together.

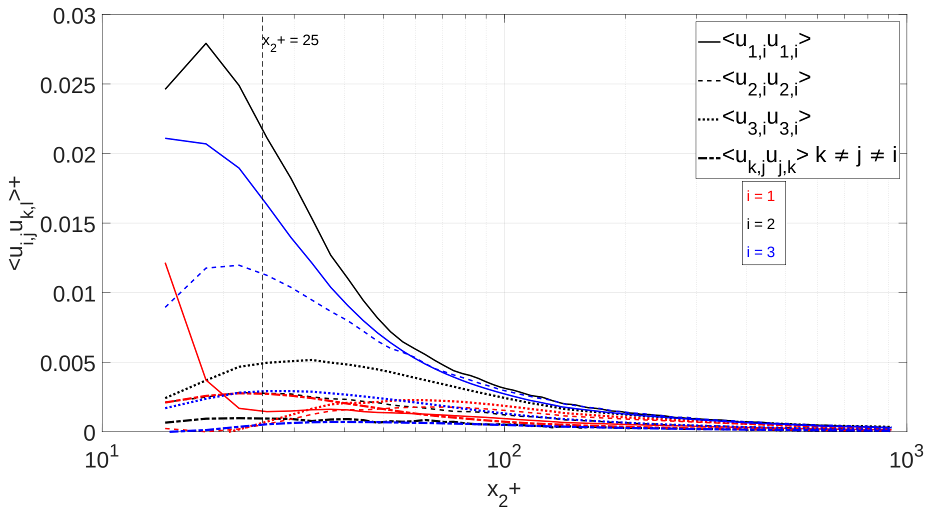

Figure 2: Linear-logarithmic plot of all dissipation derivatives moments deduced from the PIV data plotted together.

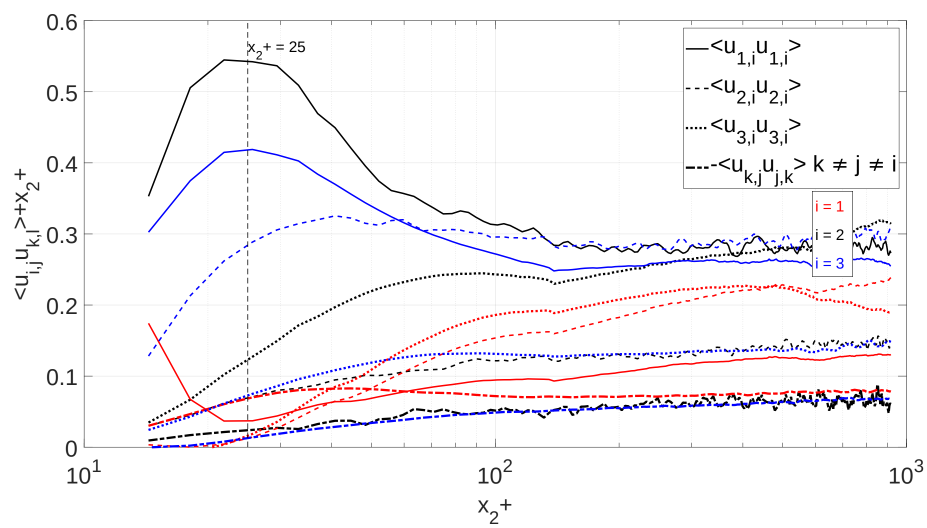

Figure 3: Linear-log plot of all the SPIV dissipation derivatives moments premultiplied by x_2^+ as a function of wall distance.

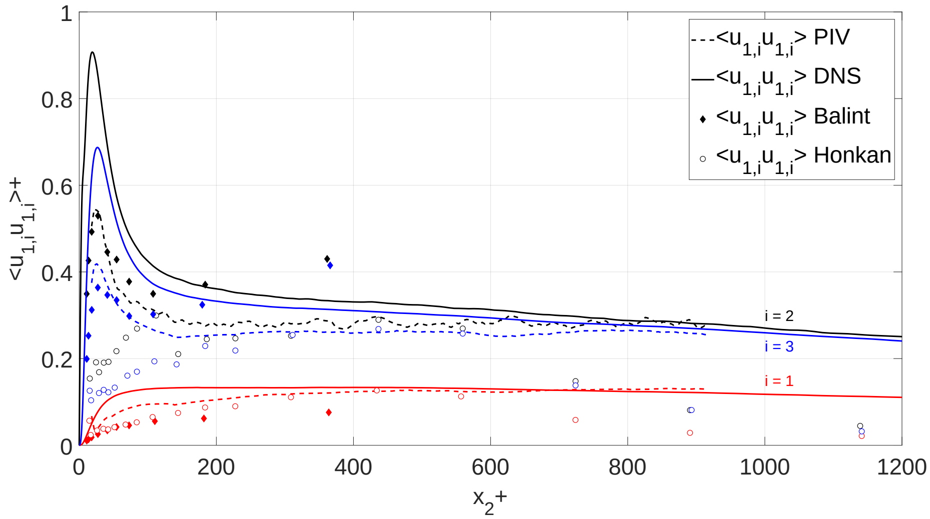

Figure 4: Comparison of SPIV derivative moments involving u_1 with DNS of [Thais et al., 2011] and hot-wire results of [Balint et al., 1991] and [Honkan and Andreopoulos, 1997].

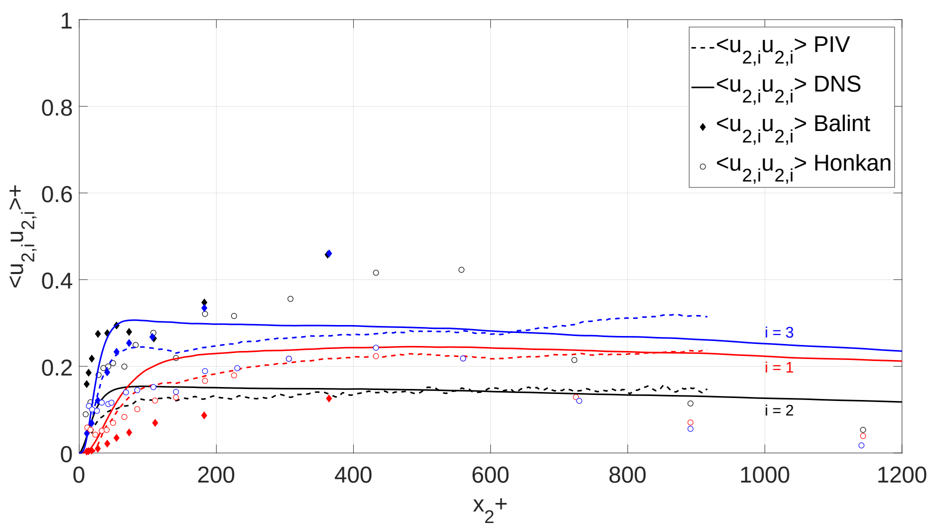

Figure 5: Comparison of SPIV derivative moments involving u_2 with DNS of [Thais et al., 2011] and hot-wire results of [Balint et al., 1991] and [Honkan and Andreopoulos, 1997].

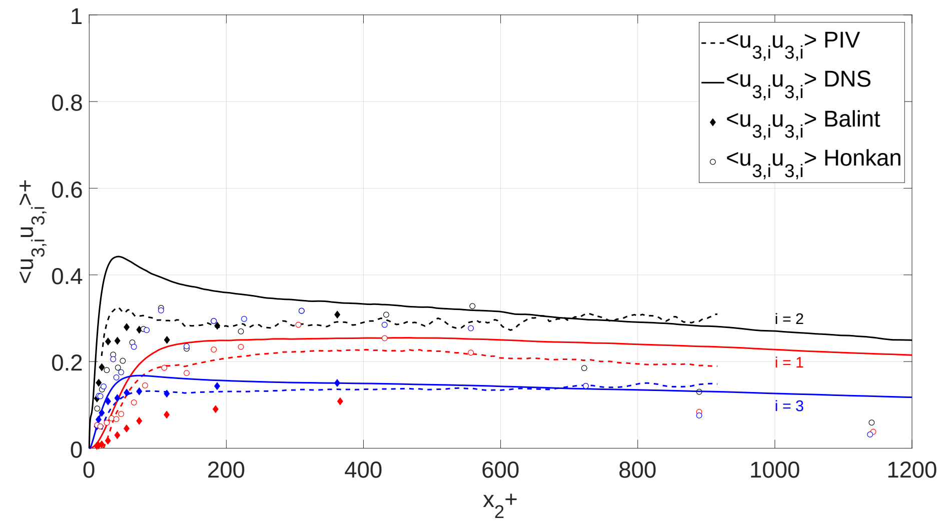

Figure 6: Comparison of SPIV derivative moments involving u_3 with DNS of [Thais et al., 2011] and hot-wire results of [Balint et al., 1991] and [Honkan and Andreopoulos, 1997].

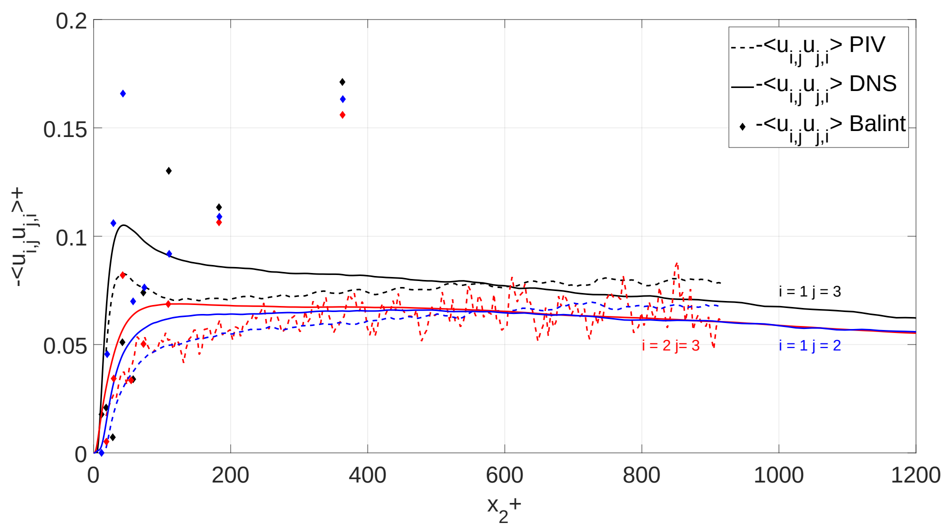

Figure 7: Comparison of SPIV derivative cross-products with channel DNS of [Thais et al., 2011] and hot-wire results of [Balint et al., 1991] and [Honkan and Andreopoulos, 1997].

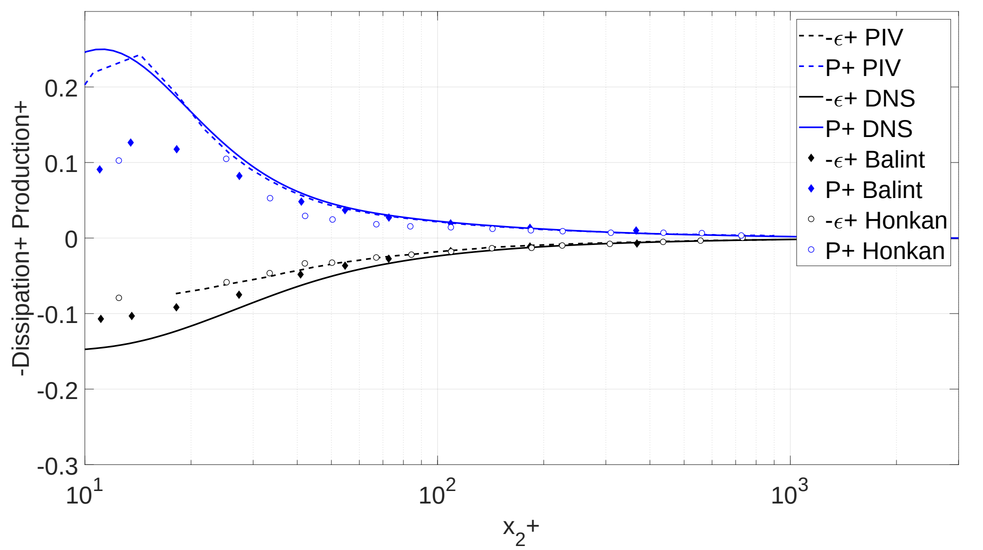

Figure 8: The dissipation rate ϵ and production rate in inner variables (linear-linear) for [Thais et al., 2011] DNS and SPIV along with [Balint et al., 1991, Andreopoulos and Honkan, 2001]
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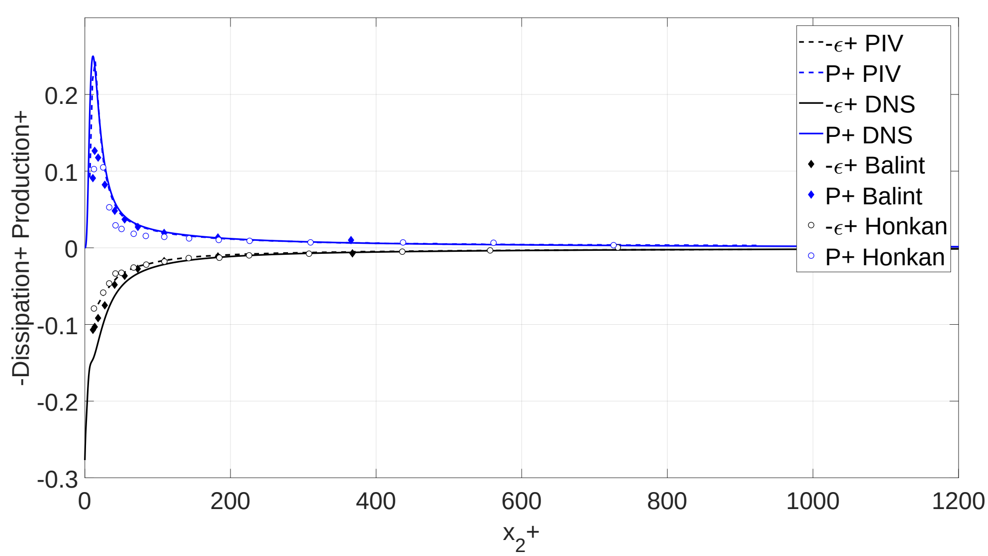

Figure 9: The dissipation rate ϵ and production rate in inner variables (linear-logarithmic) for [Thais et al., 2011] DNS and SPIV along with [Balint et al., 1991, Andreopoulos and Honkan, 2001].

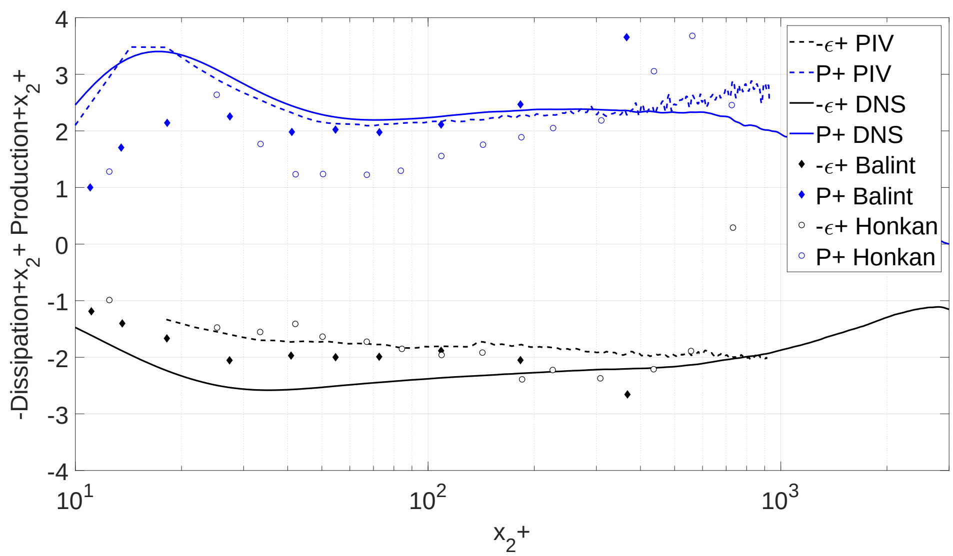

Figure 10: The dissipation rate ϵ and production rate in inner variables (linear-logarithmic) for [Thais et al., 2011] DNS and SPIV along with [Balint et al., 1991, Andreopoulos and Honkan, 2001]. The data are pre-multiplied by x_2^+.

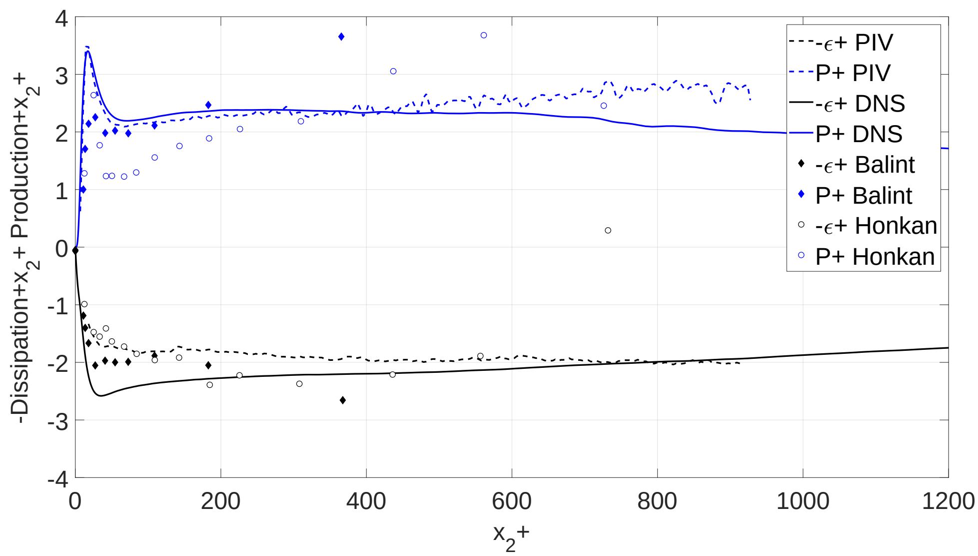

Figure 11: The dissipation rate ϵ and production rate in inner variables (linear-linear) for [Thais et al., 2011] DNS and SPIV along with [Balint et al., 1991, Andreopoulos and Honkan, 2001]. The data are pre-multiplied by x_2^+.

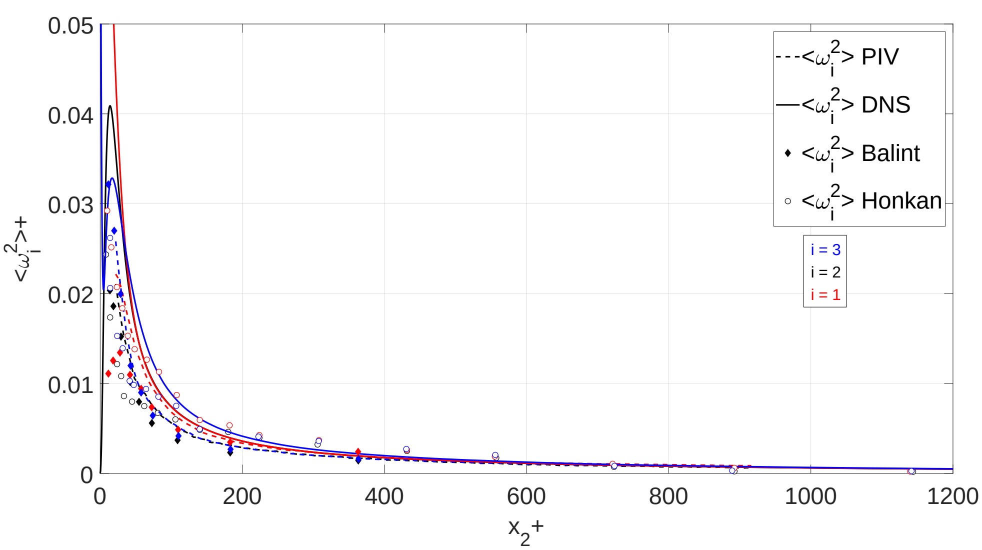

Figure 12: Mean square vorticity (enstrophy) components scaled in wall units (lin-lin plot).

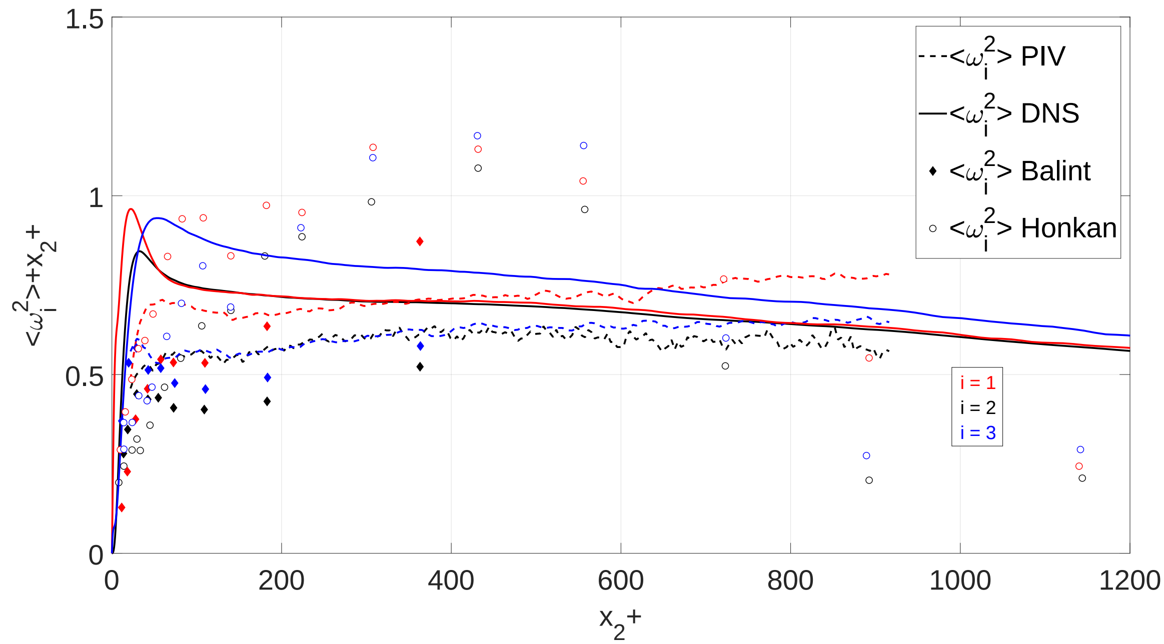

Figure 13: Mean square vorticity (enstrophy) components scaled in wall units and multiplied by y^+ (lin-lin plot).

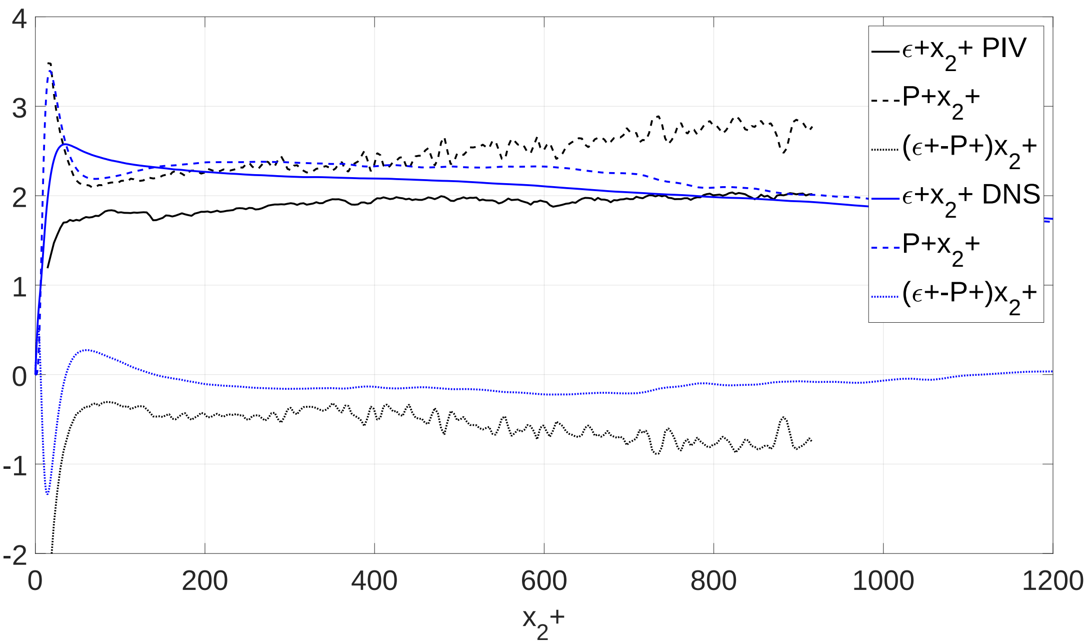

Figure 14: Linear plot of the SPIV energy balance premultiplied by x_2 in inner variables; i.e., production, $x_2^+ \langle uv \rangle^+ dU^+/dx_2^+$ (broken), dissipation $x_2^+ \varepsilon^+$ (plain) and difference between these two terms.

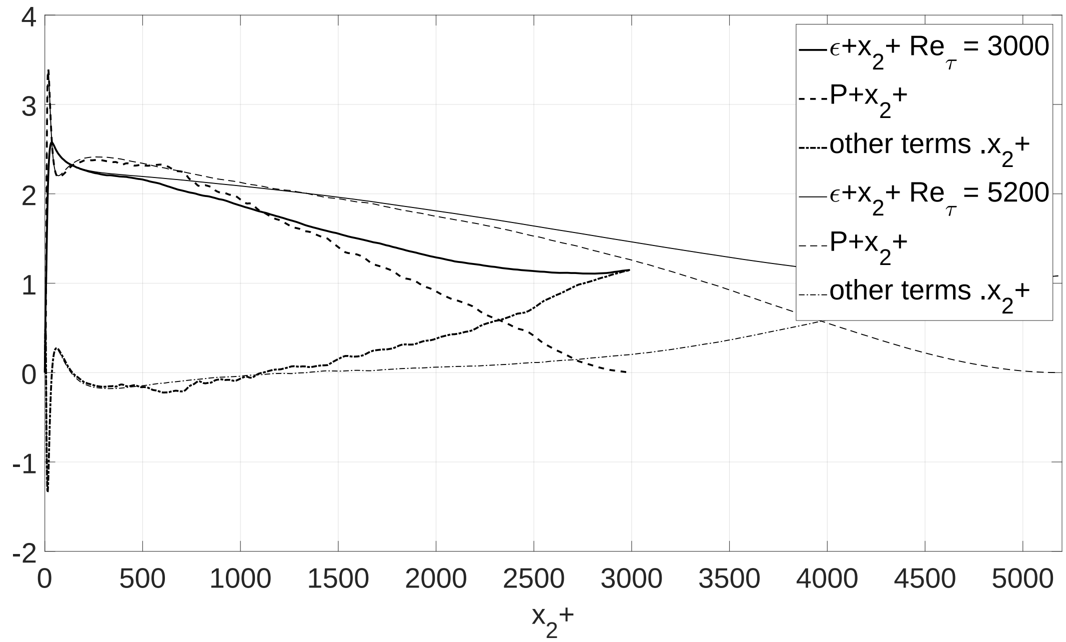

Figure 15: Linear plot of the [Thais et al., 2011] and [Lee and Moser, 2015] DNS energy balance times y in inner variables; i.e., production, $\langle uv\rangle^+ dU^+/dy^+$ (broken), dissipation $y^+\varepsilon^+$ (plain) and difference between these two terms (doted), for two values of the Reynolds number.

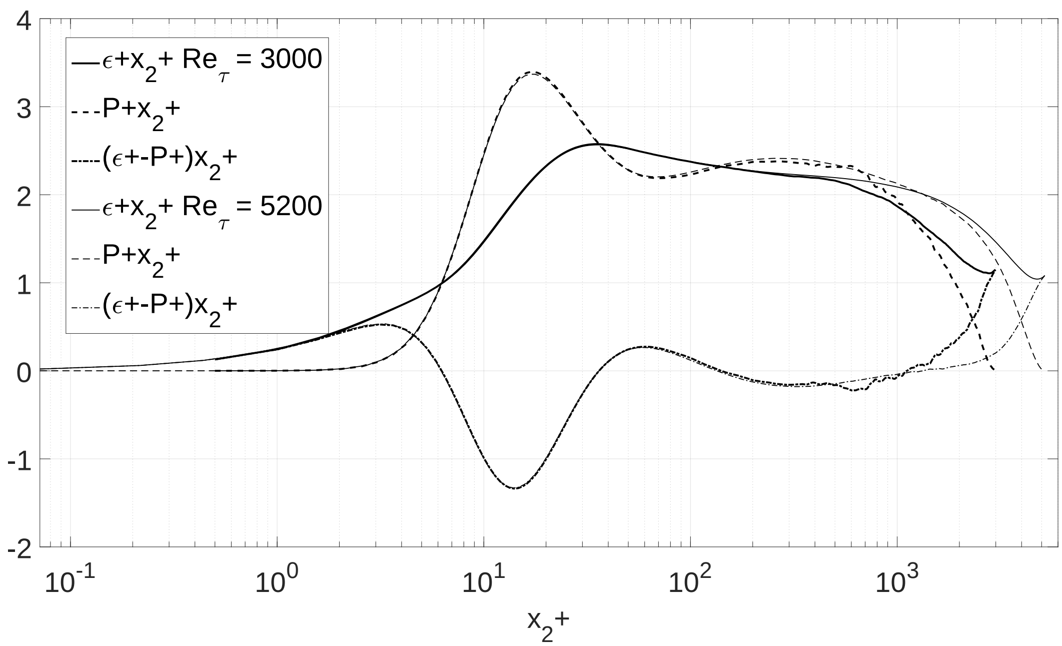

Figure 16: Logarithmic plot of the DNS energy balance premultiplied by x_2 in inner variables; i.e., production, $\langle uv\rangle^+ dU^+/dx_2^+$ (broken), dissipation $x_2^+\varepsilon^+$ (plain) and difference between these two terms (doted), for two values of the Reynolds number by [Thais et al., 2011] at $Re_\tau = 3000$ and [Lee and Moser, 2015] at $Re_\tau = 5200$.

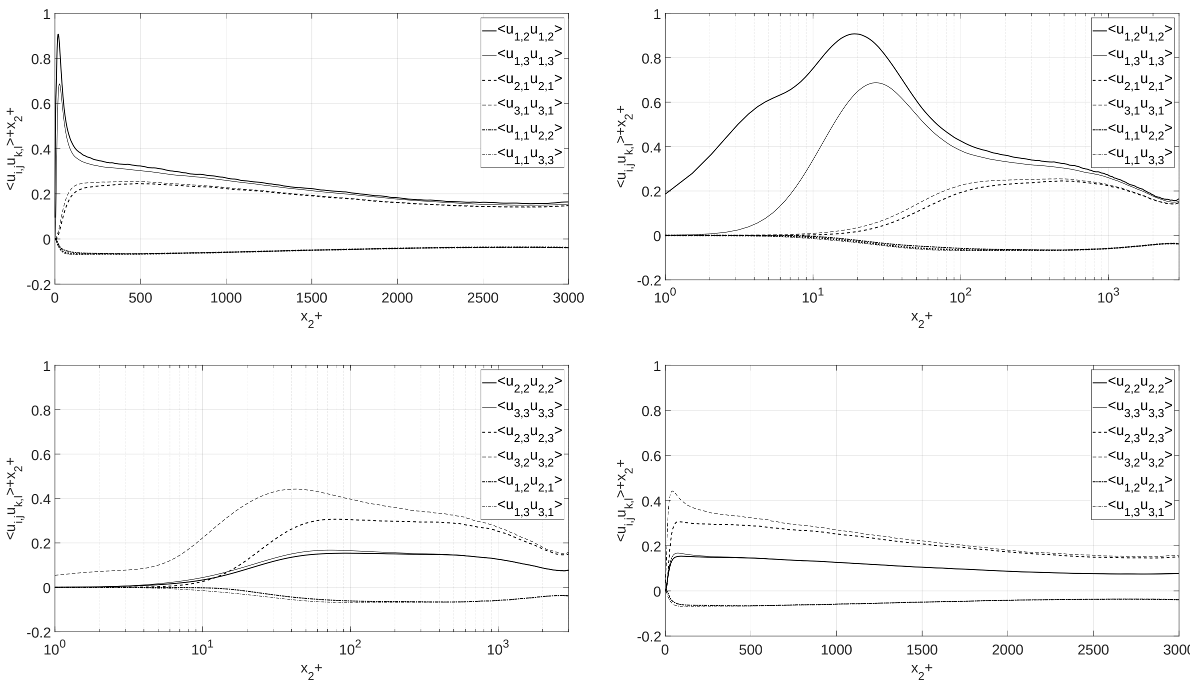

Figure 17: Comparison of derivative moments from DNS showing support for local axisymmetry outside of $y^+ = 30$.

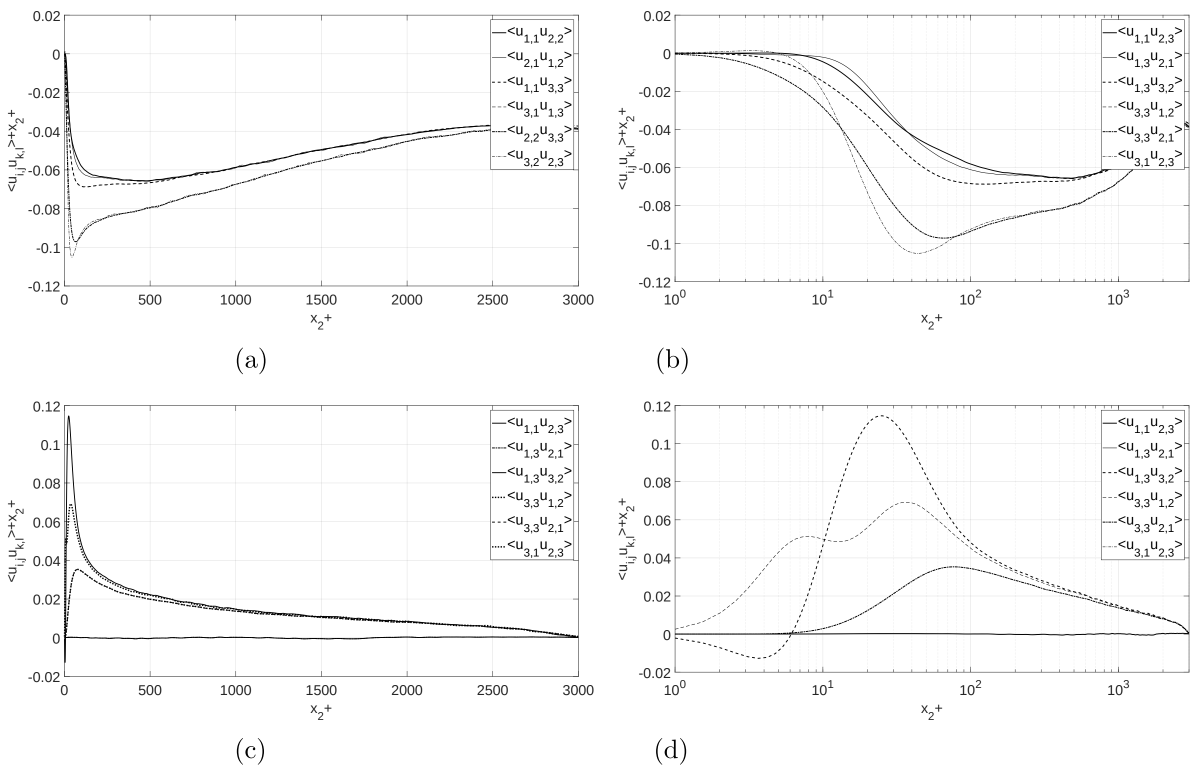

Figure 18: Comparison of derivative moments for DNS showing support for local homogeneity outside of $y^+ = 100$.