Supplementary Material for "Fluid Deformation in Isotropic Darcy Flow"

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1 Differential Operators in Streamfunction Coordinates

The gradient ∇, divergence $\nabla\cdot$ and Laplacian ∇^2 operators in orthogonal streamfunction coordinates are respectively

$$\nabla = \frac{\hat{\mathbf{g}}_1}{h_1}\frac{\partial}{\partial\xi^1} + \frac{\hat{\mathbf{g}}_2}{h_2}\frac{\partial}{\partial\xi^2} + \frac{\hat{\mathbf{g}}_3}{h_3}\frac{\partial}{\partial\xi^3}, \tag{1}$$

$$\nabla\cdot\mathbf{a} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\xi^1}(h_2h_3\hat{a}_1) + \frac{\partial}{\partial\xi^2}(h_1h_3\hat{a}_2) + \frac{\partial}{\partial\xi^3}(h_1h_2\hat{a}_3)\right], \tag{2}$$

$$\nabla^2 = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\xi^1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial\xi^1}\right) + \frac{\partial}{\partial\xi^2}\left(\frac{h_1h_3}{h_2}\frac{\partial}{\partial\xi^2}\right) + \frac{\partial}{\partial\xi^3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial\xi^3}\right)\right], \tag{3}$$

where the physical vector $\mathbf{a} = \hat{a}_1\hat{\mathbf{g}}_1 + \hat{a}_2\hat{\mathbf{g}}_2 + \hat{a}_3\hat{\mathbf{g}}_3$, and (1) simplifies the Darcy equation to

$$\mathbf{v}(\phi,\psi_1,\psi_2) = -k\nabla\phi = \frac{k}{h_1}\hat{\mathbf{g}}_1 = v\hat{\mathbf{g}}_1, \tag{4}$$

$$= \nabla\psi_1\times\nabla\psi_2 = \frac{1}{h_2}\hat{\mathbf{g}}_2\times\frac{1}{h_3}\hat{\mathbf{g}}_3 = \frac{1}{h_2h_3}\hat{\mathbf{g}}_1. \tag{5}$$

The derivatives of the unit basis vectors $\hat{\mathbf{g}}_j$ in the orthogonal streamfunction coordinate system are then

$$\frac{\partial\mathbf{g}_j}{\partial\xi^k} = \Gamma^i_{kj}\mathbf{g}_i \Rightarrow \Gamma^i_{kj}\frac{1}{h_i^2}\frac{\partial\mathbf{g}_k}{\partial\xi^j}\cdot\mathbf{g}_i,\ i=1:3, j=1:3, k=1:3, \tag{6}$$

where Γ^i_{kj} is the Christoffel symbol of the second kind. For orthogonal coordinate systems, the six Christoffel symbols with distinct indices are zero

$$\Gamma^k_{ij} = 0, \quad i\neq j\neq k\neq i, \tag{7}$$

as well as the three symbols with the same indices

$$\Gamma^i_{ii} = 0. \tag{8}$$

There are only six distinct Christoffel symbols symbols of the remaining 18 (from an original of 27) due to symmetry relations $\Gamma^i_{ij} = \Gamma^i_{ji}$. Six of these are related as

$$\Gamma^i_{ij} = \Gamma^i_{ji} = -\frac{h_j^2}{h_i^2}\Gamma^j_{ii}, \quad i \neq j, \tag{9}$$

and the remaining three symbols are Γ^i_{ii}. For orthogonal coordinate systems, $\mathbf{g}_i \cdot \mathbf{g}_i = h_i^2$, hence

$$\frac{1}{2}\frac{\partial}{\partial \xi^j}(\mathbf{g}_i \cdot \mathbf{g}_i) = \frac{\partial \mathbf{g}_i}{\partial \xi^j} \cdot \mathbf{g}_i = h_i^2 \Gamma^i_{ij} = h_i \frac{\partial h_i}{\partial \xi^j}, \tag{10}$$

and so the non-zero Christoffel symbols are explicitly

$$\Gamma^i_{ij} = \frac{1}{h_i}\frac{\partial h_i}{\partial \xi^j} \qquad\qquad \Gamma^i_{jj} = -\frac{h_j}{h_i}\frac{\partial h_j}{\partial \xi^i}. \tag{11}$$

The vector gradient in the streamfunction covariant and physical basis is then

$$\nabla \mathbf{a} = \frac{1}{h_j^2}\left(\frac{\partial a^i}{\partial \xi^j} + a^k \Gamma^i_{kj}\right)\mathbf{g}_i \otimes \mathbf{g}_j = \frac{1}{h_j}\left(\frac{\partial \hat{a}_i}{\partial \xi^j} - \hat{a}_i \Gamma^i_{ij} + \frac{h_i}{h_k}\hat{a}_k \Gamma^i_{kj}\right)\hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}_j, \tag{12}$$

which we use in Section 5 to derive en expression for the velocity gradient in streamfunction coordinates.

2 Proof That Orthogonal Streamline Coordinates Correspond to Zero Helicity Density Flow

In §§3.1 of the main paper, we assumed orthogonality of the streamfunction pair (ψ_1, ψ_2) admitted by isotropic Darcy flow. Here we prove that orthogonal streamlines coordinates correspond to zero helicity density flow. A streamfunction representation of the vorticity is obtained by taking the curl of equation (2.8) in the main paper to yield

$$\boldsymbol{\omega} = \nabla\psi_2 \cdot \nabla\nabla\psi_1 - \nabla\psi_1 \cdot \nabla\nabla\psi_2 + \nabla^2\psi_2 \nabla\psi_1 - \nabla^2\psi_1 \nabla\psi_2, \tag{13}$$

which upon taking the dot product with velocity yields the helicity-free condition for the streamfunctions as

$$\begin{aligned} h = \boldsymbol{\omega} \cdot \mathbf{v} &= (\nabla\psi_2 \cdot \nabla\nabla\psi_1 - \nabla\psi_1 \cdot \nabla\nabla\psi_2) \cdot (\nabla\psi_1 \times \nabla\psi_2) \\ &= (\nabla\psi_2 \cdot \nabla\nabla\psi_1 - \nabla\psi_1 \cdot \nabla\nabla\psi_2) \cdot \nabla\phi = 0. \end{aligned} \tag{14}$$

We now show that if we assume the existence of the orthogonal streamline coordinate system, then this helicity-free condition is automatically satisfied. Taking the gradient of the orthogonality condition $\nabla\psi_1 \cdot \nabla\psi_2 = 0$ yields $\nabla\psi_1 \cdot \nabla\nabla\psi_2 + \nabla\psi_2 \cdot \nabla\nabla\psi_1 = 0$, hence the helicity-free condition (14) simplifies to

$$\nabla\psi_1 \cdot \nabla\nabla\psi_2 \cdot \nabla\phi = \nabla\psi_2 \cdot \nabla\nabla\psi_1 \cdot \nabla\phi = 0. \tag{15}$$

as $k \neq 0$ everywhere. From (1), (12), the Hessian operator $\nabla\nabla$ in the streamline coordinate system is

$$\nabla\nabla = \left(\frac{\partial^2}{\partial\xi^i\partial\xi^j} - \Gamma^k_{ij}\frac{\partial}{\partial\xi^k}\right)\hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}_j, \tag{16}$$

where Γ^k_{ij} are the second Christoffel symbols of the streamline coordinate system. In orthogonal coordinates $\Gamma^k_{ij} = 0$ for distinct i, j, k, hence the $\hat{\mathbf{g}}_k \otimes \hat{\mathbf{g}}_j$ and $\hat{\mathbf{g}}_j \otimes \hat{\mathbf{g}}_k$ components (where $i \neq j \neq k \neq i$) of $\nabla\nabla\xi^i$ are both zero. As such, the helicity-free condition (15)

$$\nabla\xi^j \cdot \nabla\nabla\xi^i \cdot \nabla\xi^k = 0 \text{ for } i \neq j \neq k \neq i, \tag{17}$$

is satisfied if the streamfunction pair (ψ_1, ψ_2) is orthogonal.

3 Derivation of Streamfunction Governing Equations

To derive governing equations for the streamfunctions ψ_1, ψ_2 we take the curl of equation (2.8) in the main paper to yield

$$\nabla \times \mathbf{v} = \nabla\phi \times \nabla k = \nabla\psi_2 \cdot \nabla\nabla\psi_1 - \nabla\psi_1 \cdot \nabla\nabla\psi_2 + \nabla^2\psi_2\nabla\psi_1 - \nabla^2\psi_1\nabla\psi_2. \tag{18}$$

Re-writing $\nabla\phi$ as $-1/k(\nabla\psi_1 \times \nabla\psi_2)$ and using $\nabla(\nabla\psi_1 \cdot \nabla\psi_2) = 0$ then gives

$$\begin{aligned}\left(\nabla^2\psi_1 - \nabla f \cdot \nabla\psi_1\right)\nabla\psi_2 - \left(\nabla^2\psi_2 - \nabla f \cdot \nabla\psi_2\right)\nabla\psi_1 =& 2\nabla\psi_2 \cdot \nabla\nabla\psi_1 \\ =& -2\nabla\psi_1 \cdot \nabla\nabla\psi_2,\end{aligned} \tag{19}$$

where $f = \ln k$. From (15), we may write without loss of generality

$$\nabla\psi_1 \cdot \nabla\nabla\psi_2 = a_1\nabla\psi_1 - a_2\nabla\psi_2, \tag{20}$$

and from equation (3.4) in the main paper, the scalars satisfy

$$a_1 = \frac{\nabla\psi_1 \cdot \nabla\nabla\psi_2 \cdot \nabla\psi_1}{\nabla\psi_1 \cdot \nabla\psi_1} = -\frac{\nabla\psi_2 \cdot \nabla\nabla\psi_1 \cdot \nabla\psi_1}{\nabla\psi_1 \cdot \nabla\psi_1}, \tag{21}$$

$$a_2 = \frac{\nabla\psi_2 \cdot \nabla\nabla\psi_1 \cdot \nabla\psi_2}{\nabla\psi_2 \cdot \nabla\psi_2} = -\frac{\nabla\psi_1 \cdot \nabla\nabla\psi_2 \cdot \nabla\psi_2}{\nabla\psi_2 \cdot \nabla\psi_2}. \tag{22}$$

Taking the dot product of (19) with respect to $\nabla\psi_1$ and $\nabla\psi_2$ yields the coupled equations for the streamfunctions

$$\nabla^2\psi_1 - \nabla f \cdot \nabla\psi_1 = -2a_2, \tag{23}$$

$$\nabla^2\psi_2 - \nabla f \cdot \nabla\psi_2 = -2a_1. \tag{24}$$

Using (20)-(22) along with $\nabla(\nabla\psi_i \cdot \nabla\psi_i) = 2\nabla\nabla\psi_i \cdot \nabla\psi_i$ then yields equations (3.6), (3.7) in the main paper.

4 Demonstration Governing Equations Generate Orthogonal Streamfunctions

To show that equations (3.6), (3.7) in the main paper give rise to a pair of orthogonal streamfunctions, we take the cross product of $\nabla\psi_1$ with equation (2.8) in the main paper to yield an expression for $\nabla\psi_2$ which is orthogonal to both $\nabla\psi_1$ and $\nabla\phi$:

$$\nabla\psi_2 = \frac{k}{|\nabla\psi_1|^2}(\nabla\psi_1 \times \nabla\phi). \tag{25}$$

We can show that this gradient satisfies equation (3.7) in the main paper. by substituting (25):

$$\begin{aligned}\nabla^2\psi_2 - \nabla f \cdot \nabla\psi_2 =& \frac{1}{|\nabla\psi_1|^2}\nabla \cdot (\nabla\psi_1 \times k\nabla\phi) + (\nabla\psi_1 \times k\nabla\phi) \cdot \nabla \left(|\nabla\psi_1|^{-2}\right) \\ &- \frac{1}{|\nabla\psi_1|^2}\nabla f \cdot (\nabla\psi_1 \times k\nabla\phi).\end{aligned} \tag{26}$$

We note that $\nabla\cdot(\nabla\psi_1\times k\nabla\phi) = \nabla f\cdot(\nabla\psi_1\times k\nabla\phi)$ and $k\nabla\phi\times\nabla\psi_1 = \nabla\psi_2|\nabla\psi_1|^2$, then

$$\nabla^2\psi_2 - \nabla f \cdot \nabla\psi_2 = |\nabla\psi_1|^2\nabla\left(|\nabla\psi_1|^{-2}\right) \cdot \nabla\psi_2, \tag{27}$$

which is equivalent to equation (3.7) in the main paper.

5 Derivation of Velocity Gradient in Streamfunction Coordinates

To derive an expression for the *advected* velocity gradient $\boldsymbol{\epsilon}(t)$ in streamfunction coordinates (i.e., where $\boldsymbol{\epsilon}(t) \equiv \nabla\mathbf{v}(\boldsymbol{\xi}(\boldsymbol{\Xi}, t), t)^\top$, we first use (12) to derive an expression for the *stationary* velocity gradient $\mathbf{l}(\boldsymbol{\xi}) \equiv \nabla\mathbf{v}(\boldsymbol{\xi})^\top$ (i.e. at fixed coordinates $\boldsymbol{\xi}$) and then transform this into the moving frame $\boldsymbol{\xi}(\boldsymbol{\Xi}, t)$. As the velocity is $\mathbf{v} = v\hat{\mathbf{g}}_1$, then from the components of $\mathbf{l}(\boldsymbol{\xi}) = l_{ij}\hat{\mathbf{g}}_i \otimes \hat{\mathbf{g}}_j$ are then

$$l_{11} = \frac{1}{h_1}\left(\frac{\partial v}{\partial \phi} - v\Gamma^1_{11} + v\Gamma^1_{11}\right) = \frac{1}{h_1}\frac{\partial v}{\partial \xi^1} = \frac{\partial v}{\partial s}, \tag{28}$$

$$l_{22} = \frac{1}{h_2}\left(0 - 0 + \frac{h_2}{h_1}v\Gamma^2_{12}\right) = \frac{v}{h_1 h_2}\frac{\partial h_2}{\partial \xi^1} = -\frac{1}{2}\frac{\partial v}{\partial s} + \frac{v}{2}\frac{\partial \ln m}{\partial s}, \tag{29}$$

$$l_{33} = \frac{1}{h_3}\left(0 - 0 + \frac{h_3}{h_1}v\Gamma^3_{13}\right) = \frac{v}{h_1 h_3}\frac{\partial h_3}{\partial \xi^1} = -\frac{1}{2}\frac{\partial v}{\partial s} - \frac{v}{2}\frac{\partial \ln m}{\partial s}, \tag{30}$$

$$l_{12} = \frac{1}{h_2}\left(\frac{\partial v}{\partial \xi^2} - v\Gamma^1_{12} + \frac{h_1}{h_2}v\Gamma^1_{12}\right) = \frac{1}{h_2}\frac{\partial v}{\partial \xi^2} = \frac{\partial v}{\partial \psi_1}, \tag{31}$$

$$l_{13} = \frac{1}{h_3}\left(\frac{\partial v}{\partial \xi^2} - v\Gamma^1_{13} + \frac{h_1}{h_3}v\Gamma^1_{13}\right) = \frac{1}{h_3}\frac{\partial v}{\partial \xi^3} = \frac{\partial v}{\partial \psi_2}, \tag{32}$$

$$l_{21} = \frac{1}{h_1}\left(0 - 0 + \frac{h_2}{h_1}v\Gamma^2_{11}\right) = -\frac{1}{h_1}\frac{\partial h_1}{\partial \xi^2} = \frac{\partial v}{\partial \psi_1} - v\frac{\partial \ln k}{\partial \psi_1}, \tag{33}$$

$$l_{31} = \frac{1}{h_1}\left(0 - 0 + \frac{h_3}{h_1}v\Gamma^3_{11}\right) = -\frac{1}{h_1}\frac{\partial h_1}{\partial \xi^3} = \frac{\partial v}{\partial \psi_2} - v\frac{\partial \ln k}{\partial \psi_2}, \tag{34}$$

$$l_{23} = \frac{1}{h_3}\left(0 - 0 + \frac{h_2}{h_3}v\Gamma^2_{13}\right) = 0, \tag{35}$$

$$l_{32} = \frac{1}{h_2}\left(0 - 0 + \frac{h_3}{h_2}v\Gamma^3_{12}\right) = 0. \tag{36}$$

Note the components $l_{23} = l_{32} = 0$ are a direct consequence of the helicity-free nature of the flow. The velocity gradient may also be expressed in terms of the normalised Jacobian from the identity $d\boldsymbol{\xi} = \hat{\mathbf{J}}^\top \cdot d\mathbf{x}$, and by taking the temporal derivative of this expression yields

$$\frac{d\boldsymbol{\xi}}{dt} = \hat{\mathbf{J}^\top} \cdot \frac{d\mathbf{x}}{dt} + \frac{d\hat{\mathbf{J}}^\top}{dt} \cdot d\mathbf{x} \quad \Leftrightarrow \quad \mathbf{v}(\boldsymbol{\xi}) = \hat{\mathbf{J}}^\top \cdot \mathbf{v}(\mathbf{x}) + \frac{d\hat{\mathbf{J}}^\top}{dt} \cdot d\mathbf{x}, \tag{37}$$

where the Jacobian may be considered time-dependent if the frame of reference is moving with the fluid particle with trajectory $\boldsymbol{\xi}(\boldsymbol{\Xi}, t)$ or in Cartesian coordinates $\mathbf{x}(\mathbf{X}, t)$. Further differentiating with respect to $\boldsymbol{\xi}$ yields the velocity gradient in the moving frame as

$$\nabla \mathbf{v}(\boldsymbol{\xi}(\boldsymbol{\Xi}, t)) = \hat{\mathbf{J}}^\top \cdot \nabla \mathbf{v}(\mathbf{x}(\mathbf{X}, t)) \cdot \hat{\mathbf{J}} + \frac{d\hat{\mathbf{J}}^\top}{dt} \cdot \hat{\mathbf{J}}, \tag{38}$$

whereas in the stationary frame

$$\mathbf{l}(\boldsymbol{\xi})^\top = \nabla \mathbf{v}(\boldsymbol{\xi}) = \hat{\mathbf{J}}^\top \cdot \nabla \mathbf{v}(\mathbf{x}) \cdot \hat{\mathbf{J}}, \tag{39}$$

hence

$$\boldsymbol{\epsilon}(\boldsymbol{\Xi}, t) = \mathbf{l}(\boldsymbol{\xi}) + \mathbf{A}(t), \quad \mathbf{A}(t) \equiv \frac{d\hat{\mathbf{J}}^\top}{dt} \cdot \hat{\mathbf{J}}, \tag{40}$$

and since $\hat{\mathbf{J}}^\top \cdot \hat{\mathbf{J}} = \mathbf{I}$, then $\mathbf{A}(t) = -\mathbf{A}(t)^\top$. As $\hat{\mathbf{J}} = [\hat{\mathbf{g}}_1, \hat{\mathbf{g}}_2, \hat{\mathbf{g}}_3]$, then the skew-symmetric matrix $\mathbf{A}(t)$ has components

$$\mathbf{A}(t) \equiv \frac{d\hat{\mathbf{J}}^\top}{dt} \cdot \hat{\mathbf{J}} = \begin{pmatrix} 0 & \hat{\mathbf{g}}_2 \cdot \dot{\hat{\mathbf{g}}}_1 & \hat{\mathbf{g}}_3 \cdot \dot{\hat{\mathbf{g}}}_1 \\ -\hat{\mathbf{g}}_2 \cdot \dot{\hat{\mathbf{g}}}_1 & 0 & \hat{\mathbf{g}}_3 \cdot \dot{\hat{\mathbf{g}}}_2 \\ -\hat{\mathbf{g}}_3 \cdot \dot{\hat{\mathbf{g}}}_1 & -\hat{\mathbf{g}}_3 \cdot \dot{\hat{\mathbf{g}}}_2 & 0 \end{pmatrix}. \tag{41}$$

The components of $\mathbf{A}(t)$ can be related to $\mathbf{l}(\boldsymbol{\xi})$ via the relationships $\dot{\mathbf{v}} = \boldsymbol{\epsilon}(t) \cdot \mathbf{v}$, $\hat{\mathbf{g}}_1 = \mathbf{v}/v$, $l_{ij} = \hat{\mathbf{g}}_i \cdot \mathbf{l}(\boldsymbol{\xi}) \cdot \hat{\mathbf{g}}_j$, yielding

$$\hat{\mathbf{g}}_2 \cdot \dot{\hat{\mathbf{g}}}_1 = \hat{\mathbf{g}}_2 \cdot \mathbf{l}(\boldsymbol{\xi}) \cdot \hat{\mathbf{g}}_1 = l_{21}, \tag{42}$$

$$\hat{\mathbf{g}}_3 \cdot \dot{\hat{\mathbf{g}}}_1 = \hat{\mathbf{g}}_3 \cdot \mathbf{l}(\boldsymbol{\xi}) \cdot \hat{\mathbf{g}}_1 = l_{31}. \tag{43}$$

Furthermore, as the helicity is an invariant quantity, then the advected velocity gradient satisfies

$$h \equiv \mathbf{v} \cdot (\nabla \times \mathbf{v}) = \mathbf{v} \cdot (\boldsymbol{\varepsilon} : \nabla \mathbf{v}) = v_i \varepsilon_{ijk} \hat{\epsilon}_{jk} = 0, \tag{44}$$

where $\boldsymbol{\varepsilon}$ is the Levi-Civita tensor

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even permutation,} \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation,} \\ 0 & \text{otherwise,} \end{cases} \tag{45}$$

and so as $\mathbf{v} = v\hat{\mathbf{g}}_1$, then

$$h = v(\hat{\epsilon}_{23} - \hat{\epsilon}_{32}) = 0, \tag{46}$$

and from (40), $\hat{\mathbf{g}}_3 \cdot \dot{\hat{\mathbf{g}}}_2 = 0$. From these results the components of the advected velocity gradient tensor are then

$$\boldsymbol{\epsilon}[\boldsymbol{\xi}(\boldsymbol{\Xi}, t)] = \begin{pmatrix} \hat{\epsilon}_{11} & \hat{\epsilon}_{12} & \hat{\epsilon}_{13} \\ 0 & \hat{\epsilon}_{22} & 0 \\ 0 & 0 & \hat{\epsilon}_{33} \end{pmatrix} = \begin{pmatrix} l_{11} & l_{12} + l_{21} & l_{13} + l_{31} \\ 0 & l_{22} & 0 \\ 0 & 0 & l_{33} \end{pmatrix}, \tag{47}$$

as is given in equations (4.15), (4.16) in the main paper.

6 Calculation of Transverse Stretching Directions and Magnitudes

To compute the transverse stretching directions and magnitudes along the reference streamline shown in Figure 1(c) in the main paper, the Cartesian deformation tensor $F(\mathbf{X}, t)$ is first computed from equation (4.4) in the main paper. At any point along the reference streamline, this tensor may then rotated to align with the velocity vector via the rotation matrix

$$\mathbf{Q}(t) = \cos\theta(t)\,\mathbf{I} + \sin\theta(t)(\mathbf{q}(t))_\times^\top - \cos\theta(t)\mathbf{q}(t) \otimes \mathbf{q}(t), \tag{48}$$

where $()_\times$is the cross product matrix, and $\mathbf{q}(t)$, $\theta(t)$ are the rotation axis and angle that rotates the $\mathbf{e}_1$ vector of the Cartesian frame to align with the local velocity vector $\mathbf{v}(t) = \{v_1, v_2, v_3\}^\top = v_i \mathbf{e}_i$:

$$\mathbf{q}(t) = \frac{\mathbf{e}_1 \times \mathbf{v}}{||\mathbf{e}_1 \times \mathbf{v}||} = \frac{1}{\sqrt{v_2^2 + v_3^2}} \{0, v_3, -v_2\}^\top, \tag{49}$$

$$\cos\theta(t) = \frac{\mathbf{e}_1 \cdot \mathbf{v}}{||\mathbf{v}||} = \frac{v_1}{v}. \tag{50}$$

This reoriented coordinate frame is denoted as $\mathbf{x}' = \{x'_1, x'_2, x'_3\}^\top = x'_i \mathbf{e}'_i$, which is related to the Cartesian frame as Truesdell (1954)

$$\mathbf{x}' = \mathbf{x}_0(t) + \mathbf{Q}^\top(t) \cdot \mathbf{x}, \tag{51}$$

where the translation vector $\mathbf{x}_0(t)$ is the position vector along the reference streamline. Note that this rotation into the streamwise direction does not necessarily align with the streamfunction coordinate directions as the transverse coordinates $\mathbf{e}'_2$, $\mathbf{e}'_3$ do not align with the coordinates ψ_1, ψ_2. The deformation gradient tensor then transforms between these frames as

$$\mathbf{F}'(t) = \mathbf{Q}^\top \mathbf{F}(t) \mathbf{Q}(0), \tag{52}$$

and the fluid deformation transverse to the streamwise direction is given by the projection $\mathbf{F}'_{2\text{D}}(t) = \mathbf{P}\mathbf{F}'(t)\mathbf{P}$, where $\mathbf{P} = \text{diag}(0, 1, 1)$, yielding

$$\mathbf{F}'_{2\text{D}}(t) = \begin{pmatrix} F'_{22} & F'_{23} \\ F'_{32} & F'_{33} \end{pmatrix}. \tag{53}$$

Conversely, the corresponding deformation tensor $\mathbf{F}_{2\text{D}}(t)$ in streamfunction coordinates is diagonal due to the helicity-free condition. Thus the principal stretches and associated stretching directions of $\mathbf{F}'_{2\text{D}}(t)$ give both the components $\hat{F}_{22}$, $\hat{F}_{33}$ and the corresponding coordinate directions of the streamfunction coordinate frame. As $\mathbf{F}'_{2\text{D}}(t)$ is in general not diagonalizable, we perform the singular value decomposition (SVD)

$$\mathbf{F}'_{2\text{D}}(t) = \mathbf{A}\mathbf{\Sigma}\mathbf{B}^\top, \tag{54}$$

where $\mathbf{A}$ and $\mathbf{B}$ are unitary (diagonalizable) matrices and $\mathbf{\Sigma}$ is diagonal. The SVD allows representation of $\mathbf{F}'_{2\text{D}}(t)$ in terms of the right $\mathbf{U}$ and left $\mathbf{V}$ stretch tensors via the polar decomposition theorem as

$$\mathbf{F}'_{2\text{D}}(t) = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}, \tag{55}$$

where the rotation matrix $\mathbf{R} = \mathbf{A}\mathbf{B}$, and the stretch tensors are $\mathbf{U} = \mathbf{B}^\top\mathbf{\Sigma}\mathbf{B}$, $\mathbf{V} = \mathbf{A}^\top\mathbf{\Sigma}\mathbf{A}$. Thus the eigenvectors of $\mathbf{U}$, $\mathbf{V}$ are respectively given by $\mathbf{A} = (\mathbf{s}_1, \mathbf{s}_2)^\top$, $\mathbf{B}$ and the eigenvectors of both $\mathbf{U}$ and $\mathbf{V}$ are given by the diagonal components of $\mathbf{\Sigma} = \text{diag}(\Sigma_1, \Sigma_1)$, where $\Sigma_1 = \hat{F}_{22}$, $\Sigma_2 = \hat{F}_{33}$. Hence the vectors $\Sigma_1\mathbf{s}_1$, $\Sigma_2\mathbf{s}_2$ correspond to the principal axes of the transverse deformation in the

$\mathbf{R}^{\top}\mathbf{x}'$ frame. These vectors can then be rotated back into the Cartesian frame to give the principal axes as

$$\mathbf{d}_r(t) = \Sigma_1 \mathbf{Q}(t)\mathbf{R}_{3\mathrm{D}}\mathbf{s}_1, \tag{56}$$

$$\mathbf{d}_q(t) = \Sigma_2 \mathbf{Q}(t)\mathbf{R}_{3\mathrm{D}}\mathbf{s}_2, \tag{57}$$

where

$$\mathbf{R}_{3\mathrm{D}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & R_{11} & R_{12} \\ 0 & R_{21} & R_{22} \end{pmatrix}. \tag{58}$$

Thus the vectors $\mathbf{d}_r(t)$, $\mathbf{d}_q(t)$ indicates the directions of the streamfunction coordinates ψ_1, ψ_2, and their magnitude quantifies the relative streamfunction gradients $|\mathbf{d}_r(t)| = |\nabla\psi_1(t)|/|\nabla\psi_1(0)|$, $|\mathbf{d}_q(t)| = |\nabla\psi_2(t)|/|\nabla\psi_2(0)|$.

References

Truesdell, C.A. 1954 *The Kinematics of Vorticity*. Indiana University Press.