Dynamic stabilisation of Rayleigh-Plateau modes on a liquid cylinder

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1. Floquet analysis

The base-state studied earlier in Patankar *et al.* (2018) (indicated with a subscript) comprises of the following alongwith the pressure jump condition at the free-surface

$$\mathbf{u}_b = 0, \quad -\frac{1}{\rho}\boldsymbol{\nabla} p_b + \mathcal{F}(r,t)\hat{\mathbf{e}}_r = 0 \quad \text{with} \quad \mathcal{F}(r,t) \equiv -h\cos\left(\Omega t\right)\left(\frac{r}{R_0}\right), \quad 0 \leqslant r \leqslant R_0$$
$$\text{and} \quad p_b(R_0,t) = \frac{T}{R_0}. \tag{1.1}$$

Here T is the coefficient of surface tension. Integrating eqn. 1.1 from r to R_0 and using the boundary condition at $r = R_0$, we obtain the pressure variation in the base state viz.

$$p_b(r,t) = \frac{\rho h}{2R_0}\left(R_0^2 - r^2\right)\cos(\Omega t) + \frac{T}{R_0} \tag{1.2}$$

We express all quantities as sum of base plus perturbation

$$\hat{p} = p_b + p, \quad \hat{\mathbf{u}} = \mathbf{0} + \mathbf{u} \tag{1.3}$$

Substituting 1.3 in the incompressible Navier-Stokes equations and linearing about the base state, we obtain

$$\left(\frac{\partial}{\partial t} - \nu\Delta\right)\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p, \quad \nabla\cdot\mathbf{u} = 0 \tag{1.4a,b}$$

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where $\mathbf{\Delta u}$ is the vector Laplacian defined as $\Delta\mathbf{u} \equiv \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) - \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{u} = -\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{u}$. The linearised boundary conditions are

$$\frac{\partial\eta}{\partial t}(\theta, z, t) = u_r(R_0, \theta, z, t) \tag{1.5}$$

$$\tau_{zr}(R_0, \theta, z, t) = \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)(R_0, \theta, z, t) = 0 \tag{1.6}$$

$$\tau_{r\theta}(R_0, \theta, z, t) = \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right](R_0, \theta, z, t) = 0 \tag{1.7}$$

$$p - \rho\left[-\mathcal{F}(r,t)\eta + 2\nu\left(\frac{\partial u_r}{\partial r}\right)_{r=R_0}\right] = -\frac{T}{R_0^2}\left[\eta + \left(\frac{\partial^2\eta}{\partial\theta^2}\right) + R_0^2\left(\frac{\partial^2\eta}{\partial z^2}\right)\right] \tag{1.8}$$

In order to eliminate pressure from 1.8, there are a number of steps. First, we take the divergence of eqn. 1.4a with $\boldsymbol{\nabla}_O \equiv \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_z\frac{\partial}{\partial z}$ leading to

$$\boldsymbol{\nabla}_O\cdot\left(\frac{\partial}{\partial t} - \nu\Delta\right)\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla}_O\cdot\boldsymbol{\nabla}p = -\frac{1}{\rho}\left(\Delta_0 + \frac{1}{r}\frac{\partial}{\partial r}\right)p, \tag{1.9}$$

$$\text{where}\quad \Delta_O = \boldsymbol{\nabla}_O\cdot\boldsymbol{\nabla}_O \equiv \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}.$$

An expression for $\frac{\partial p}{\partial r}$ is obtained from the radial component of equation 1.4a. This is

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = -\left[\frac{\partial u_r}{\partial t} - \nu\left\{\Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\left(\frac{\partial u_\theta}{\partial\theta}\right)\right\}\right] \tag{1.10}$$

where $\Delta \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$ is the scalar Laplacian expressed in cylindrical coordinates. Using incompressibility, the divergence of the *left hand side* of eqn. 1.4a is

$$\boldsymbol{\nabla}\cdot\left(\frac{\partial}{\partial t} - \nu\Delta\right)\mathbf{u} = 0 \tag{1.11}$$

Rewriting $\boldsymbol{\nabla} = \boldsymbol{\nabla}_0 + \mathbf{e}_r\frac{\partial}{\partial r}$ and using eqn. 1.10, eqn. 1.13 may be written as

$$\boldsymbol{\nabla}_O\cdot\left(\frac{\partial}{\partial t} - \nu\Delta\right)\mathbf{u} + \mathbf{e}_r\cdot\left[\frac{\partial}{\partial r}\left\{\left(\frac{\partial}{\partial t} - \nu\Delta\right)\mathbf{u}\right\}\right] = 0$$

$$\Rightarrow \boldsymbol{\nabla}_O\cdot\left(\frac{\partial}{\partial t} - \nu\Delta\right)\mathbf{u} = -\frac{\partial}{\partial r}\left[\frac{\partial u_r}{\partial t} - \nu\left\{\Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\left(\frac{\partial u_\theta}{\partial\theta}\right)\right\}\right] \tag{1.12}$$

We operate on eqn. 1.8 with $\frac{1}{\rho}\Delta_O$ and replace $\frac{1}{\rho}\Delta_O p$ in the resultant expression using eqns. 1.9, leading to

$$\frac{1}{\rho r}\frac{\partial p}{\partial r} + \boldsymbol{\nabla}_O\cdot\left(\frac{\partial}{\partial t} - \nu\Delta\right)\mathbf{u} - \mathcal{F}(r,t)\Delta_0\eta + 2\nu\Delta_0\left(\frac{\partial u_r}{\partial r}\right) = \frac{T}{\rho R_0^2}\Delta_0\left[\eta + \left(\frac{\partial^2\eta}{\partial\theta^2}\right) + R_0^2\left(\frac{\partial^2\eta}{\partial z^2}\right)\right]$$

$$\text{at } r = R_0 \tag{1.13}$$

Finally, using 1.10 and 1.12 in eqn. 1.13 we obtain

$$\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\left[\frac{\partial u_r}{\partial t}-\nu\left\{\Delta u_r-\frac{u_r}{r^2}-\frac{2}{r^2}\left(\frac{\partial u_\theta}{\partial \theta}\right)\right\}\right]+\mathcal{F}(r,t)\Delta_0\eta-2\nu\Delta_0\left(\frac{\partial u_r}{\partial r}\right)$$
$$=-\frac{T}{\rho R_0^2}\Delta_0\left[\eta+\left(\frac{\partial^2\eta}{\partial\theta^2}\right)+R_0^2\left(\frac{\partial^2\eta}{\partial z^2}\right)\right]\text{ at } r=R_0 \tag{1.14}$$

Analogous to the elimination of pressure from the normal stress boundary condition, we shift to a streamfunction-vorticity formulation in three dimensions which naturally satisfies the continuity equation. For this we employ the toroidal-poloidal decomposition on a cylinder

$$\mathbf{u}=\boldsymbol{\nabla}\times(\psi\hat{\mathbf{e}}_z)+\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times(\xi\hat{\mathbf{e}}_z) \tag{1.15}$$

where $\psi(r,\theta,z,t),\xi(r,\theta,z,t)$ are scalar fields and $\hat{\mathbf{e}}_z$ is unit vector along the axial direction of the cylinder. The components of the velocity field components written in terms of ψ and ξ are

$$u_r=\left(\frac{\partial^2\xi}{\partial z\partial r}\right)+\frac{1}{r}\left(\frac{\partial\psi}{\partial\theta}\right),\quad u_\theta=\frac{1}{r}\left(\frac{\partial^2\xi}{\partial z\partial\theta}\right)-\left(\frac{\partial\psi}{\partial r}\right),\quad u_z=-\Delta_H\xi \tag{1.16}$$
$$\text{where,}\quad \Delta_H\equiv\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

We now derive equations for the scalar fields ψ and ξ by taking the curl and double curl of eqn. 1.4a. It is useful to note the following components

$$(\boldsymbol{\nabla}\times\xi\hat{\mathbf{e}}_z)_r=\frac{1}{r}\frac{\partial\xi}{\partial\theta},\quad(\boldsymbol{\nabla}\times\xi\hat{\mathbf{e}}_z)_\theta=-\frac{\partial\xi}{\partial r},\quad(\boldsymbol{\nabla}\times\xi\hat{\mathbf{e}}_z)_z=0 \tag{1.17a,b,c}$$

$$(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\xi\hat{\mathbf{e}}_z)_r=\frac{\partial^2\xi}{\partial z\partial r},\quad(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\xi\hat{\mathbf{e}}_z)_\theta=\frac{1}{r}\frac{\partial^2\xi}{\partial z\partial\theta},\quad(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\xi\hat{\mathbf{e}}_z)_z=-\Delta_{\mathrm{H}}\xi \tag{1.18a,b,c}$$

where $\Delta_{\mathrm{H}}\equiv\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$ and

$$(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\xi\hat{\mathbf{e}}_z)_r=-\frac{1}{r}\frac{\partial}{\partial\theta}(\Delta_{\mathrm{H}}\xi)-\frac{1}{r}\frac{\partial^2}{\partial z^2}\left(\frac{\partial\xi}{\partial\theta}\right),$$
$$(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\xi\hat{\mathbf{e}}_z)_\theta=\frac{\partial^3\xi}{\partial z^2\partial r}+\frac{\partial}{\partial r}(\Delta_{\mathrm{H}}\xi),\quad(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\xi\hat{\mathbf{e}}_z)_z=0 \tag{1.19a,b,c}$$

The curl of eqn. 1.4a leads to the diffusion equation for vorticity,

$$\frac{\partial\boldsymbol{\omega}}{\partial t}=\nu\boldsymbol{\nabla}\times\boldsymbol{\Delta}\mathbf{u}=-\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\omega}=\nu\boldsymbol{\Delta}\boldsymbol{\omega} \tag{1.20}$$

where in the second equality in 1.21, we have used $\boldsymbol{\nabla}\cdot\boldsymbol{\omega}=0$. Similarly, the double curl of eqn. 1.4a implies

$$\frac{\partial}{\partial t}\boldsymbol{\Delta}\mathbf{u}=\nu\boldsymbol{\Delta}\boldsymbol{\Delta}\mathbf{u} \tag{1.21}$$

From 1.15, we note that

$$\boldsymbol{\omega}=\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times(\psi\hat{\mathbf{e}}_z)+\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times(\xi\hat{\mathbf{e}}_z) \tag{1.22}$$

The z-component of eqn. 1.22 alongwith eqns. 1.18c and 1.19c imply,

$$(\boldsymbol{\omega})_z \equiv \omega_z = -\Delta_{\rm H}\psi \tag{1.23}$$

Similarly, eqns. 1.15 alongwith eqns. 1.17c and 1.18c imply

$$(\mathbf{u})_z \equiv u_z = -\Delta_{\rm H}\xi \tag{1.24}$$

The z-component of eqn. 1.20 is (noting that $(\boldsymbol{\Delta\omega})_z = \Delta\omega_z$)

$$\left(\frac{\partial}{\partial t} - \nu\Delta\right)\Delta_{\rm H}\psi = 0 \tag{1.25}$$

Similarly the z component of 1.21 is

$$\left(\frac{\partial}{\partial t} - \nu\Delta\right)\Delta\Delta_{\rm H}\xi = 0 \tag{1.26}$$

Note that Δ is related to $\Delta_{\rm H}$ through $\Delta - \Delta_{\rm H} = \frac{\partial^2}{\partial z^2}$.

In order to determine the scalar fields $\psi(r,\theta,z,t), \xi(r,\theta,z,t)$, we need to solve equations 1.251.26. Analogous to the inviscid analysis in Patankar *et al.* (2018) we seek three dimensional standing wave solutions of the form

$$\psi(r,\theta,z,t) = \Psi_m(r,t;k)\sin(m\theta)\cos(kz), \quad \xi(r,\theta,z,t) = \Xi_m(r,t;k)\cos(m\theta)\sin(kz),$$
$$\eta(\theta,z,t) = a_m(t;k)\cos(m\theta)\cos(kz), \tag{1.27a,b,c}$$

where $k \in \mathbb{R}^+$ and $m \in \mathbb{Z}^+$. Substituting eqns 1.27 (a,b) into eqns. 1.25 and 1.26 we obtain the equations governing $\Psi_m(r,t;k)$ and $\Xi_m(r,t;k)$ viz.

$$\left(\frac{\partial}{\partial t} - \nu\mathcal{L}\right)\mathcal{L}_{\mathcal{H}}\Psi = 0, \quad \left(\frac{\partial}{\partial t} - \nu\mathcal{L}\right)\mathcal{L}\mathcal{L}_{\mathcal{H}}\Xi = 0 \tag{1.28a,b}$$
$$\text{where} \quad \mathcal{L}_{\mathcal{H}} \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2}, \quad \mathcal{L} \equiv \mathcal{L}_H - k^2.$$

Using the Floquet ansatz for time periodic base states, we assume the following forms for $\Psi_m(r,t;k), \Xi_m(r,t;k)$ and $a_m(t;k)$

$$\Psi_m(r,t;k) = e^{\lambda_m(k)t}\sum_{n=-\infty}^{\infty}\tilde{\psi}_n^{(m)}(r;k)e^{in\Omega t}, \ \Xi_m(r,t;k) = e^{\lambda_m(k)t}\sum_{n=-\infty}^{\infty}\tilde{\xi}_n^{(m)}(r;k)e^{in\Omega t},$$
$$a_m(t;k) = e^{\lambda_m(k)t}\sum_{n=-\infty}^{\infty}\mathcal{M}_n e^{in\Omega t} \tag{1.29a,b,c}$$

with $\lambda_m(k)$ being the Floquet exponent and $\tilde{\psi}_n^{(m)}(r;k)$ and $\tilde{\xi}_n^{(m)}(r;k)$ are the complex eigenfunctions for each Fourier mode (k,m).

We substitute 1.29(a,b) into 1.28(a,b) respectively yielding fourth and sixth order differential equations governing $\tilde{\psi}_n^{(m)}(r;k)$ and $\tilde{\xi}_n^{(m)}(r;k)$ for each n in the expansion 1.29(a,b).

$$\mathbf{O}^{(k,m)}\cdot\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right)\tilde{\psi}_n^{(m)}(r;k) = 0,$$
$$\mathbf{O}^{(k,m)}\cdot\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} - k^2\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right)\tilde{\xi}_n^{(m)}(r;k) = 0, \tag{1.30a,b}$$

where the linear operator $\mathbf{O}^{(k,m)} \equiv \left[\lambda_m(k) + in\Omega - \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} - k^2\right)\right]$. Equations 1.30(a,b) are solved with the finiteness condition at $r \to 0$. We show the details for solution to $\tilde{\psi}_n^{(m)}(r;k)$, with $\mathcal{L}_{\mathcal{H}}\tilde{\psi}_n^{(m)}(r;k) \equiv \mathcal{G}_n^{(m)}(r;k)$, eqns. 1.30a may be re-written as,

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left(\frac{m^2}{r^2} + j_n^2\right)\right]\mathcal{G}_n^{(m)}(r;k) = 0, \quad j_n^2 \equiv k^2 + \frac{\lambda_m(k) + in\Omega}{\nu} \tag{1.31}$$

Eq. 1.31 is the modified Bessel equation whose solutions are

$$\mathcal{G}_n^{(m)}(r;k) = A_{1n}\mathrm{I}_m(j_n r) + A_{2n}\mathrm{K}_m(j_n r) \tag{1.32}$$

Due to boundedness as $r \to 0$, $A_{2n} = 0$ and we obtain from 1.32,

$$\mathcal{L}_{\mathcal{H}}\tilde{\psi}_n^{(m)}(r;k) = A_{1n}\mathrm{I}_m(j_n r), \qquad \mathrm{Re}(j_n) > 0 \tag{1.33}$$

The final solution to eqn. 1.33 is

$$\tilde{\psi}_n^{(m)}(r;k) = \mathcal{A}_n\mathrm{I}_m(lr) + \mathcal{B}_n r^m, \qquad A_n \equiv \frac{A_{1n}}{j_n^2} \tag{1.34}$$

Similarly we obtain solution for $\tilde{\xi}_n^{(m)}(r;k)$,

$$\tilde{\xi}_n^{(m)}(r;k) = C_n\mathrm{I}_m(j_n r) + \mathcal{D}_n\mathrm{I}_m(kr) + \mathcal{E}_n r^m \tag{1.35}$$

The Floquet ansatz in eqn. 1.29(a,b) implies that the velocity components may be written as

$$(u_r, u_\theta, u_z) = \sum_{n=-\infty}^{\infty}\left(\tilde{u}_{r,n}(r)\cos(m\theta)\cos(kz), \tilde{u}_{\theta,n}(r)\sin(m\theta)\cos(kz), \tilde{u}_{z,n}(r)\cos(m\theta)\sin(kz)\right) \times \exp\left[(in\Omega + \lambda_m(k))\,t\right] \tag{1.36}$$

where the (complex) eigenmodes $\tilde{u}_{r,n}(r), \tilde{u}_{\theta,n}(r)$ and $\tilde{u}_{z,n}(r)$ are determined using expressions 1.16. The components of the velocity field components written in terms of ψ and ξ are

$$\tilde{u}_{r,n}(r) = \frac{1}{r}\frac{\partial\tilde{\psi}_n^{(m)}(r;k)}{\partial\theta} + \frac{\partial^2\tilde{\Xi}_n^{(m)}(r;k)}{\partial z\partial r}, \quad \tilde{u}_{\theta,n}(r) = -\frac{\partial\tilde{\psi}_n^{(m)}(r;k)}{\partial r} + \frac{1}{r}\frac{\partial^2\tilde{\Xi}_n^{(m)}(r;k)}{\partial z\partial\theta} \tag{1.37}$$

$$\tilde{u}_{z,n}(r) = -\Delta_{\mathrm{H}}\tilde{\Xi}_n^{(m)}(r;k) \tag{1.38}$$

Substituting 1.34 & 1.35 into above equation leads to

$$\begin{aligned}
\tilde{u}_{r,n}(r) &= \frac{m}{r}\mathrm{I}_m(j_n r)\mathcal{A}_n + m\mathcal{B}_n r^{m-1} + kj_n\mathrm{I}_m^{'}(j_n r)C_n + k^2\mathrm{I}_m^{'}(kr)\mathcal{D}_n + km\mathcal{E}_n r^{m-1}\\
\tilde{u}_{\theta,n}(r) &= -\left\{j_n\mathrm{I}_m^{'}(j_n r)\mathcal{A}_n + m\mathcal{B}_n r^{m-1} + \frac{km}{r}\left(\mathrm{I}_m(j_n r)C_n + \mathrm{I}_m(kr)\mathcal{D}_n\right) + km\mathcal{E}_n r^{m-1}\right\}\\
\tilde{u}_{z,n}(r) &= -\left\{j_n^2\mathrm{I}_m(j_n r)C_n + k^2\mathrm{I}_m(kr)\mathcal{D}_n\right\},
\end{aligned} \tag{1.39a,b,c}$$

prime indicating differentiation with respect to the argument e.g. $\mathrm{I}_m^{'}(z) \equiv \dfrac{d\mathrm{I}_m}{dz}$. The compatibility condition is

$$\mathbf{e}_r \cdot \left(\frac{\partial}{\partial t} - \nu\Delta\right)\boldsymbol{\omega} = 0 \quad \text{at } r = R_0 \tag{1.40}$$

$$\frac{\partial\omega_r}{\partial t} = \nu(\Delta\boldsymbol{\omega})_r = \Delta\omega_r - \frac{\omega_r}{r^2} - \frac{2}{r^2}\frac{\partial\omega_\theta}{\partial\theta} \tag{1.41}$$

The components of vorticity field is given by following expressions,

$$\omega_r = \frac{\partial^2\psi}{\partial z\partial r} - \frac{1}{r}\frac{\partial\Delta\xi}{\partial\theta}$$
$$\omega_\theta = \frac{1}{r}\frac{\partial^3\psi}{\partial z\partial\theta^2} + \frac{\partial^2\Delta\xi}{\partial r\partial\theta}$$
$$\omega_z = -\Delta_{\mathrm{H}}\psi$$

Also,

$$\Delta\omega_r = \frac{\partial^2\Delta\psi}{\partial z\partial r} + \frac{2}{r^3}\frac{\partial^3\psi}{\partial z\partial\theta^2} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial z\partial r} - \frac{1}{r}\frac{\partial\Delta\Delta\xi}{\partial\theta} - \frac{1}{r^3}\frac{\partial\Delta\xi}{\partial\theta} + \frac{2}{r^2}\frac{\partial^2\Delta\xi}{\partial r\partial\theta} \tag{1.43}$$

Using the above expressions into 1.41, we obtain

$$\frac{\partial}{\partial t}\left[\frac{\partial^2\psi}{\partial z\partial r} - \frac{1}{r}\frac{\partial\Delta\xi}{\partial\theta}\right] = \frac{\partial^2\Delta\psi}{\partial z\partial r} - \frac{1}{r}\frac{\partial\Delta\Delta\xi}{\partial\theta} \tag{1.44}$$

After some algebra we finally arrive at one simple condition from the above equation,

$$k\mathcal{E}(s) + \mathcal{B}(s) = 0 \tag{1.45}$$

By using 1.45 in equation 1.39(a,b), we get rid of $\mathcal{B}_n$ & $\mathcal{E}_n$ which leads to

$$\tilde{u}_{r,n}(r) = \frac{m}{r}\mathrm{I}_m(j_nr)\mathcal{A}_n + kj_n\mathrm{I}_m^{'}(j_nr)\mathcal{C}_n + k^2\mathrm{I}_m^{'}(kr)\mathcal{D}_n$$
$$\tilde{u}_{\theta,n}(r) = -\left\{j_n\mathrm{I}_m^{'}(j_nr)\mathcal{A}_n + \frac{km}{r}\Big(\mathrm{I}_m(j_nr)\mathcal{C}_n + \mathrm{I}_m(kr)\mathcal{D}_n\Big)\right\}$$
$$\tilde{u}_{z,n}(r) = -\left\{j_n^2\mathrm{I}_m(j_nr)\mathcal{C}_n + k^2\mathrm{I}_m(kr)\mathcal{D}_n\right\}, \tag{1.46a,b,c}$$

1.1. *Boundary conditions*

The kinematic boundary condition in eqn. 1.5 leads to

$$\frac{m}{R_0}\mathrm{I}_m(j_nR_0)\mathcal{A}_n + kj_n\mathrm{I}_m'(j_nR_0)\mathcal{C}_n + k^2\mathrm{I}_m'(kR_0)\mathcal{D}_n = [\lambda_m(k) + in\Omega]\mathcal{M}_n \tag{1.47}$$

The zero shear stress conditions eqns. 1.6 and 1.7 are

$$\frac{mk}{R_0}\mathrm{I}_m(j_nR_0)\mathcal{A}_n + \left(j_n^2 + k^2\right)j_n\mathrm{I}_m'(j_nR_0)\mathcal{C}_n + 2k^3\mathrm{I}_m'(kR_0)\mathcal{D}_n = 0 \tag{1.48}$$
$$\frac{m}{R_0}\mathrm{I}_m(j_nR_0)\Lambda_{1n}\mathcal{A}_n + 2kj_n\mathrm{I}_m'(j_nR_0)\Lambda_{2n}\mathcal{C}_n + 2k^2\mathrm{I}_m'(kR_0)\Lambda_3\mathcal{D}_n = 0 \tag{1.49}$$

$$\text{where}\quad \Lambda_{1n} \equiv 1 - \frac{j_nR_0}{m^2}\frac{\mathrm{I}_m'(j_nR_0)}{\mathrm{I}_m(j_nR_0)} + \frac{j_n^2R_0^2}{m^2}\frac{\mathrm{I}_m''(j_nR_0)}{\mathrm{I}_m(j_nR_0)},$$
$$\Lambda_{2n} \equiv 1 - \frac{1}{j_nR_0}\frac{\mathrm{I}_m(j_nR_0)}{\mathrm{I}_m'(j_nR_0)} \quad\text{and}\quad \Lambda_3 \equiv 1 - \frac{1}{kR_0}\frac{\mathrm{I}_m(kR_0)}{\mathrm{I}_m'(kR_0)}.$$

We rewrite the normal stress jump condition 1.14 as,

$$\mu\left[\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\left(\Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\left(\frac{\partial u_\theta}{\partial\theta}\right) - \frac{1}{\nu}\frac{\partial u_r}{\partial t}\right) - \frac{\mathcal{F}(r,t)\Delta_0\eta}{\nu} + 2\Delta_0\left(\frac{\partial u_r}{\partial r}\right)\right]$$
$$= \frac{T}{R_0^2}\Delta_0\left[\eta + \left(\frac{\partial^2\eta}{\partial\theta^2}\right) + R_0^2\left(\frac{\partial^2\eta}{\partial z^2}\right)\right] \text{ at } r = R_0 \tag{1.50}$$

We substitute the solution obtained previously for u_r, u_θ and η with their Floquet ansatz into above expression. We show simple expansion of above equation term by term below for any, n.

$$\Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\left(\frac{\partial u_\theta}{\partial \theta}\right) - \frac{1}{\nu}\frac{\partial u_r}{\partial t} = -\frac{(\lambda_m(k)+in\Omega)}{\nu} k^2 \mathrm{I}'_m(kr) D_n e^{(\lambda_m(k)+in\Omega)t}\cos(m\theta)\cos(kz) \tag{1.51}$$

$$\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\left(\Delta u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\left(\frac{\partial u_\theta}{\partial \theta}\right) - \frac{1}{\nu}\frac{\partial u_r}{\partial t}\right) = -\frac{(\lambda_m(k)+in\Omega)}{\nu}\left(k^3 \mathrm{I}''_m(kr) + \frac{k^2}{r}\mathrm{I}'_m(kr)\right) D_n e^{(\lambda_m(k)+in\Omega)t}\cos(m\theta)\cos(kz) \tag{1.52}$$

Now, rest of the terms are pretty straight forward in eqn 1.50. Substituting 1.51 & 1.52 back in 1.50 leads to,

$$\Bigg[\mu\Bigg\{\left[(k^2 - j_n^2)\tfrac{k\mathrm{I}'_m(kR_0)}{R_0} - \left(k^2 + j_n^2 + \tfrac{2m^2}{R_0^2}\right)k^2\mathrm{I}''_m(kR_0)\right]k\mathcal{D}_n - 2\left(k^2 + \tfrac{m^2}{R_0^2}\right) j_n^2 \mathrm{I}''_m(j_nR_0)kC_n$$
$$-2\left(k^2 + \tfrac{m^2}{R_0^2}\right)\tfrac{m}{R_0}\left(j_n\mathrm{I}'_m(j_nR_0) - \tfrac{\mathrm{I}_m(j_nR_0)}{R_0}\right)\mathcal{A}_n\Bigg\}$$
$$-\tfrac{T}{R_0^2}\left(k^2 + \tfrac{m^2}{R_0^2}\right)\left(k^2R_0^2 + m^2 - 1\right)\mathcal{M}_n\Bigg]\left(\tfrac{2R_0^2}{\rho\left(k^2R_0^2+m^2\right)}\right) = h\left[\mathcal{M}_{n-1} + \mathcal{M}_{n+1}\right] \tag{1.53}$$

Using eqns. 1.47, 1.48 and 1.49, we solve for $\mathcal{A}_n, C_n$ and $\mathcal{D}_n$ in terms of $(\lambda_m(k)+in\Omega)\mathcal{M}_n$ in Mathematica software and substituted back in 1.53. We re-write the above as a generalised eigenvalue problem with $A_n = A_n^r + \mathrm{i}A_n^i$ complex.

$$A_n\mathcal{M}_n = h\left(\mathcal{M}_{n+1} + \mathcal{M}_{n-1}\right) \tag{1.54}$$

We obtain the stability charts in a similar manner described in our inviscid study Patankar *et al.* (2018) following the idea from Kumar & Tuckerman (1994) by doing the numerical floquet analysis. The coefficients A_n depend on the Floquet exponent $\lambda_m(k) = \mu + \mathrm{i}\alpha$. So, for fixed Floquet exponent, 1.54 can be considered to be an eigenvalue problem with eigenvalues h and eigenvectors whose components are the real and imaginary parts of the $\mathcal{M}_n$. That is, we write 1.54 as the generalized eigenvalue problem.

$$\mathbf{A}\cdot\mathbf{M} = h\,\mathbf{Q}\cdot\mathbf{M} \qquad\qquad n = 0, 1, 2, ...N \tag{1.55}$$

The reality conditions: $\mathcal{M}_{-1} = \mathcal{M}_1^*$ (harmonic) or $\mathcal{M}_{-1} = \mathcal{M}_0^*$ (subharmonic)

In 1.55, $\mathbf{A}$ is a diagonal complex matrix and $\mathbf{Q}$ is a banded matrix whose structure depends on α. In the harmonic case, ($\alpha = 0$) we have

$$\begin{pmatrix} A_0^r & -A_0^i & 0 & 0 & 0 & 0 & \cdots \\ A_0^i & A_0^r & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & A_1^r & -A_1^i & 0 & 0 & \cdots \\ 0 & 0 & A_1^i & A_1^r & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & A_2^r & -A_2^i & \cdots \\ 0 & 0 & 0 & 0 & A_2^i & A_2^r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \mathcal{M}_0^r \\ \mathcal{M}_0^i \\ \mathcal{M}_1^r \\ \mathcal{M}_1^i \\ \mathcal{M}_2^r \\ \mathcal{M}_2^i \\ \vdots \end{pmatrix} = h \begin{pmatrix} 0 & 0 & 2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \mathcal{M}_0^r \\ \mathcal{M}_0^i \\ \mathcal{M}_1^r \\ \mathcal{M}_1^i \\ \mathcal{M}_2^r \\ \mathcal{M}_2^i \\ \vdots \end{pmatrix} \tag{1.56}$$

and in the subharmonic case, ($\alpha = \frac{\Omega}{2}$) we have

$$\begin{pmatrix} A_0^r & -A_0^i & 0 & 0 & 0 & 0 & \cdots \\ A_0^i & A_0^r & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & A_1^r & -A_1^i & 0 & 0 & \cdots \\ 0 & 0 & A_1^i & A_1^r & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & A_2^r & -A_2^i & \cdots \\ 0 & 0 & 0 & 0 & A_2^i & A_2^r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \mathcal{M}_0^r \\ \mathcal{M}_0^i \\ \mathcal{M}_1^r \\ \mathcal{M}_1^i \\ \mathcal{M}_2^r \\ \mathcal{M}_2^i \\ \vdots \end{pmatrix} = h \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \mathcal{M}_0^r \\ \mathcal{M}_0^i \\ \mathcal{M}_1^r \\ \mathcal{M}_1^i \\ \mathcal{M}_2^r \\ \mathcal{M}_2^i \\ \vdots \end{pmatrix} \tag{1.57}$$

The subharmonic ($\alpha = \frac{\Omega}{2}$) and harmonic cases ($\alpha = 0$) are solved using the matlab generalised eigenvalue solver *eig(,)*, MATLAB (2015) considering $N = 30$ in the Fourier series. In the present method, we fix $\mu + i\alpha$, with $\mu = 0$ for the stability curves and with $\alpha = \frac{\Omega}{2}$ or $\alpha = 0$ for the sub-harmonic and harmonic cases respectively. We then solve 1.55 for given fluid parameters and (k, m), to obtain eigenvalues h. Only real and positive values of are selected in this context. This gives us the marginal stability curves in $h - k$ plane.

2. IVP-equation for $a_m(t;k)$

In this section, we derive an equation governing the amplitude of Faraday waves $a_m(t;k)$. The derivation is totally adapted from the previous section of Floquet analysis. We only show the key equations to obtain the final amplitude equation. The starting point of the present derivation are eqns. 1.25 & 1.26. We employ Laplace transforms defined as

$$\left[\tilde{\Psi}^{(m)}(r,s;k), \tilde{\Xi}^{(m)}(r,s;k), \tilde{a}_m(s;k)\right] = \int_0^\infty \exp(-st) \left[\Psi_m(r,t;k), \Xi_m(r,t;k), a_m(t;k)\right] dt \tag{2.1}$$

The Laplace transform operator is indicated as $\hat{\mathbf{L}}(\cdot)$ in further algebra and variables in the Laplace domain are indicated with a tilde on top. Laplace transforming eqns. 1.28 with the initial conditions $\Psi_m(r,0;k) = \Xi_m(r,0;k) = 0$, $\dot{a}_m(0;k) = 0$ and $a_m(0;k) = a_0$ which correspond to deformation of the free surface and zero perturbation velocity initially, we obtain

$$(s - \nu\mathcal{L})\,\mathcal{L}_H\tilde{\Psi}^{(m)}(r,s;k) = 0, \quad (s - \nu\mathcal{L})\,\mathcal{L}\mathcal{L}_H\tilde{\Xi}^{(m)}(r,s;k) = 0 \tag{2.2a,b}$$

The solution to eqns. 2.2(a,b) which stay finite as $r \longrightarrow 0$ are the counterparts of expressions 1.34 and 1.35. These are

$$\tilde{\Psi}^{(m)}(r,s;k) = \mathcal{A}(s)\mathrm{I}_m(lr) + \mathcal{B}(s)r^m, \qquad \tilde{\Xi}^{(m)}(r,s) = \mathcal{C}(s)\mathrm{I}_m(lr) + \mathcal{D}(s)\mathrm{I}_m(kr) + \mathcal{E}(s)r^m \tag{2.3a}$$

$$\text{where} \quad l^2(s) \equiv k^2 + \frac{s}{\nu}, \quad Re(l) > 0.$$

and $\mathcal{A}(s), \mathcal{B}(s), \mathcal{C}(s), \mathcal{D}(s)$ and $\mathcal{E}(s)$ are unknown functions to be determined subsequently. The algebra which follows is enormously simplified by recognising that the set of variables $\left[\mathcal{A}(s), \mathcal{B}(s), \mathcal{C}(s), \mathcal{D}(s), l^2\right]$ in this section are the analogues of the corresponding set $\left[\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n, \mathcal{D}_n, j_n^2\right]$ used in the previous section. The compatibility condition is thus

$$\mathcal{B}(s) + k\mathcal{E}(s) = 0 \tag{2.4}$$

Using the above, the expressions of the three components of velocity in Laplace domain are obtained as

$$\begin{aligned}
\tilde{u}_r(r,\theta,z,s) &= \left[\frac{m}{r}\mathrm{I}_m(lr)\mathcal{A}(s) + kl\mathrm{I}'_m(lr)\mathcal{C}(s) + k^2\mathrm{I}'_m(kr)\mathcal{D}(s)\right]\cos(m\theta)\cos(kz)\\
\tilde{u}_\theta(r,\theta,z,s) &= -\left[l\mathrm{I}'_m(lr)\mathcal{A}(s) + \frac{km}{r}\left(\mathrm{I}_m(lr)\mathcal{C}(s) + \mathrm{I}_m(kr)\mathcal{D}(s)\right)\right]\sin(m\theta)\cos(kz)\\
\tilde{u}_z(r,\theta,z,s) &= -\left[l^2\mathrm{I}_m(lr)\mathcal{C}(s) + k^2\mathrm{I}_m(kr)\mathcal{D}(s)\right]\cos(m\theta)\sin(kz)\\
\tilde{\eta}(z,\theta,s) &= \tilde{a}_m(s;k)\cos(m\theta)\cos(kz)
\end{aligned} \tag{2.5a,b,c,d}$$

Using 2.5, we express boundary conditions in the Laplace transformed domain. These are,

$$\tilde{u}_r(R_0,\theta,z,s) = s\tilde{\eta}(z,\theta,s) - \eta(z,\theta,0), \tag{2.6a}$$

$$\mu\left(\frac{\partial\tilde{u}_r}{\partial z} + \frac{\partial\tilde{u}_z}{\partial r}\right)_{r=R_0} = 0, \quad \mu\left(r\frac{\partial}{\partial r}\left(\frac{\tilde{u}_\theta}{r}\right) + \frac{1}{r}\frac{\partial\tilde{u}_r}{\partial\theta}\right)_{r=R_0} = 0, \tag{2.6b,c}$$

$$\begin{aligned}
&\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\left[s\tilde{u}_r - \nu\left\{\Delta\tilde{u}_r - \frac{\tilde{u}_r}{r^2} - \frac{2}{r^2}\left(\frac{\partial\tilde{u}_\theta}{\partial\theta}\right)\right\}\right] - \left(\frac{m^2}{r^2} + k^2\right)\tilde{\mathcal{F}}(r,s) * \tilde{\eta}(z,\theta,s)\\
&- 2\nu\Delta_O\left(\frac{\partial\tilde{u}_r}{\partial r}\right) = \frac{T}{\rho R_0^2}\left(1 - m^2 - k^2R_0^2\right)\left(\frac{m^2}{r^2} + k^2\right)\tilde{\eta}(z,\theta,s) \quad \text{at } r = R_0,
\end{aligned} \tag{2.6d}$$

$$\text{with} \quad \Delta_O \equiv \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2},$$

Note the convolution like integral in eqn. 2.6d arises from the Laplace transform of the product of two functions (see Prosperetti (2011), eqn. 11.2.24)

$$\hat{\mathbf{L}}\left(\mathcal{F}(r,t)\cdot\eta(z,\theta,t)\right) = \tilde{\mathcal{F}}(r,s) * \tilde{\eta}(z,\theta,s) \tag{2.7}$$

One may combine eqns. 2.6(a,b,c) with the expressions in 2.5(a,b,c,d) to obtain linear, inhomogenous equations governing $\mathcal{A}(s), C(s)$ and $\mathcal{D}(s)$. These are

$$\frac{m}{R_0}\mathrm{I}_m(lR_0)\mathcal{A}(s) + kl\mathrm{I}'_m(lR_0)C(s) + k^2\mathrm{I}'_m(kR_0)\mathcal{D}(s) = s\tilde{a}_m(s;k) - a(0)$$

$$\frac{mk}{R_0}\mathrm{I}_m(lR_0)\mathcal{A}(s) + \left(k^2+l^2\right) l\mathrm{I}'_m(lR_0)C(s) + 2k^3\mathrm{I}'_m(kR_0)\mathcal{D}(s) = 0,$$

$$\frac{m}{R_0}\Lambda_1(s)\mathrm{I}_m(lR_0)\mathcal{A}(s) + 2kl\mathrm{I}'_m(lR_0)\Lambda_2(s)C(s) + 2k^2\mathrm{I}'_m(kR_0)\Lambda_3\mathcal{D}(s) = 0, \quad \text{(2.8a,b,c)}$$

$$\text{where,}\quad \Lambda_1(s) = 1 - \frac{lR_0}{m^2}\frac{\mathrm{I}'_m(lR_0)}{\mathrm{I}_m(lR_0)} + \frac{R_0^2 l^2}{m^2}\frac{\mathrm{I}''_m(lR_0)}{\mathrm{I}_m(lR_0)}$$

$$\Lambda_2(s) = 1 - \frac{1}{lR_0}\frac{\mathrm{I}_m(lR_0)}{\mathrm{I}'_m(lR_0)}, \quad \text{and } \Lambda_3 = 1 - \frac{1}{kR_0}\frac{\mathrm{I}_m(kR_0)}{\mathrm{I}'_m(kR_0)}.$$

Analogous to the previous section, we obtain expressions for $\mathcal{A}(s), C(s)$ and $\mathcal{D}(s)$ from eqns. 2.8(a,b,c) in terms of $\tilde{a}_m$. These are

$$\mathcal{A}(s) = 2k^2 l\mathrm{I}'_m(lR_0)\mathrm{I}'_m(kR_0)\left\{\frac{\left(l^2+k^2\right)\Lambda_3 - 2k^2\Lambda_2(s)}{\beta(s)}\right\}\left[s\tilde{a}_m - a(0)\right] \quad (2.9)$$

$$C(s) = \frac{2mk^3}{R_0}\mathrm{I}_m(lR_0)\mathrm{I}'_m(kR_0)\left(\frac{\Lambda_1(s) - \Lambda_3}{\beta(s)}\right)\left[s\tilde{a}_m - a(0)\right] \quad (2.10)$$

$$\mathcal{D}(s) = \frac{ml}{R_0}\mathrm{I}_m(lR_0)\mathrm{I}'_m(lR_0)\left\{\frac{2k^2\Lambda_2(s) - \left(l^2+k^2\right)\Lambda_1(s)}{\beta(s)}\right\}\left[s\tilde{a}_m(s;k) - a(0)\right] \quad (2.11)$$

$$\text{where,}\quad \beta(s) = \mathrm{Det}\begin{bmatrix} \frac{m}{R_0}\mathrm{I}_m(lR_0) & kl\mathrm{I}'_m(lR_0) & k^2\mathrm{I}'_m(kR_0) \\ \frac{mk}{R_0}\mathrm{I}_m(lR_0) & \left(l^2+k^2\right) l\mathrm{I}'_m(lR_0) & 2k^3\mathrm{I}'_m(kR_0) \\ \frac{m}{R_0}\mathrm{I}_m(lR_0)\Lambda_1(s) & 2kl\mathrm{I}'_m(lR_0)\Lambda_2(s) & 2k^2\mathrm{I}'_m(kR_0)\Lambda_3 \end{bmatrix}$$

$$= \frac{mlk^2}{R_0}\mathrm{I}_m(lR_0)\mathrm{I}'_m(lR_0)\mathrm{I}'_m(kR_0)\Lambda(s) \quad (2.12)$$

$$\text{and}\quad \Lambda(s) \equiv \left(k^2 - l^2\right)\Lambda_1(s) - 2k^2\Lambda_2(s) + 2l^2\Lambda_3 \quad (2.13)$$

We simplify the Laplace transformed normal stress condition 2.6d in a similar mannner done before using 2.5(a,b,d) and the expressions for $\mathcal{A}(s), C(s)$ and $\mathcal{D}(s)$ calculated above.

$$\left(k^2 + \frac{m^2}{R_0^2}\right)\left[\frac{T}{\rho}\left(k^2 + \frac{m^2-1}{R_0^2}\right)\tilde{a}_m - \tilde{\mathcal{F}}(R_0;s) * \tilde{a}_m\right]$$
$$+2\nu\frac{m}{R_0}\left(k^2 + \frac{m^2}{R_0^2}\right)\Lambda_2(s)l\mathrm{I}'_m(lR_0)\mathcal{A}(s) + 2\nu k\left(k^2 + \frac{m^2}{R_0^2}\right)l^2\mathrm{I}''_m(lR_0)C(s)$$
$$+k\left[2\nu k^2\left(k^2 + \frac{m^2}{R_0^2}\right)\mathrm{I}''_m(kR_0) + s\left\{k^2\mathrm{I}''_m(kR_0) + \frac{k\mathrm{I}'_m(kR_0)}{R_0}\right\}\right]\mathcal{D}(s) = 0 \quad (2.14)$$

$$\Rightarrow \left[\frac{T}{\rho}\left(k^2 + \frac{m^2-1}{R_0^2}\right)\tilde{a}_m - \tilde{\mathcal{F}}(R_0;s) * \tilde{a}_m\right] + 2\nu\frac{m}{R_0}\Lambda_2(s)l\mathrm{I}'_m(lR_0)\mathcal{A}(s)$$
$$+2\nu kl^2\mathrm{I}''_m(lR_0)C(s) + 2\nu k^3\mathrm{I}''_m(kR_0)\mathcal{D}(s) + k\mathrm{I}_m(kR_0)s\mathcal{D}(s) = 0 \quad (2.15)$$

and using the expressions for $\mathcal{A}(s), C(s)$ and $\mathcal{D}(s)$, we obtain

$$\frac{\Lambda(s)}{2k^2\Lambda_2(s)-\left(l^2+k^2\right)\Lambda_1(s)}\left[\frac{T}{\rho R_0^2}\frac{\left(k^2R_0^2+m^2-1\right)}{\frac{\mathrm{I}_m(kR_0)}{k\mathrm{I}'_m(kR_0)}}\tilde{a}_m-\frac{k\mathrm{I}'_m(kR_0)}{\mathrm{I}_m(kR_0)}\tilde{\mathcal{F}}(R_0;s)*\tilde{a}_m\right]$$

$$+4\nu\frac{k\mathrm{I}'_m(kR_0)}{\mathrm{I}_m(kR_0)}\left[\frac{l\mathrm{I}'_m(lR_0)}{\mathrm{I}_m(lR_0)}\left\{\frac{\left(l^2+k^2\right)\Lambda_3-2k^2\Lambda_2(s)}{2k^2\Lambda_2(s)-\left(l^2+k^2\right)\Lambda_1(s)}\right\}\Lambda_2(s)+k^2l\frac{\mathrm{I}''_m(lR_0)}{\mathrm{I}'_m(lR_0)}\left\{\frac{\Lambda_1(s)-\Lambda_3}{2k^2\Lambda_2(s)-\left(l^2+k^2\right)\Lambda_1(s)}\right\}\right]\left[s\tilde{a}_m-a(0)\right]$$

$$+2\nu k^2\frac{\mathrm{I}''_m(kR_0)}{\mathrm{I}_m(kR_0)}\left[s\tilde{a}_m-a(0)\right]+\left[s^2\tilde{a}_m-sa(0)-\dot{a}(0)\right]=0 \tag{2.16}$$

which maybe expressed as

$$\left[s^2\tilde{a}_m-sa(0)-\dot{a}(0)\right]+2\nu k^2\frac{\mathrm{I}''_m(kR_0)}{\mathrm{I}_m(kR_0)}\left[s\tilde{a}_m-a(0)\right]$$

$$+\tilde{\chi}(s)\left[\frac{T}{\rho R_0^2}\frac{\left(k^2R_0^2+m^2-1\right)}{\frac{\mathrm{I}_m(kR_0)}{k\mathrm{I}'_m(kR_0)}}\tilde{a}_m-\frac{k\mathrm{I}'_m(kR_0)}{\mathrm{I}_m(kR_0)}\tilde{\mathcal{F}}(R_0;s)*\tilde{a}_m\right]+4\nu k\frac{\mathrm{I}'_m(kR_0)}{\mathrm{I}_m(kR_0)}\tilde{\zeta}(s)\left[s\tilde{a}_m-a(0)\right]=0 \tag{2.17}$$

The resultant expression maybe inverted into the time domain to obtain the integro-differential equation governing $a_m(t;k)$ i.e.

$$\frac{d^2a_m}{dt^2}+2\nu k^2\frac{\mathrm{I}''_m(kR_0)}{\mathrm{I}_m(kR_0)}\frac{da_m}{dt}+\int_0^t\hat{\mathbf{L}}^{-1}\left(\tilde{\chi}(s)\right)\frac{\mathrm{I}'_m(kR_0)}{\mathrm{I}_m(kR_0)}\left[\frac{T}{\rho R_0^3}kR_0\left(k^2R_0^2+m^2-1\right)\right.$$

$$\left.+hk\cos\left[\Omega(t-\tau)\right]\right]a_m(t-\tau)d\tau+4\nu k\frac{\mathrm{I}'_m(kR_0)}{\mathrm{I}_m(kR_0)}\int_0^t\hat{\mathbf{L}}^{-1}\left[\zeta(s)\right]\frac{da_m}{d\tau}(t-\tau)d\tau=0 \tag{2.18}$$

$$\text{where }\tilde{\chi}(s)\equiv\frac{\left(k^2-l^2\right)\Lambda_1(s)-2k^2\Lambda_2(s)+2l^2\Lambda_3}{2k^2\Lambda_2(s)-\left(l^2+k^2\right)\Lambda_1(s)},$$

$$\tilde{\zeta}(s)\equiv l\frac{\mathrm{I}'_m(lR_0)}{\mathrm{I}_m(lR_0)}\left\{\frac{2k^2\Lambda_2(s)-\left(l^2+k^2\right)\Lambda_3}{\left(l^2+k^2\right)\Lambda_1(s)-2k^2\Lambda_2(s)}\right\}\Lambda_2(s)-k^2l\frac{\mathrm{I}''_m(lR_0)}{\mathrm{I}'_m(lR_0)}\left\{\frac{\Lambda_1(s)-\Lambda_3}{\left(l^2+k^2\right)\Lambda_1(s)-2k^2\Lambda_2(s)}\right\},$$

Note that since inversion of $\tilde{\chi}(s)$ and $\tilde{\zeta}(s)$ is not possible analytically without further approximations, these inversions are only indicated formally as $\hat{\mathbf{L}}^{-1}(\cdot)$ in eqn. 2.18.

2.1. *Axisymmetric case $m = 0$*

For the axisymmetric case ($m = 0$), expressions for $\tilde{\chi}(s)$ and $\tilde{\zeta}(s)$ become very simple viz.

$$\tilde{\chi}(s)=\frac{l^2-k^2}{l^2+k^2}=\frac{s}{2\nu k^2+s},\quad\tilde{\zeta}(s)=-\frac{k^2}{l^2+k^2}\frac{\mathrm{I}''_0(lR_0)}{\mathrm{I}'_0(lR_0)}=-\frac{\nu lk^2}{s+2\nu k^2}\frac{\mathrm{I}''_0(lR_0)}{\mathrm{I}'_0(lR_0)} \tag{2.19}$$

Using this in 2.17 leads to,

$$\left[s^2\tilde{a}_0-sa(0)\right]+2\nu k^2\frac{\mathrm{I}''_0(kR_0)}{\mathrm{I}_0(kR_0)}\left[s\tilde{a}_0-a(0)\right]$$

$$+\frac{s}{(s+2\nu k^2)}\frac{\mathrm{I}'_0(kR_0)}{\mathrm{I}_0(kR_0)}\left[\frac{T}{\rho R_0^3}kR_0\left(k^2R_0^2-1\right)\tilde{a}_0-k\tilde{\mathcal{F}}(R_0;s)*\tilde{a}_0\right]$$

$$-4\nu k\frac{\mathrm{I}'_0(kR_0)}{\mathrm{I}_0(kR_0)}\frac{\nu lk^2}{s+2\nu k^2}\frac{\mathrm{I}''_0(lR_0)}{\mathrm{I}'_0(lR_0)}\left[s\tilde{a}_0-a(0)\right]=0$$

Multiplying the above equation by factor of $\frac{(2\nu k^2+s)}{s}$

$$\left[s^2\tilde{a}_0 - sa(0)\right] + 2\nu k^2\left(1+\frac{\mathrm{I}_0''(kR_0)}{\mathrm{I}_0(kR_0)}\right)\left[s\tilde{a}_0 - a(0)\right]$$
$$+\frac{\mathrm{I}_0'(kR_0)}{\mathrm{I}_0(kR_0)}\left[\frac{T}{\rho R_0^3}kR_0\left(k^2R_0^2-1\right)\tilde{a}_0 - k\tilde{\mathcal{F}}(R_0;s)*\tilde{a}_0\right]$$
$$+\frac{4\nu^2k^4}{\mathrm{I}_0(kR_0)}\left(\frac{\mathrm{I}_0''(kR_0)}{s} - \frac{l}{k}\frac{\mathrm{I}_0'(kR_0)\mathrm{I}_0''(lR_0)}{s\mathrm{I}_0'(lR_0)}\right)\left[s\tilde{a}_0 - a(0)\right] = 0$$

The above equation inverted to time domain is obtained as,

$$\frac{d^2a_0}{dt^2} + 2\nu k^2\left(1+\frac{\mathrm{I}_0''(kR_0)}{\mathrm{I}_0(kR_0)}\right)\frac{da_0}{dt} + \frac{\mathrm{I}_0'(kR_0)}{\mathrm{I}_0(kR_0)}\left[\frac{T}{\rho R_0^3}kR_0\left(k^2R_0^2-1\right) + hk\cos(\Omega t)\right]a_0(t)$$
$$+\frac{4\nu^2k^4}{\mathrm{I}_0(kR_0)}\int_0^t \hat{\mathbf{L}}^{-1}\left[\mathcal{K}(s)\right]\frac{da_0}{d\tau}(t-\tau)d\tau = 0$$
$$\text{where,}\quad \mathcal{K}(s) = \left(\frac{\mathrm{I}_0''(kR_0)}{s} - \frac{l}{k}\frac{\mathrm{I}_0'(kR_0)\mathrm{I}_0''(lR_0)}{s\mathrm{I}_0'(lR_0)}\right) \tag{2.20}$$

It is shown easily that the above equation reduces to that derived by Berger (1988) in the unforced limit $h \to 0$. For this we note that eqn. 73 in the study by Berger (1988) is (written in our notation)

$$\frac{d^2a_0}{dt^2} + \left[4\nu k^2\left(1 - \frac{1}{2kR_0}\frac{\mathrm{I}_1(kR_0)}{\mathrm{I}_0(kR_0)}\right)\right]\frac{da_0}{dt}(t) + \left[\frac{T}{\rho R_0^3}kR_0\left(k^2R_0^2-1\right)\frac{\mathrm{I}_1(kR_0)}{\mathrm{I}_0(kR_0)}\right]a_0(t)$$
$$= -\frac{2i\nu k^2}{R_0\mathrm{I}_0(kR_0)}\hat{\mathbf{L}}^{-1}\left\{(s\tilde{a}_0 - a(0))\left(-2i\nu k^2R_0\mathrm{I}_0(kR_0)\frac{1}{s} + 2i\nu kR_0\mathrm{I}_1(kR_0)\frac{l\mathrm{I}_0(lR_0)}{s\mathrm{I}_0(lR_0)}\right)\right\}$$
$$= -\frac{4\nu^2k^4}{\mathrm{I}_0(kR_0)}\hat{\mathbf{L}}^{-1}\left\{(s\tilde{a}_0 - a(0))\left(\frac{\mathrm{I}_0(kR_0)}{s} - \frac{l}{ks}\mathrm{I}_1(kR_0)\frac{\mathrm{I}_0(lR_0)}{\mathrm{I}_1(lR_0)}\right)\right\} \tag{2.21}$$

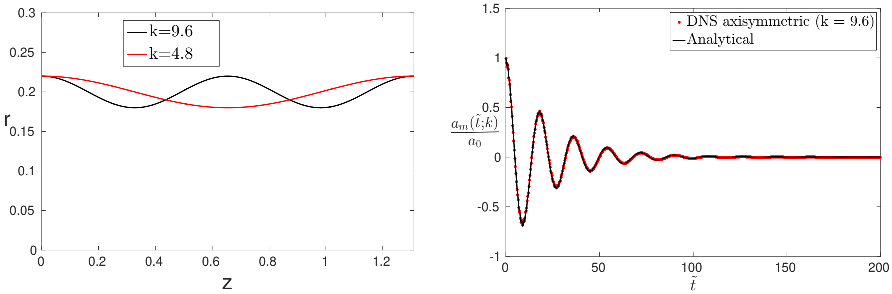
Using the identities $\mathrm{I}_0'(z) = \mathrm{I}_1(z)$ and $\mathrm{I}_0''(z) = \mathrm{I}_0(z) - \frac{\mathrm{I}_1(z)}{z}$, it maybe shown that

$$\mathcal{K}(s) = \left(\frac{\mathrm{I}_0(kR_0)}{s} - \frac{l}{ks}\mathrm{I}_1(kR_0)\frac{\mathrm{I}_0(lR_0)}{\mathrm{I}_1(lR_0)}\right) \tag{2.22}$$

where $\mathcal{K}(s)$ is given by eqn. 2.20. Using the convolution theorem, it is thus seen that eqn. 2.20 and 2.22 are the same (the coefficient of the second terms are also same using the identity for $\mathrm{I}_0''(z)$).

3. Effectiveness of stabilisation

In our proposed stabilisation method, at *any frequency* (e.g. 600π or 2200π), the optimal level of forcing h is chosen such that simultaneously *all* Fourier modes (both axisymmetric or three dimensional) accessible to the cylinder are rendered stable. For the case of $\Omega = 600\pi$ (case 1 in table 3), this is be seen from the stability plots in figure 5*a* and figure 5*b* in the manuscript. The choice of the forcing frequency 600π is dictated by the length of the liquid cylinder L and the existence of the desired ordering $h_{\mathrm{cr1}} < h < h_{\mathrm{cr2}}$ at this chosen frequency. In DNS, we excite the longest possible axisymmetric, RP unstable



(a) $L = 1.31$ cm (b) Time signal

Figure 1: a) Demonstration of Fourier modes which are stabilised by forcing at $\Omega = 600\pi$ and $h = 1.8 \times 10^4$ cm/s^2. The cylinder is of length $L = 1.31$ cm and stabilisation of two Fourier modes are demonstrated. The mode with $k = 4.8$ cm^{-1}, $m = 0$ is the longest possible mode that can fit on this cylinder, this is an RP unstable mode (at $h = 0$) and is shown to be stabilised via forcing in the manuscript, fig. 8. Here we excite a mode of wavelength equal to half of the cylinder length i.e. we choose $k = 9.6$ cm^{-1}, $m = 0$ in DNS showing that this mode remains stable at the chosen level of forcing, see time signal in panel b). Note that $k = 9.6$ cm^{-1} is an RP stable mode (at $h = 0$) and remains stable under forcing, as predicted by the stability chart in fig. 5a and 5b in the manuscript.

mode on the cylinder of length $L = 1.31$ cm viz. $k_0 = \frac{2\pi}{L} = 4.8$ cm^{-1} and impose an optimal forcing of forcing $h = 1.8 \times 10^4$ cm/s^2. At any time t, nonlinear interactions in DNS may produce multiple higher modes of k_0 (axisymmetric or three dimensional). However our choice of the level of forcing viz. $h = 1.8 \times 10^4$ cm/s^2 ensures that *all* modes accessible to the system (axisymmetric or three-dimensional) are simultaneously stabilised at this forcing. Due to periodic nature of boundary conditions, any wavelength exceeding the length of the cylinder viz. $\lambda > 1.31$ cm cannot appear in the simulations. As an additional validation, we now verify through DNS that for case 1 in table 3 with $\Omega = 600\pi$, not only is the longest possible axisymmetric perturbation with $\lambda = 1.31$ cm stabilised in DNS due to forcing, but an axisymmetric perturbation with wavelength half as long as the cylinder i.e. with $\lambda = 0.655$ cm also remains stable at the imposed level of forcing (see fig. 1). This is consistent with what is predicted in fig. 5, where the RP mode k_0 and all its higher integer multiples for $m = 0$ and $m > 0$ are predicted to be stabilised.

The need for changing frequency to $\Omega = 2200\pi$ (case 2 in table 3) is that, we wish to demonstrate stabilisation of longer cylinders compared to the aforementioned case (holding it's radius constant). For the second case (Case 2 in table 3), the cylinder radius is the same as earlier but this is now a longer cylinder of $L = 1.8$ cm. If we had chosen the forcing frequency to be $\Omega = 600\pi$ for this case as before, the pink star in figure 5a ($k_0 = 3.48 = \frac{2\pi}{1.8}$ corresponding to the longest RP unstable mode which can appear on this cylinder) clearly shows that the ordering $h_{\mathrm{cr1}} < h < h_{\mathrm{cr2}}$ cannot be achieved at this frequency. This is verified by imagining a horizontal line drawn through the pink star in figure 5a in the manuscript which will pass through the instability tongues. Thus forcing this cylinder at a frequency of 600π and $h \approx 4 \times 10^4$ cm/s^2 allows the possibility that some of the parametrically unstable modes may appear in DNS and de-stabilize the cylinder. This possibility can be eliminated by increasing the forcing frequency to 2200π which shifts the stability tongue upwards restoring the ordering $h_{\mathrm{cr1}} < h < h_{\mathrm{cr2}}$. As seen in fig. 6$a$ and 6b in the manuscript, at this frequency if one forces the system at $h = 1.65 \times 10^5$ cm/s^2, all Fourier modes accessible to the system (both axisymmetric and three dimensional) are stabilised. Here too, any wavelength exceeding the length of the cylinder viz. $\lambda > 1.8$ cm cannot appear in simulations. We note here that this case 2 has been chosen to correspond to the fastest growing RP mode viz. $k_0 R_0 \approx 0.69$.

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