

# Kinetic theory of granular particles immersed in a molecular gas. Supplementary material

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## 1. DSMC method. Some technical details

As described in section 1, we treat in this paper the molecular gas as a thermostat in the sense that its state is not perturbed by the presence of grains. Therefore, the collision stage in the DSMC method must be slightly modified to accurately reproduce Eq. (3.1). We follow similar steps as proposed by Montanero & Garzó (2002) to numerically solve the Boltzmann–Enskog equation of a homogeneous granular mixture.

The simulation is initiated by drawing the particle velocities from a Maxwellian distribution at temperature  $T_g$  following the Box–Muller transform (Box & Muller 1958). Since the granular gas is assumed to be spatially homogeneous, only the collision stage is described here. The procedure can be summarised as follows:

(i) A required number of  $N_j^{\delta t}$  candidate pairs to collide in a time  $\delta t$  is selected. This number is given by† (Montanero & Garzó 2002)

$$N_j^{\delta t} = j \frac{2^{d-3} d \Gamma\left(\frac{d}{2}\right) (\sigma + \sigma_j)^2}{\pi^{\frac{d}{2}-1} \sigma^d} N_j \phi g_j^{\max} \delta t, \quad (1.1)$$

where  $j = 1$  ( $j = 2$ ) refers to a granular (molecular) particle. Namely,  $N_1^{\delta t}$  refers to granular–granular collisions, while  $N_2^{\delta t}$  refers to granular–molecular collisions. Here,  $N_j$  is the total number of particles of species  $j$  and  $g_j^{\max}$  is an upper bound of the average relative velocity. A good estimate is  $g_j^{\max} = C v_j^{\text{th}}$ , where  $v_j^{\text{th}} = \sqrt{2T_g/\bar{m}}$  is the mean thermal velocity,  $\bar{m} = (m + m_j)/2$ , and  $C$  is a constant, e.g.,  $C = 5$  (Bird 1994). Note that in Eq. (1.1) collisions among molecular particles themselves have been neglected.

(ii) A colliding direction  $\hat{\boldsymbol{\sigma}}_j$  is randomly selected with equiprobability.

(iii) The collision is accepted if

$$|\hat{\boldsymbol{\sigma}}_j \cdot \mathbf{g}_{12}| = |\hat{\boldsymbol{\sigma}}_j \cdot (\mathbf{v}_1 - \mathbf{v}_2)| > U(0, 1) g_j^{\max}, \quad (1.2)$$

where  $U(0, 1)$  is a random number uniformly distributed in  $[0, 1]$ .

(iv) If the collision is accepted, only the velocities of the granular particles are updated according to the relationships (2.4) (for  $j = 1$ ) and (2.6) (for  $j = 2$ ) of the main text.

The above procedure constitutes an intermediate method between Bird’s (Bird 1994) and Nanbu’s (Nanbu 1986) schemes since in the latter only one of the colliding particles changes its velocity. However, as pointed out by Montanero & Santos (1997), both

† In contrast to the work of Montanero & Garzó (2002), we consider here a very dilute system and so, the pair correlation functions are set equal to 1.

schemes are equivalent and equally useful to solve the Boltzmann equation since in both techniques momentum is conserved on average. Thus, we do not need to account for collisions among molecular particles themselves because  $n/n_g \ll 1$  and the computational cost would be very expensive.

Moreover, in the theory all the mechanical information of the molecular gas (with the exception the mass ratio  $m/m_g$ ) is enclosed in the (reduced) drift coefficient  $\gamma^*$  throughout the reduced (bath) temperature  $T_g^*$ . Let us denote by  $N_g$  and  $N$  the total number of granular and molecular particles, respectively. Since  $N/N_g = n/n_g$ , then  $\sigma$  and  $\sigma_g$  are related by

$$\sigma_g = \left[ \left( \frac{\sqrt{\pi}}{4\sqrt{2}} \frac{N}{N_g} \sqrt{\frac{m}{m_g}} \frac{1}{\phi \sqrt{T_g^*}} \right)^{1/(d-1)} - 1 \right] \sigma. \quad (1.3)$$

While deriving Eq. (1.3) use has been made of the relationships

$$n = \frac{2^{d-1} d \Gamma(\frac{d}{2})}{\pi^{d/2}} \sigma^{-d} \phi, \quad n_g = \frac{d \Gamma(\frac{d}{2})}{4\pi^{(d-1)/2}} \left( \frac{m}{m_g} \right)^{1/2} \left( \frac{m}{2T_g} \right)^{1/2} \bar{\sigma}^{1-d} \gamma. \quad (1.4)$$

Equation (1.3) establishes a constraint in the inputs regarding the molecular gas. For this reason, once the inputs appearing in the theory ( $d, m/m_g, T_g^*, \phi$ ) are fixed, then we choose  $N_g$  in such a way that  $N_g/N \gg 1$  and  $\sigma_g/\sigma > 0$ .

To carry on the simulations corresponding to the effective suspension model employed by Gómez González & Garzó (2019), the influence of the external fluid on grains is taken into account by updating the velocity of every single grain at each time step  $\delta t$  according to (Khalil & Garzó 2014; Gómez González *et al.* 2021):

$$\mathbf{v} \rightarrow e^{-\gamma \delta t} \mathbf{v} + \left( \frac{6\gamma T_g \delta t}{m} \right)^{1/2} \mathbf{U}[-1, 1]. \quad (1.5)$$

Here,  $\mathbf{U}$  is an uniformly distributed random vector in  $[-1, 1]^3$ . Equation (1.5) converges to the Fokker–Planck operator [Eq. (2.14) of the main text] when a time step  $\delta t$  much smaller than the mean free time between collisions is considered (Khalil & Garzó 2014).

## 2. Zeroth- and first-order distribution functions

To zeroth-order in the expansion, the distribution  $f^{(0)}$  verifies the kinetic equation

$$\partial_t^{(0)} f^{(0)} = J[f^{(0)}, f^{(0)}] + J_g[f^{(0)}, f_g^{(0)}]. \quad (2.1)$$

The conservation laws at this order give

$$\partial_t^{(0)} n = 0, \quad \partial_t^{(0)} \mathbf{U} = \mathbf{0}, \quad \partial_t^{(0)} T = -T \left( \zeta^{(0)} + \zeta_g^{(0)} \right). \quad (2.2)$$

Upon obtaining the second relation in Eq. (2.2), we have accounted for that the distributions  $f^{(0)}$  and  $f_g^{(0)}$  are isotropic in  $\mathbf{V}$  and so, the zeroth-order contribution to the production of momentum vanishes ( $\mathcal{F}^{(0)}[f^{(0)}] = \mathbf{0}$ ).

Since the zeroth-order distribution  $f^{(0)}$  qualifies as a normal solution, then  $\partial_t^{(0)} f^{(0)} = (\partial_T f^{(0)})(\partial_t^{(0)} T)$ , and Eq. (2.1) can be rewritten as

$$-\left( \zeta^{(0)} + \zeta_g^{(0)} \right) T \frac{\partial f^{(0)}}{\partial T} = J[f^{(0)}, f^{(0)}] + J_g[f^{(0)}, f_g^{(0)}]. \quad (2.3)$$

Equation (2.3) has the same form as the time-dependent homogeneous Boltzmann equation, except that  $f^{(0)}(\mathbf{r}, \mathbf{v}; t)$  is the *local* version of the above distribution. Dimensional analysis requires that  $f^{(0)}$  has the scaled form

$$f^{(0)}(\mathbf{r}, \mathbf{v}; t) = n(\mathbf{r}; t) v_{\text{th}}(\mathbf{r}; t)^{-d} \varphi(\mathbf{c}, T/T_g), \quad (2.4)$$

where  $\mathbf{c} = \mathbf{V}/v_{\text{th}}$ ,  $v_{\text{th}}(\mathbf{r}; t) = \sqrt{2T(\mathbf{r}; t)/m}$  being the local thermal velocity. As expected, in contrast to the so-called homogeneous cooling state for (dry) granular gases (van Noije & Ernst 1998; Garzó 2019), the time dependence of the scaled distribution  $\varphi$  does not only occur through the scaled velocity  $\mathbf{c}$  but also through the temperature ratio  $T/T_g$ .

As mentioned before, since  $f^{(0)}$  is isotropic in  $\mathbf{V}$ , the heat flux vanishes ( $\mathbf{q}^{(0)} = \mathbf{0}$ ) and the pressure tensor  $P_{ij}^{(0)} = p\delta_{ij}$ , where  $p = nT$  is the hydrostatic pressure.

To first order in the spatial gradients, the Boltzmann kinetic equation for  $f^{(1)}(\mathbf{r}, \mathbf{v}; t)$  is

$$\partial_t^{(0)} f^{(1)} + \mathcal{L}f^{(1)} - J_g[f^{(1)}, f_g^{(0)}] = -\left(D_t^{(1)} + \mathbf{V} \cdot \nabla\right) f^{(0)} - \frac{m_g}{T_g} \Delta \mathbf{U} \cdot J_g[f^{(0)}, \mathbf{V} f_g^{(0)}], \quad (2.5)$$

where  $D_t^{(1)} \equiv \partial_t^{(1)} + \mathbf{U} \cdot \nabla$  and

$$\mathcal{L}X = -\left(J[f^{(0)}, X] + J[X, f^{(0)}]\right) \quad (2.6)$$

is the linearised Boltzmann collision operator. To first order, the macroscopic balance equations read

$$D_t^{(1)} n = -n \nabla \cdot \mathbf{U}, \quad D_t^{(1)} T = -\frac{2p}{dn} \nabla \cdot \mathbf{U} - T \left(\zeta^{(1)} + \zeta_g^{(1)}\right), \quad (2.7)$$

$$D_t^{(1)} \mathbf{U} = -\rho^{-1} \nabla p - \xi \Delta \mathbf{U} + \rho^{-1} \mathcal{K}[f^{(1)}], \quad (2.8)$$

where  $\zeta^{(1)}$  and  $\zeta_g^{(1)}$  are the first-order contributions to the production rates, the operator  $\mathcal{K}[X]$  is

$$\mathcal{K}_i[X] = \int d\mathbf{v} m V_i J_g[X, f_g^{(0)}], \quad (2.9)$$

and

$$\xi = \frac{1}{d} \frac{m_g}{T_g} \int d\mathbf{v} m \mathbf{V} \cdot J_g[f^{(0)}, \mathbf{V} f_g^{(0)}]. \quad (2.10)$$

The production rates are defined in terms of the distribution  $f^{(1)}$  and their explicit forms are given by Eqs. (2.27)–(2.29).

The use of the balance equations (2.7) and (2.8) allows one to compute the first term on the right side of Eq. (2.5). The result is

$$\begin{aligned} -\left(D_t^{(1)} + \mathbf{V} \cdot \nabla\right) f^{(0)} &= \mathbf{A} \cdot \nabla \ln T + \mathbf{B} \cdot \nabla \ln n + C_{ij} \frac{1}{2} \left(\partial_i U_j + \partial_j U_i - \frac{2}{d} \delta_{ij} \nabla \cdot \mathbf{U}\right) \\ &\quad + D \nabla \cdot \mathbf{U} + \mathbf{E} \cdot \Delta \mathbf{U} + \rho^{-1} \frac{\partial f^{(0)}}{\partial \mathbf{V}} \cdot \mathcal{K}[f^{(1)}] \\ &\quad + T \left(\zeta^{(1)} + \zeta_g^{(1)}\right) \frac{\partial f^{(0)}}{\partial T}, \end{aligned} \quad (2.11)$$

where  $\partial_i \equiv \partial/\partial r_i$  and the quantities  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $C_{ij}$ ,  $D$ , and  $\mathbf{E}$  are given by

$$\mathbf{A}(\mathbf{V}) = -\mathbf{V} T \frac{\partial f^{(0)}}{\partial T} - \frac{p}{\rho} \frac{\partial f^{(0)}}{\partial \mathbf{V}}, \quad (2.12)$$

$$\mathbf{B}(\mathbf{V}) = -\mathbf{V}n \frac{\partial f^{(0)}}{\partial n} - \frac{p}{\rho} \frac{\partial f^{(0)}}{\partial \mathbf{V}}, \quad (2.13)$$

$$C_{ij}(\mathbf{V}) = V_i \frac{\partial f^{(0)}}{\partial V_j}, \quad (2.14)$$

$$D(\mathbf{V}) = \frac{1}{d} \frac{\partial}{\partial \mathbf{V}} \cdot (\mathbf{V}f^{(0)}) + \frac{2}{d} T \frac{\partial f^{(0)}}{\partial T} - f^{(0)} + n \frac{\partial f^{(0)}}{\partial n}, \quad (2.15)$$

$$\mathbf{E}(\mathbf{V}) = -\rho^{-1} \frac{\partial f^{(0)}}{\partial \mathbf{V}} \xi. \quad (2.16)$$

Substitution of Eq. (2.11) into Eq. (2.5) yields

$$\begin{aligned} & \partial_t^{(0)} f^{(1)} - T \left( \zeta^{(1)} + \zeta_g^{(1)} \right) \frac{\partial f^{(0)}}{\partial T} + \mathcal{L}f^{(1)} - J_g[f^{(1)}, f_g^{(0)}] - \rho^{-1} \frac{\partial f^{(0)}}{\partial \mathbf{V}} \cdot \boldsymbol{\kappa}[f^{(1)}] \\ & = \mathbf{A} \cdot \nabla \ln T + \mathbf{B} \cdot \nabla \ln n + C_{ij} \frac{1}{2} \left( \partial_i U_j + \partial_j U_i - \frac{2}{d} \delta_{ij} \nabla \cdot \mathbf{U} \right) + D \nabla \cdot \mathbf{U} + \mathbf{E}' \cdot \Delta \mathbf{U}, \end{aligned} \quad (2.17)$$

where

$$\mathbf{E}'(\mathbf{V}) = \mathbf{E} - \frac{m_g}{T_g} J_g[f^{(0)}, \mathbf{V}f_g^{(0)}]. \quad (2.18)$$

The solution of Eq. (2.17) is of the form

$$\begin{aligned} f^{(1)}(\mathbf{V}) &= \mathcal{A}(\mathbf{V}) \cdot \nabla \ln T + \mathcal{B}(\mathbf{V}) \cdot \nabla \ln n + C_{ij} \frac{1}{2} \left( \partial_i U_j + \partial_j U_i - \frac{2}{d} \delta_{ij} \nabla \cdot \mathbf{U} \right) \\ &+ \mathcal{D}(\mathbf{V}) \nabla \cdot \mathbf{U} + \mathcal{E}(\mathbf{V}) \cdot \Delta \mathbf{U}. \end{aligned} \quad (2.19)$$

As mentioned in the main text, the presence of the term proportional to the velocity difference  $\Delta \mathbf{U}$  in the distribution  $f^{(1)}$  is also common in driven granular mixtures (Khalil & Garzó 2013; Gómez González *et al.* 2020). Since the gradients of the hydrodynamic fields and the term  $\Delta \mathbf{U}$  are assumed to be independent, substitution of the expression (2.19) into Eq. (2.17) yields the following set of coupled linear integral equations:

$$\begin{aligned} & - \left( \zeta^{(0)} + \zeta_g^{(0)} \right) T \partial_T \mathcal{A}_i - \frac{1}{2} \left[ \zeta^{(0)} + \zeta_g^{(0)} \left( 1 + 2\chi \frac{\partial \ln \zeta_g^*}{\partial \chi} \right) \right] \mathcal{A}_i + \mathcal{L} \mathcal{A}_i - \rho^{-1} \frac{\partial f^{(0)}}{\partial V_j} \mathcal{K}_j[\mathcal{A}_i] \\ & - J_g[\mathcal{A}_i, f_g^{(0)}] = A_i, \end{aligned} \quad (2.20)$$

$$\begin{aligned} & - \left( \zeta^{(0)} + \zeta_g^{(0)} \right) T \partial_T \mathcal{B}_i + \mathcal{L} \mathcal{B}_i - J_g[\mathcal{B}_i, f_g^{(0)}] - \rho^{-1} \frac{\partial f^{(0)}}{\partial V_j} \mathcal{K}_j[\mathcal{B}_i] = B_i + \left[ \zeta^{(0)} + \zeta_g^{(0)} \right. \\ & \left. \times \left( 1 - \varepsilon \frac{\partial \ln \zeta_g^*}{\partial \varepsilon} \right) \right] \mathcal{A}_i, \end{aligned} \quad (2.21)$$

$$- \left( \zeta^{(0)} + \zeta_g^{(0)} \right) T \partial_T C_{ij} + \mathcal{L} C_{ij} - \rho^{-1} \frac{\partial f^{(0)}}{\partial V_\ell} \mathcal{K}_\ell[C_{ij}] - J_g[C_{ij}, f_g^{(0)}] = C_{ij}, \quad (2.22)$$

$$- \left( \zeta^{(0)} + \zeta_g^{(0)} \right) T \partial_T \mathcal{D} + \mathcal{L} \mathcal{D} - (\zeta_U + \zeta_{U_g}) T \frac{\partial f^{(0)}}{\partial T} - \rho^{-1} \frac{\partial f^{(0)}}{\partial V_i} \mathcal{K}_i[\mathcal{D}] - J_g[\mathcal{D}, f_g^{(0)}] = D, \quad (2.23)$$

$$- \left( \zeta^{(0)} + \zeta_g^{(0)} \right) T \partial_T \mathcal{E}_i + \mathcal{L} \mathcal{E}_i - \rho^{-1} \frac{\partial f^{(0)}}{\partial V_j} \mathcal{K}_j[\mathcal{E}_i] - J_g[\mathcal{E}_i, f_g^{(0)}] = E'_i. \quad (2.24)$$

While obtaining Eqs. (2.20)–(2.24), we have accounted for the results

$$\partial_t^{(0)} \{\mathbf{A}, \mathbf{B}, C_{ij}, D, \mathcal{E}\} = - \left( \zeta^{(0)} + \zeta_g^{(0)} \right) T \partial_T \{\mathbf{A}, \mathbf{B}, C_{ij}, D, \mathcal{E}\}, \quad (2.25)$$

and

$$\begin{aligned} \partial_t^{(0)} \nabla \ln T = \nabla \partial_t^{(0)} \ln T = -\nabla \left( \zeta^{(0)} + \zeta_g^{(0)} \right) &= - \left[ \zeta^{(0)} + \zeta_g^{(0)} \left( 1 - \varepsilon \frac{\partial \ln \zeta_g^*}{\partial \varepsilon} \right) \right] \nabla \ln n \\ &- \frac{1}{2} \left[ \zeta^{(0)} + \zeta_g^{(0)} \left( 1 + 2\chi \frac{\partial \ln \zeta_g^*}{\partial \chi} \right) \right] \nabla \ln T. \end{aligned} \quad (2.26)$$

In Eq. (2.23), we have taken into account that since the production rates  $\zeta$  and  $\zeta_g$  are scalar quantities, then their first-order corrections in spatial gradients  $\zeta^{(1)}$  and  $\zeta_g^{(1)}$  must be proportional to  $\nabla \cdot \mathbf{U}$  since  $\nabla n$ ,  $\nabla T$ , and  $\Delta \mathbf{U}$  are vectors and the tensor  $\partial_i U_j + \partial_j U_i - \frac{2}{d} \delta_{ij} \nabla \cdot \mathbf{U}$  is traceless. Thus,

$$\zeta^{(1)} = \zeta_U \nabla \cdot \mathbf{U}, \quad \zeta_g^{(1)} = \zeta_{Ug} \nabla \cdot \mathbf{U}, \quad (2.27)$$

where (Garzó 2019)

$$\zeta_U = \frac{\pi^{(d-1)/2}}{2d\Gamma\left(\frac{d+3}{2}\right)} (1 - \alpha^2) \frac{m\sigma^{d-1}}{nT} \int d\mathbf{v}_1 \int d\mathbf{v}_2 f^{(0)}(\mathbf{V}_1) \mathcal{D}(\mathbf{V}_1) g_{12}^3, \quad (2.28)$$

$$\zeta_{Ug} = -\frac{m}{dnT} \int d\mathbf{v} V^2 J_g[\mathcal{D}, f_g^{(0)}]. \quad (2.29)$$

The necessary conditions for the solution to the integral equations (2.20)–(2.24) to exist [Fredholm alternative (Margeneau & Murphy 1956)] is that

$$\int d\mathbf{v} \left\{ 1, \mathbf{v}, \frac{m}{2} V^2 \right\} f^{(1)}(\mathbf{V}) = \{0, \mathbf{0}, 0\}. \quad (2.30)$$

The conditions (2.30) on the first-order distribution  $f^{(1)}(\mathbf{V})$  are used later to establish the existence of a unique solution of Eqs. (2.20)–(2.24). The fulfilment of conditions (2.30) necessarily requires that the right sides of the integral equations (2.20)–(2.24) are orthogonal to the set  $\{1, \mathbf{V}, \frac{m}{2} V^2\}$ , namely,

$$\int d\mathbf{v} \left\{ 1, \mathbf{V}, \frac{m}{2} V^2 \right\} \begin{pmatrix} \mathbf{A}(\mathbf{V}) \\ \mathbf{B}(\mathbf{V}) \\ C_{ij}(\mathbf{V}) \\ D(\mathbf{V}) \\ \mathbf{E}'(\mathbf{V}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (2.31)$$

It is straightforward to prove fulfilment of the conditions (2.31) by direct integration using the definitions (2.12)–(2.16) of  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $C_{ij}$ ,  $D$ , and  $\mathbf{E}'$ , respectively.

In steady-state conditions, the constraint  $\zeta^{(0)} + \zeta_g^{(0)} = 0$  applies locally and so, the first term of the left-hand side of Eqs. (2.20)–(2.24) vanish. This yields the set of integral equations

$$-\chi \frac{\partial \ln \zeta_g^*}{\partial \chi} \mathcal{A}_i + \mathcal{L} \mathcal{A}_i - \rho^{-1} \frac{\partial f^{(0)}}{\partial V_j} \mathcal{K}_j[\mathcal{A}_i] - J_g[\mathcal{A}_i, f_g^{(0)}] = A_i, \quad (2.32)$$

$$\mathcal{L} \mathcal{B}_i - J_g[\mathcal{B}_i, f_g^{(0)}] - \rho^{-1} \frac{\partial f^{(0)}}{\partial V_j} \mathcal{K}_j[\mathcal{B}_i] = B_i - \varepsilon \frac{\partial \ln \zeta_g^*}{\partial \varepsilon} \mathcal{A}_i, \quad (2.33)$$

$$\mathcal{L}\mathcal{C}_{ij} - \rho^{-1} \frac{\partial f^{(0)}}{\partial V_\ell} \mathcal{K}_\ell[\mathcal{C}_{ij}] - J_g[\mathcal{C}_{ij}, f_g^{(0)}] = C_{ij}, \quad (2.34)$$

$$\mathcal{L}\mathcal{D} - (\zeta_U + \zeta_{Ug}) T \frac{\partial f^{(0)}}{\partial T} - \rho^{-1} \frac{\partial f^{(0)}}{\partial V_i} \mathcal{K}_i[\mathcal{D}] - J_g[\mathcal{D}, f_g^{(0)}] = D, \quad (2.35)$$

$$\mathcal{L}\mathcal{E}_i - \rho^{-1} \frac{\partial f^{(0)}}{\partial V_j} \mathcal{K}_j[\mathcal{E}_i] - J_g[\mathcal{E}_i, f_g^{(0)}] = E'_i. \quad (2.36)$$

Here, all the quantities appearing in Eqs. (2.32)–(2.36) are evaluated in the steady state.

### 3. Leading Sonine approximations to the Navier–Stokes–Fourier transport coefficients

This section addresses the approximations made to achieve the expressions of the Navier–Stokes–Fourier transport coefficients. First, regarding the explicit form of  $f^{(0)}$ , the results obtained in section 3 have shown that the magnitude of the cumulant  $a_2$  is in general very small. Therefore,  $f^{(0)}(\mathbf{V})$  can be well represented by the Maxwellian distribution, namely,

$$f^{(0)}(\mathbf{V}) \rightarrow n \left( \frac{m}{2\pi T} \right)^{d/2} \exp \left( - \frac{mV^2}{2T} \right). \quad (3.1)$$

The use of the Maxwellian distribution (3.1) allows us to get simple but accurate expressions for the Navier–Stokes–Fourier transport coefficients. With the Maxwellian approximation (3.1), the collision integral (2.10) can be easily obtained from the results derived by Garzó & Montanero (2007) for arbitrary coefficients of restitution. Particularising to elastic collisions we get

$$\xi = \rho \mu \theta^{-1/2} (1 + \theta)^{1/2} \gamma, \quad (3.2)$$

where

$$\theta = \frac{mT_g}{m_g T} \quad (3.3)$$

is the ratio of the mean square velocities of granular and molecular gas particles. The zeroth-contributions to the production rates are  $\zeta^{(0)} = (v_{\text{th}} \zeta^*)/\ell$  and  $\zeta_g^{(0)} = (v_{\text{th}} \zeta_g^*)/\ell$ , where

$$\zeta^* = \frac{\sqrt{2\pi}^{(d-1)/2}}{d\Gamma\left(\frac{d}{2}\right)} (1 - \alpha^2), \quad \zeta_g^* = 2x(1 - x^2)\mu^{1/2}\varepsilon, \quad (3.4)$$

and  $x$  and  $\varepsilon$  are defined by Eqs. (3.8) and (3.9) of the main text. The Maxwellian approximation to the steady temperature ratio  $T/T_g$  can be obtained by inserting the expressions (3.4) of  $\zeta^*$  and  $\zeta_g^*$  into the (exact) steady-state condition  $\zeta^* + \zeta_g^* = 0$ . This yields the following cubic equation for  $x$ :

$$2x(x^2 - 1) = \vartheta, \quad \vartheta = \frac{\sqrt{2\pi}^{(d-1)/2}}{d\Gamma\left(\frac{d}{2}\right)} \mu^{-1/2} \varepsilon^{-1} (1 - \alpha^2). \quad (3.5)$$

The physical root of Eq. (3.5) can be written as (Santos 2003)

$$x = \begin{cases} \frac{\sqrt{3}}{3} \left\{ \sqrt{3} \cos \left[ \frac{1}{3} \sin^{-1} \left( \frac{3\sqrt{3}}{4} \vartheta \right) \right] + \sin \left[ \frac{1}{3} \sin^{-1} \left( \frac{3\sqrt{3}}{4} \vartheta \right) \right] \right\}, & \vartheta \leq \frac{4\sqrt{3}}{9} \\ \frac{2\sqrt{3}}{3} \cosh \left[ \frac{1}{3} \cosh^{-1} \left( \frac{3\sqrt{3}}{4} \vartheta \right) \right], & \vartheta \geq \frac{4\sqrt{3}}{9}. \end{cases} \quad (3.6)$$

The temperature ratio  $T/T_g$  in the steady state is then given by

$$\frac{T}{T_g} = \frac{m/m_g}{\left(1 + \frac{m}{m_g}\right)x^2 - 1}. \quad (3.7)$$

With respect to the functions  $(\mathcal{A}, \mathcal{B}, \mathcal{C}_{ij}, \mathcal{D}, \mathcal{E})$ , it is useful to write them in a series expansion of Sonine (Laguerre) polynomials. In practice only the leading terms in these expansions are retained; they provide a quite accurate description over a wide range of inelasticity. In addition, when the cumulants  $a_2$  are neglected, it is straightforward to prove that Eq. (2.15) yields  $D = 0$  and so, the production rates  $\zeta_U = \zeta_{U_g} = 0$ . Non-vanishing contributions to both production rates (which arise from  $a_2$ ) are expected to be very small (Gómez González & Garzó 2019). Thus, we will focus here our attention in the Navier–Stokes–Fourier transport coefficients  $\eta$ ,  $\kappa$ ,  $\bar{\mu}$ , and  $\kappa_U$ . To obtain them, we consider the leading Sonine approximations to the unknowns  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}_{ij}$ ,  $\mathcal{D}$ , and  $\mathcal{E}$  and neglect non-Gaussian corrections to the zeroth-order distribution  $f^{(0)}$  (i.e., we take  $a_2 = 0$ ). Since the procedure to obtain these expressions is quite similar to the one employed in some previous works on granular binary mixtures (Garzó & Dufty 2002; Garzó & Montanero 2007), only some partial results will be displayed in this section.

### 3.1. Leading Sonine approximation to $\eta$

In the case of the shear viscosity  $\eta$ , the leading Sonine approximation to  $\mathcal{C}_{ij}(\mathbf{V})$  (lowest degree polynomial) is

$$\mathcal{C}_{ij}(\mathbf{V}) \rightarrow -f^{(0)}(\mathbf{V})R_{ij}(\mathbf{V})\frac{\eta}{nT^2}. \quad (3.8)$$

Since  $R_{ij}(\mathbf{V})$  is a traceless tensor, then  $\mathcal{K}_\ell[R_{ij}] = 0$ , and the integral equation (2.34) reads

$$-\frac{\eta}{nT^2} \left\{ \mathcal{L}[f^{(0)}R_{ij}] - J_g[f^{(0)}R_{ij}, f_g^{(0)}] \right\} = C_{ij}. \quad (3.9)$$

To determine  $\eta$ , we multiply both sides of Eq. (3.9) by  $R_{ij}(\mathbf{V})$  and integrate over velocity. The result can be written as

$$\frac{1}{(d-1)(d+2)} \frac{\eta}{nT^2} \left\{ \int d\mathbf{v} R_{ij}(\mathbf{V}) \mathcal{L} [f^{(0)}R_{ij}] - \int d\mathbf{v} R_{ij}(\mathbf{V}) J_g [f^{(0)}R_{ij}, f_g^{(0)}] \right\} = p, \quad (3.10)$$

where use has been made of the result

$$\int d\mathbf{v} R_{ij}(\mathbf{V}) C_{ij}(\mathbf{V}) = -\frac{d(d+1)-2}{d} \int d\mathbf{v} mV^2 f^{(0)}(\mathbf{V}) = -(d+2)(d-1)p. \quad (3.11)$$

The collision integral involving the linearised Boltzmann collision operator  $\mathcal{L}$  is given by (Brey *et al.* 1998)

$$\frac{1}{(d-1)(d+2)} \frac{1}{nT^2} \int d\mathbf{v} R_{ij}(\mathbf{V}) \mathcal{L} [f^{(0)}R_{ij}] = \nu_\eta^* \nu_0, \quad (3.12)$$

where  $\nu_\eta^*$  is defined by Eq. (5.2) of the main text and  $\nu_0 = p/\eta_0$ ,  $\eta_0$  being the shear viscosity of a dilute gas of elastic hard spheres. The collision integral involving the Boltzmann–Lorentz operator  $J_g$  can be obtained from previous works on granular mixtures (Garzó & Dufty 2002; Garzó & Montanero 2007) when one particularises to elastic collisions. In terms of the drift coefficient  $\gamma$ , the result is

$$-\frac{1}{(d-1)(d+2)} \frac{1}{nT^2} \int d\mathbf{v} R_{ij}(\mathbf{V}) J_g [f^{(0)}R_{ij}, f_g^{(0)}] = \tilde{\nu}_\eta \gamma, \quad (3.13)$$

where  $\tilde{\nu}_\eta$  is given by Eq. (5.3) of the main text. The expression for  $\eta$  can be easily obtained when one takes into account Eqs. (3.12) and (3.13) in Eq. (3.10).

### 3.2. Leading Sonine approximation to $\kappa$ , $\bar{\mu}$ , and $\kappa_U$

The heat flux transport coefficients  $\kappa$ ,  $\bar{\mu}$ , and  $\kappa_U$  are defined in terms of  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{E}$ , respectively. Their leading Sonine approximations are given by

$$\mathbf{A}(\mathbf{V}) \rightarrow c_\kappa \mathbf{S}(\mathbf{V}) f^{(0)}(\mathbf{V}), \quad \mathbf{B}(\mathbf{V}) \rightarrow c_\mu \mathbf{S}(\mathbf{V}) f^{(0)}(\mathbf{V}), \quad \mathbf{E}(\mathbf{V}) \rightarrow c_{\kappa_U} \mathbf{S}(\mathbf{V}) f^{(0)}(\mathbf{V}), \quad (3.14)$$

where the Sonine coefficients  $c_\kappa$ ,  $c_\mu$ , and  $c_{\kappa_U}$  are defined, respectively, as

$$\begin{pmatrix} c_\kappa \\ c_\mu \\ c_{\kappa_U} \end{pmatrix} = \frac{2}{d(d+2)} \frac{m}{nT^3} \int d\mathbf{V} \begin{pmatrix} \mathbf{A}(\mathbf{V}) \cdot \mathbf{S}(\mathbf{V}) \\ \mathbf{B}(\mathbf{V}) \cdot \mathbf{S}(\mathbf{V}) \\ \mathbf{E}(\mathbf{V}) \cdot \mathbf{S}(\mathbf{V}) \end{pmatrix} = - \begin{pmatrix} \frac{2}{d+2} \frac{m}{nT^2} \kappa \\ \frac{2}{d+2} \frac{m}{T^3} \bar{\mu} \\ \frac{2}{d+2} \frac{m}{nT^3} \kappa_U \end{pmatrix}. \quad (3.15)$$

Using the Sonine approximations (3.14), the collision integral (2.9) [when  $X = S_i(\mathbf{V})$ ] is

$$\mathcal{K}_i[S_j] = \int d\mathbf{v} mV_i J_g[S_j f^{(0)}, f_g^{(0)}] = -\delta_{ij} \frac{1}{2} nT^2 \mu \theta^{-1/2} (1+\theta)^{-1/2} \gamma. \quad (3.16)$$

Taking into account Eq. (3.16), the integral equation (2.32) becomes

$$\begin{aligned} & -\frac{2}{d+2} \frac{m}{nT^2} \kappa \left\{ -\zeta_g^{(0)} \chi \frac{\partial \ln \zeta_g^*}{\partial \chi} f^{(0)} \mathbf{S} + \mathcal{L}[f^{(0)} \mathbf{S}] - J_g[f^{(0)} \mathbf{S}, f_g^{(0)}] \right\} = \mathbf{A} \\ & + \frac{1}{d+2} \frac{\mu \gamma}{n} \theta^{-1/2} (1+\theta)^{-1/2} \frac{\partial f^{(0)}}{\partial \mathbf{V}} \kappa, \end{aligned} \quad (3.17)$$

where use has been made of the Sonine approximation (3.14) to  $\mathbf{A}$ . As in the case of the shear viscosity,  $\kappa$  is determined by multiplying both sides of Eq. (3.17) by  $\mathbf{S}(\mathbf{V})$  and integrating over  $\mathbf{V}$ . After some algebra, one achieves

$$-\zeta_g^{(0)} \chi \frac{\partial \ln \zeta_g^*}{\partial \chi} \kappa + \frac{2}{d(d+2)} \frac{m}{nT^3} \kappa \left\{ \int d\mathbf{V} \mathbf{S} \cdot \mathcal{L}[f^{(0)} \mathbf{S}] - \int d\mathbf{V} \mathbf{S} \cdot J_g[f^{(0)} \mathbf{S}, f_g^{(0)}] \right\} = \frac{d+2}{2} \frac{p}{m}, \quad (3.18)$$

where we have accounted for the results

$$-\frac{1}{dT} \int d\mathbf{V} \mathbf{S}(\mathbf{V}) \cdot \mathbf{A}(\mathbf{V}) = \frac{d+2}{2m} p, \quad \int d\mathbf{V} \mathbf{S}(\mathbf{V}) \cdot \frac{\partial f^{(0)}}{\partial \mathbf{V}} = 0. \quad (3.19)$$

The corresponding collision integrals can be written as (Brey *et al.* 1998; Garzó & Dufty 2002; Garzó & Montanero 2007)

$$\frac{2}{d(d+2)} \frac{m}{nT^3} \int d\mathbf{V} \mathbf{S} \cdot \mathcal{L}[f^{(0)} \mathbf{S}] = \nu_\kappa^* \nu_0, \quad (3.20)$$

$$-\frac{2}{d(d+2)} \frac{m}{nT^3} \int d\mathbf{V} \mathbf{S} \cdot J_g[f^{(0)} \mathbf{S}, f_g^{(0)}] = \tilde{\nu}_\kappa \gamma, \quad (3.21)$$

where  $\nu_\kappa^*$  and  $\tilde{\nu}_\kappa$  are given by Eqs. (5.6) and (5.7), respectively, of the main text. In addition, according to the expression of  $\tilde{\zeta}_g^{(0)}$ , one gets the relation

$$-\zeta_g^{(0)} \chi \frac{\partial \ln \zeta_g^*}{\partial \chi} = \beta \gamma, \quad \beta = (x^{-1} - 3x) \mu^{3/2} \chi^{-1/2}. \quad (3.22)$$

Substitution of Eqs. (3.20)–(3.22) into Eq. (3.18) leads to the expression for  $\kappa$ .

The evaluation of the diffusive heat conductivity  $\bar{\mu}$  follows similar steps to those carried



out in the evaluation of  $\kappa$ . Taking into account the leading Sonine approximations (3.14) to  $\mathcal{A}$  and  $\mathcal{B}$ , the integral equation (2.33) reads

$$\begin{aligned} -\frac{2}{d+2} \frac{m}{T^3} \bar{\mu} \left\{ \mathcal{L}[f^{(0)} \mathbf{S}] - J_g[f^{(0)} \mathbf{S}, f_g^{(0)}] \right\} &= \mathbf{B} + \frac{1}{d+2} \frac{\mu\gamma}{T} \theta^{-1/2} (1+\theta)^{-1/2} \frac{\partial f^{(0)}}{\partial \mathbf{V}} \bar{\mu} \\ -\frac{2}{d+2} \frac{m}{nT^2} \zeta_g^{(0)} \kappa f^{(0)} \mathbf{S}, & \end{aligned} \quad (3.23)$$

where use has been made of Eq. (3.16) and the result

$$\zeta_g^{(0)} \varepsilon \frac{\partial \ln \zeta_g^*}{\partial \varepsilon} = \zeta_g^{(0)}. \quad (3.24)$$

Multiplying both sides of Eq. (3.23) by  $\mathbf{S}(\mathbf{V})$  and integrating over velocity, one gets

$$(\nu_0 \nu_\kappa^* + \tilde{\nu}_\kappa \gamma) \bar{\mu} = \frac{T}{n} \zeta^{(0)} \kappa, \quad (3.25)$$

where the steady state condition  $\zeta_g^{(0)} = -\zeta^{(0)}$  has been employed. The solution to Eq. (3.25) yields the final expression for  $\bar{\mu}$ .

Finally, we consider the coefficient  $\kappa_U$ . As said in the main text, it is a new transport coefficient not present for dry granular monocomponent gases. Taking into account the expression (3.2) of  $\xi$ , Eq. (3.16), and the leading Sonine approximation (3.14) to  $\mathcal{E}$ , the integral equation (2.36) reads

$$\begin{aligned} -\frac{2}{d+2} \frac{m}{nT^3} \kappa_U \left\{ \mathcal{L}[f^{(0)} \mathbf{S}] - J_g[f^{(0)} \mathbf{S}, f_g^{(0)}] \right\} &= -\mu \theta^{-1/2} (1+\theta)^{1/2} \gamma \frac{\partial f^{(0)}}{\partial \mathbf{V}} \\ + \frac{1}{d+2} \frac{\mu\gamma}{nT} \theta^{-1/2} (1+\theta)^{-1/2} \frac{\partial f^{(0)}}{\partial \mathbf{V}} \kappa_U - \frac{m_g}{T_g} J_g[f^{(0)}, \mathbf{V} f_g^{(0)}]. & \end{aligned} \quad (3.26)$$

As in the cases of  $\kappa$  and  $\bar{\mu}$ , one multiplies both sides of Eq. (3.16) by  $\mathbf{S}(\mathbf{V})$  and integrates over velocity to get

$$(\nu_0 \nu_\kappa^* + \tilde{\nu}_\kappa \gamma) \kappa_U = -\frac{1}{2} nT \mu (1+\theta)^{-1/2} \theta^{-1/2} H \gamma, \quad (3.27)$$

where use has been made of the result

$$\frac{m_g}{T_g} \int d\mathbf{V} \mathbf{S} \cdot J_g[f^{(0)}, \mathbf{V} f_g^{(0)}] = -\frac{dnT}{2} \mu \gamma (1+\theta)^{-1/2} \theta^{-1/2} H. \quad (3.28)$$

The expression of the quantity  $H$  is displayed in Eq. (5.12) of the main text. The final expression for  $\kappa_U$  can be easily obtained from Eq. (3.27).

#### 4. Linear stability analysis

Some intermediate mathematical steps for the linear stability analysis around the HSS are offered in this section. We assume that we slightly perturb the HSS by small spatial gradients and hence, the Navier–Stokes hydrodynamic equations are linearised around the HSS. This state describes a homogeneous state ( $\nabla n_H = \nabla T_H = 0$ ) with vanishing flow velocity fields ( $\mathbf{U} = \mathbf{U}_g = \mathbf{0}$ ). In addition, the steady condition is  $\zeta_H^{(0)} + \zeta_{gH}^{(0)} = 0$ . Here, the subscript  $H$  denotes quantities evaluated in the HSS. We suppose that the deviations

$$\delta y_\beta(\mathbf{r}, t) = y_\beta(\mathbf{r}, t) - y_{\beta, H} \quad (4.1)$$

are small. Here,  $\delta y_\beta(\mathbf{r}, t)$  denotes the deviations of the hydrodynamic fields

$$\{y_\beta; \beta = 1, \dots, d+2\} \equiv \{n, \mathbf{U}, T\} \quad (4.2)$$

from their values in the homogeneous *steady* state. Moreover, as usual in the simulations of clustering instabilities in fluid-solid systems (Fullmer *et al.* 2017), the molecular gas properties are assumed to be constant and so, they are not perturbed.

Although the reference HSS is stationary [and so, in contrast to what happens in dry granular gases, one does not have to eliminate the time dependence of the transport coefficients through adequate changes of space and time (Brey *et al.* 1998; Garzó 2005)], in order to compare the present stability analysis with the one carried out in the Brownian limit (Gómez González & Garzó 2019), we introduce the following space and time variables:

$$\tau = \frac{v_0}{2\ell}t, \quad \mathbf{r}' = \frac{\mathbf{r}}{2\ell}, \quad (4.3)$$

where  $v_0 = \sqrt{T_H/m}$  and  $\ell = 1/(n_H\sigma^{d-1})$ . The dimensionless time scale  $\tau$  measures the average number of collisions per particle in the time interval between 0 and  $t$ . The unit length  $\mathbf{r}'$  is proportional to the mean free path  $\ell$  of solid particles.

A set of Fourier transformed dimensionless variables are introduced as

$$\rho_{\mathbf{k}}(\tau) = \frac{\delta n_{\mathbf{k}}(\tau)}{n_H}, \quad \mathbf{w}_{\mathbf{k}}(\tau) = \frac{\delta \mathbf{U}_{\mathbf{k}}(\tau)}{v_0}, \quad \theta_{\mathbf{k}}(\tau) = \frac{\delta T_{\mathbf{k}}(\tau)}{T_H}, \quad (4.4)$$

where the elements of the set  $\delta y_{\mathbf{k}\beta} \equiv \{\rho_{\mathbf{k}}(\tau), \mathbf{w}_{\mathbf{k}}(\tau), \theta_{\mathbf{k}}(\tau)\}$  are defined as

$$\delta y_{\mathbf{k}\beta}(\tau) = \int d\mathbf{r}' e^{-i\mathbf{k}\cdot\mathbf{r}'} \delta y_{\beta}(\mathbf{r}', \tau). \quad (4.5)$$

Note that here the wave vector  $\mathbf{k}$  is dimensionless.

The evolution equation for the  $d-1$  transverse velocity components  $\mathbf{w}_{\mathbf{k}\perp}$  is given by

$$\frac{\partial \mathbf{w}_{\mathbf{k}\perp}}{\partial \tau} = \lambda_{\perp}(k) \mathbf{w}_{\mathbf{k}\perp}, \quad (4.6)$$

where

$$\lambda_{\perp}(k) = 2\xi^* + \frac{1}{2}\eta^*k^2 - \frac{2\sqrt{2}d}{d+2}\mu\gamma^*\kappa_U^*X, \quad (4.7)$$

where  $\xi^* = \ell\xi_H/(\rho_H v_0)$ ,  $\eta^* = \eta_H/(\sigma^{1-d}\sqrt{mT_H})$ , and  $\kappa_U^* = \kappa_{UH}/(dn_H T_H)$ . In the Brownian limit ( $m/m_g \rightarrow \infty$ ),  $\xi^* = \sqrt{2}\gamma^*$ ,  $X \rightarrow 0$ , and the expression (4.7) for  $\lambda_{\perp}(k)$  is consistent with the one obtained in previous works (Gómez González & Garzó 2019).

The matrix equation obeying the remaining three longitudinal modes is

$$\frac{\partial \delta y_{\mathbf{k}\beta}(\tau)}{\partial \tau} + M_{\beta\mu} \delta y_{\mathbf{k}\mu}(\tau) = 0, \quad (4.8)$$

where  $\delta y_{\mathbf{k}\beta}(\tau)$  denotes now the set  $\{\rho_{\mathbf{k}}, w_{\mathbf{k}\parallel}, \theta_{\mathbf{k}}\}$  and  $M$  is the square matrix

$$M = \begin{pmatrix} 0 & ik & 0 \\ ik(1 - \sqrt{2}\frac{d}{d+2}\mu\gamma^*\bar{\mu}X) & 2\xi^* - \Theta(k) & ik(1 - \sqrt{2}\frac{d}{d+2}\mu\gamma^*D_T^*X) \\ 2\sqrt{2}\zeta^* + \bar{\mu}^*k^2 & ik(\frac{2}{d} - 2\kappa_U^*) & 2\sqrt{2}(\bar{\zeta}_g\gamma^* + \frac{1}{2}\zeta^*) + D_T^*k^2 \end{pmatrix}, \quad (4.9)$$

where  $D_T^* = \kappa_H/(d\sigma^{1-d}\sqrt{T_H/m})$ ,  $\bar{\mu}^* = \rho_H\bar{\mu}_H/(d\sigma^{1-d}T_H\sqrt{mT_H})$ ,

$$\bar{\zeta}_g = \left(\frac{\mu T_H}{T_g}\right)^{1/2} \left[ x_H(1 - x_H^2) - \frac{\mu T_g}{x_H T_H}(1 - 3x_H^2) \right], \quad \Theta(k) = \frac{2\sqrt{2}d}{d+2}\mu\gamma^*\kappa_U X - \frac{d-1}{d}\eta^*k^2. \quad (4.10)$$

Here,  $x_H$  is defined by Eq. (3.8) of the main text with the replacement  $T \rightarrow T_H$ .

The longitudinal three modes have the form  $\exp[\lambda_{\ell}(k)\tau]$  for  $\ell = 1, 2, 3$ . Here, the

eigenvalues  $\lambda_\ell(k)$  of the matrix  $M$  are the solutions of the cubic equation

$$\lambda^3 + W(k)\lambda^2 + Y(k)\lambda + Z(k) = 0, \quad (4.11)$$

where

$$W(k) = \sqrt{2} \left( \zeta^* + 2\bar{\zeta}_g \gamma^* + \sqrt{2}\xi^* \right) - \frac{2\sqrt{2}d}{d+2} \mu \gamma^* \kappa_U X + k^2 \left( D_T^* + \frac{d-1}{d} \eta^* \right), \quad (4.12)$$

$$Y(k) = \frac{d-1}{d} \eta^* D_T^* k^4 + k^2 \left\{ \frac{2}{d+2} + \sqrt{2} \frac{d-1}{d} (\zeta^* + 2\bar{\zeta}_g \gamma^*) \eta^* - \frac{d}{d+2} \left( \sqrt{2} \mu X \bar{\mu}^* \gamma^* - 1 \right) \right. \\ \left. + \frac{2}{d(d+2)} (d\kappa_U^* - 1) \left[ d \left( \sqrt{2} \mu X D_T^* \gamma^* - 1 \right) - 2 \right] + \frac{2}{d+2} D_T^* [(d+2)\xi^* - \sqrt{2} d \mu X \kappa_U^* \gamma^*] \right\} + \frac{2\sqrt{2}}{d+2} (\zeta^* + 2\bar{\zeta}_g \gamma^*) \left[ (d+2)\xi^* - \sqrt{2} d \mu X \kappa_U^* \gamma^* \right], \quad (4.13)$$

$$Z(k) = k^2 \left\{ \sqrt{2} (2\bar{\zeta}_g \gamma^* - \zeta^*) + \frac{2d}{d+2} \mu X \gamma^* \left[ 2 (\zeta^* D_T^* - \bar{\zeta}_g \bar{\mu}^* \gamma^*) - \zeta^* \bar{\mu}^* \right] + k^2 (D_T^* - \bar{\mu}^*) \right\}. \quad (4.14)$$

In the Brownian limit, Eqs. (4.11)–(4.14) are consistent with those obtained by Gómez González & Garzó (2019). †

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† There is a typo in Eq. (99) of Gómez González & Garzó (2019) since the first term of  $W(k)$  should be  $\sqrt{2} (\zeta^* + 4\chi^{-1}\gamma^* + 2\gamma^*)$ .

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