Electronic supplement for the article Rousseau, G. & Ancey, C., An experimental investigation of turbulent free-surface flows over a steep permeable bed

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The objective of this electronic supplement is to provide a detailed derivation of the double-averaged Stokes and Navier-Stokes equations (§ 3 and 4, respectively), particularly equation (2.8) in the body of the article. Our derivation closely follows what was proposed by Gray (1975) and Whitaker (1996); the notation and the methods of averaging are specified in § 1 and 2.

1. Notation

We consider a steady uniform flow of water on a sloping bed, made up of stationary spherical particles of equal diameter d. This packing has a uniform depth and rests on an impervious solid boundary, whose inclination is θ ($i = \tan\theta$ is the bed slope).

2. Averaging

2.1. *Definitions*

We consider a control volume in the form of a thin parallelepiped: its length and width L_* and W_* are much longer than the particle diameter d, whereas its depth H_* is small relative to d. At any point $\boldsymbol{x} = (x, y, z)$, the integration volume is $V = (x - L_*/2, x + L_*/2) \times (y - W_*/2, y + W_*/2) \times (z - H_*/2, z + H_*/2)$. This volume V comprises a fixed solid volume V_s and fluid volume V_f. The surface bounding the control volume V can be broken fluid and solid parts $\mathcal{S}_f$ and $\mathcal{S}_p$. For the fluid volume V_f, we also define the interface (denoted by $\mathcal{A}_s$) with the solid particles contained in V_f, and the interface $\mathcal{A}_f$ that coincides the outer control surface $\mathcal{A}_f = \mathcal{S}_f$. We refer to $\boldsymbol{n}$ as the outward-pointing unit normal vector to the control surface $\mathcal{A}_f$, going from the fluid to the solid phases. We define a local porosity function:

$$\epsilon(z,t) = \frac{1}{V}\int_V \delta(\boldsymbol{x},t)\mathrm{d}V = \frac{V_f}{V}, \tag{2.1}$$

where $\delta(\boldsymbol{x},t)$ is the indicating function at point $\boldsymbol{x}$ which takes the value of 1 when $\boldsymbol{x}$ lies in the fluid phase at time t, and 0 otherwise. In the present article, we will assume the bed particles are rigid and do not move (the medium is not deformable) and, as a consequence, the dependence of porosity on time will be dropped in the following. We define two volume averages for an arbitrary function ψ related to the fluid phase (Gray 1975; Gray & Lee 1977; Whitaker 1986; Chandesris & Jamet 2006):

- The *fluid phase average* (also called *superficial average*)

$$\langle\psi\rangle_v(\boldsymbol{x},t) = \frac{1}{V}\int_V \psi(\boldsymbol{x}+\boldsymbol{\xi},t)\delta(\boldsymbol{x}+\boldsymbol{\xi},t)|\mathrm{d}\boldsymbol{\xi}|. \tag{2.2}$$

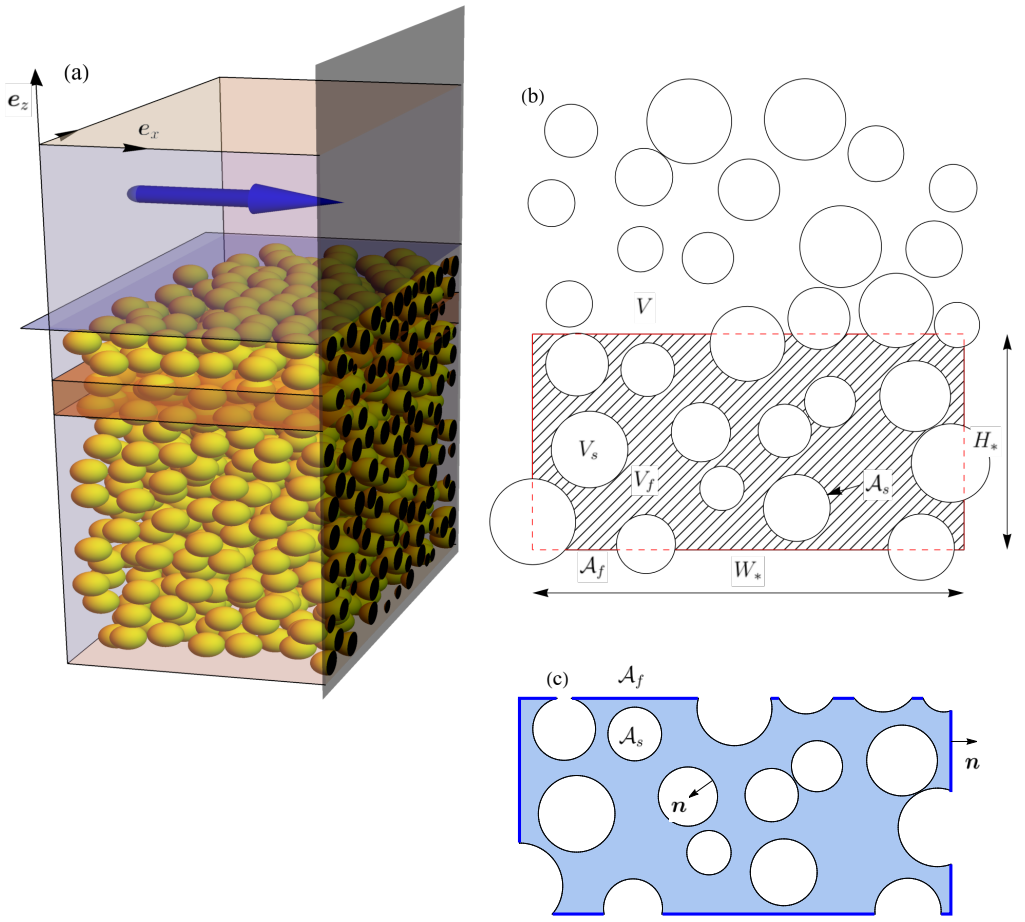

FIGURE 1. Notational scheme. (a) Three-dimensional view. (b) Front view of the control volume

- The *intrinsic phase average*

$$\langle\psi\rangle_f(\boldsymbol{x},t) = \frac{1}{V_f}\int_V \psi(\boldsymbol{x}+\boldsymbol{\xi},t)\delta(\boldsymbol{x}+\boldsymbol{\xi},t)|\mathrm{d}\boldsymbol{\xi}|, \tag{2.3}$$

where $\boldsymbol{\xi}$ denotes a local coordinate relative to the point $\boldsymbol{x}$ at which the average is calculated and $|\mathrm{d}\boldsymbol{\xi}|$ is an infinitesimal volume of integration around $\boldsymbol{\xi}$. Note that in the article, we simplified the notation by replacing $\langle\psi\rangle_f \to \langle\psi\rangle$ and not using the fluid phase average.

Caveat: In our documents, we prefer to use "phase average" used by Gray (1975), Carbonell & Whitaker (1984), Howes & Whitaker (1985), and Whitaker (1986) rather "superficial average" employed by most authors since Ochoa-Tapia & Whitaker (1995) because "superficial" is often understood as indicative of surface integrals, which is not the case here. We presume that the latter was defined by analogy with superficial velocity (also called filter velocity) used in the mechanics of fluid flow through a porous medium to refer to the artificial flow velocity of the fluid phase calculated as if it occupied the whole space. Whitaker (1996) made a cryptic comment to justify the use of "superficial average": "In some previous publications (see, for example, Whitaker, 1986) the quantities defined by Equations (1.5) [(2.2) here] and (1.7) [(2.3) here] have been referred to as the

phase average and the *intrinsic phase average* respectively. The original choice of words was based on the idea that one should distinguish between the average over *an individual phase* and the average over *multiple phases*. The latter average is typically encountered in one-equation models of heat and mass transfer (Nozad et al., 1985; Ochoa-Tapia et al., 1994); however, one must also deal with averages over individual phases in those cases (Quintard and Whitaker, 1993, 1995). A unique set of words does not seem to be available to identify all the types of averages that one encounters in the method of volume averaging; however, it is clear that errors on the order of ϵ have been made because of the failure to distinguish between the averages defined by Equations (1.5) [(2.2) here] and (1.7) [(2.3) here]. For parameters that are linear in ϵ this means an error of a factor of three, while a factor of ten results if the dependence is quadratic. To help avoid such errors the words superficial and intrinsic have been adopted."

Phase and intrinsic averages are linked because ψ is defined on the fluid phase solely and $V_f = \epsilon(z)V$:

$$\langle \psi \rangle_v = \epsilon \langle \psi \rangle_f . \tag{2.4}$$

2.2. *Theorems*

As the phase interface is a material surface, the indicating function $\delta(\boldsymbol{x}, t)$ satisfies (Drew 1983)

$$\frac{\partial}{\partial t}\delta + \boldsymbol{w} \cdot \nabla \delta = 0, \tag{2.5}$$

where $\boldsymbol{w}$ denotes the velocity of the phase interface $\mathcal{A}_s$. Differentiating the phase average (2.2) with respect to time leads to

$$\left\langle \frac{\partial \psi}{\partial t} \right\rangle_v = \frac{\partial}{\partial t} \langle \psi \rangle_v - \frac{1}{V} \int_{\mathcal{A}_s} \psi \boldsymbol{w} \cdot \boldsymbol{n} \mathrm{d}S. \tag{2.6}$$

In the paper, we assume that the phase interface is stationary ($\boldsymbol{w} = 0$), and thus

$$\left\langle \frac{\partial \psi}{\partial t} \right\rangle_v = \frac{\partial}{\partial t} \langle \psi \rangle_v . \tag{2.7}$$

Similarly, using the Gauss theorem, we can show that (Howes & Whitaker 1985; Whitaker 1999)

$$\langle \nabla \psi \rangle_v = \nabla \langle \psi \rangle_v + \frac{1}{V} \int_{\mathcal{A}_s} \psi \boldsymbol{n} \mathrm{d}S, \tag{2.8}$$

and

$$\langle \nabla \cdot \boldsymbol{\psi} \rangle_v = \nabla \cdot \langle \boldsymbol{\psi} \rangle_v + \frac{1}{V} \int_{\mathcal{A}_s} \boldsymbol{\psi} \cdot \boldsymbol{n} \mathrm{d}S. \tag{2.9}$$

Applying equation (2.8) to $\psi = 1$, we obtain the relation

$$\langle \nabla 1 \rangle_v = \nabla \langle 1 \rangle_v + \frac{1}{V} \int_{\mathcal{A}_s} \boldsymbol{n} \mathrm{d}S \Rightarrow \nabla \epsilon = -\frac{1}{V} \int_{\mathcal{A}_s} \boldsymbol{n} \mathrm{d}S. \tag{2.10}$$

Assuming that the control volume V is much larger than the particle diameter, Carbonell & Whitaker (1984) showed that the intrinsic average does not vary significantly within the control volume, and thus

$$\frac{1}{V} \int_{\mathcal{A}_s} \langle \psi \rangle_f \boldsymbol{n} \mathrm{d}S = \langle \psi \rangle_f \frac{1}{V} \int_{\mathcal{A}_s} \boldsymbol{n} \mathrm{d}S = - \langle \psi \rangle_f \nabla \epsilon. \tag{2.11}$$

3. Averaging the Stokes equations

Let us consider the Stokes equations that comprise the mass and momentum balance equations for a Newtonian fluid in the limit of vanishing Reynolds numbers:

$$\nabla \cdot \boldsymbol{u} = 0, \tag{3.1}$$

$$\varrho \boldsymbol{g} - \nabla p + \mu \nabla^2 \boldsymbol{u} = 0, \tag{3.2}$$

where $\boldsymbol{u} = (u, v, w)$ denotes the fluid velocity, ϱ is its density, p is the fluid pression and μ denotes its dynamic viscosity.

3.1. *Decomposition*

To study the effect on spatial nonuniformity, we define the spatial variation $\tilde{\psi}$ with respect to the phase average:

$$\psi = \langle \psi \rangle_f + \tilde{\psi}. \tag{3.3}$$

Although defining the spatial distribution from the phase average would make our calculations easier, it has the disadvantage of creating undesired terms in the balance equations (Gray 1975). We follow Gray (1975) who stated that "since the point associated with the averaging volume need not be in the α-phase [the fluid phase], the intrinsic phase average is still defined for all space and may be non-zero in the β-phase [the solid phase]."

As $\langle \psi \rangle_v$ is constant through the control volume V_f or V, the following relation holds true:

$$\langle \langle \psi \rangle_v \rangle_v = \langle \langle \psi \rangle_v \rangle_f = \langle \langle \psi \rangle_f \rangle_f = \langle \psi \rangle_v. \tag{3.4}$$

If we swap order, we have a slightly different relation by virtue of equations (2.4) and (3.4) (Carbonell & Whitaker 1984; Ochoa-Tapia & Whitaker 1995; Whitaker 1996; Civan 2014)

$$\langle \langle \psi \rangle_f \rangle_v = \epsilon \langle \langle \psi \rangle_f \rangle_f = \epsilon \langle \psi \rangle_f. \tag{3.5}$$

This latter relation differs from the one used by Gray (1975), who assumed commutativity between averaging operators.

3.2. *Averaged mass balance equation*

Applying equation (2.9) to the continuity equation (3.1), we deduce that

$$\langle \nabla \cdot \boldsymbol{u} \rangle_v = \nabla \cdot \langle \boldsymbol{u} \rangle_v + \frac{1}{V} \int_{\mathcal{A}_s} \boldsymbol{u} \cdot \boldsymbol{n} \mathrm{d}S. \tag{3.6}$$

Taking into account the no-slip condition on the solid boundary $\mathcal{A}_s$, we can reduce this equation

$$\langle \nabla \cdot \boldsymbol{u} \rangle_v = \nabla \cdot \langle \boldsymbol{u} \rangle_v = 0. \tag{3.7}$$

Using the relation (2.1), we also find that:

$$\nabla \cdot \langle \boldsymbol{u} \rangle_v = \nabla \cdot (\epsilon \langle \boldsymbol{u} \rangle_f) = 0. \tag{3.8}$$

Expanding the terms, we can cast this equation into

$$\nabla \cdot (\epsilon \langle \boldsymbol{u} \rangle_f) = \epsilon \nabla \cdot \langle \boldsymbol{u} \rangle_f + \nabla \epsilon \cdot \langle \boldsymbol{u} \rangle_f = 0. \tag{3.9}$$

3.3. *Averaged momentum balance equation*

Let us first consider the pressure gradient in the momentum balance equation (3.2), we get

$$\langle \nabla p \rangle_v = \nabla \langle p \rangle_v + \frac{1}{V} \int_{\mathcal{A}_s} p \boldsymbol{n} \mathrm{d}S.$$

We define the pressure disturbance as:

$$p = \langle p \rangle_f + \tilde{p}. \tag{3.10}$$

Using the relation (2.4), we get

$$\langle p \rangle_v = \epsilon \langle \nabla p \rangle_f = \nabla(\epsilon \langle p \rangle_f) + \frac{1}{V} \int_{\mathcal{A}_s} (\langle p \rangle_f + \tilde{p}) \boldsymbol{n} \mathrm{d}S,$$

$$\langle \nabla p \rangle_v = \epsilon \nabla \langle p \rangle_f + \langle p \rangle_f \nabla \epsilon + \frac{1}{V} \int_{\mathcal{A}_s} \tilde{p} \boldsymbol{n} \mathrm{d}S + \frac{\langle p \rangle_f}{V} \int_{\mathcal{A}_s} \boldsymbol{n} \mathrm{d}S,$$

and by virtue of equation (2.11), we finally end up with

$$\langle \nabla p \rangle_v = \epsilon \langle \nabla p \rangle_f = \epsilon \nabla \langle p \rangle_f + \frac{1}{V} \int_{\mathcal{A}_s} \tilde{p} \boldsymbol{n} \mathrm{d}S. \tag{3.11}$$

For the Laplacian of the velocity, we get

$$\langle \nabla \cdot \nabla \boldsymbol{u} \rangle_v = \nabla \cdot \langle \nabla \boldsymbol{u} \rangle_v + \frac{1}{V} \int_{\mathcal{A}_s} \nabla \boldsymbol{u} \cdot \boldsymbol{n} \mathrm{d}S,$$

and

$$\langle \nabla \boldsymbol{u} \rangle_v = \nabla \langle \boldsymbol{u} \rangle_v + \frac{1}{V} \int_{\mathcal{A}_s} \boldsymbol{u} \boldsymbol{n} \mathrm{d}S,$$

and because of the no-slip condition on the solid boundary $\mathcal{A}_s$, we deduce

$$\langle \nabla \boldsymbol{u} \rangle_v = \nabla \langle \boldsymbol{u} \rangle_v ,$$

and thus

$$\left\langle \nabla^2 \boldsymbol{u} \right\rangle_v = \nabla^2 \langle \boldsymbol{u} \rangle_v + \frac{1}{V} \int_{\mathcal{A}_s} \nabla \boldsymbol{u} \cdot \boldsymbol{n} \mathrm{d}S. \tag{3.12}$$

Using equation (2.4) and the decomposition

$$\boldsymbol{u} = \langle \boldsymbol{u} \rangle_f + \tilde{\boldsymbol{u}},$$

we transform equation (3.12) into

$$\left\langle \nabla^2 \boldsymbol{u} \right\rangle_v = \nabla^2 (\epsilon \langle \boldsymbol{u} \rangle_f) + \frac{1}{V} \int_{\mathcal{A}_s} \nabla (\langle \boldsymbol{u} \rangle_f + \tilde{\boldsymbol{u}}) \cdot \boldsymbol{n} \mathrm{d}S,$$

and expanding the Laplacian term, we obtain

$$\left\langle \nabla^2 \boldsymbol{u} \right\rangle_v = \epsilon \nabla^2 \langle \boldsymbol{u} \rangle_f + \langle \boldsymbol{u} \rangle_f \nabla^2 \epsilon + 2(\nabla \epsilon \cdot \nabla) \langle \boldsymbol{u} \rangle_f + \frac{\nabla \langle \boldsymbol{u} \rangle_f}{V} \cdot \int_{\mathcal{A}_s} \boldsymbol{n} \mathrm{d}S + \frac{1}{V} \int_{\mathcal{A}_s} \nabla \tilde{\boldsymbol{u}} \cdot \boldsymbol{n} \mathrm{d}S,$$

where $(\nabla \epsilon \cdot \nabla)$ is the directional derivative in the direction of $\nabla \epsilon$. By virtue of equation (2.11), we end up with

$$\left\langle \nabla^2 \boldsymbol{u} \right\rangle_v = \epsilon \nabla^2 \langle \boldsymbol{u} \rangle_f + \langle \boldsymbol{u} \rangle_f \nabla^2 \epsilon + 2 (\nabla \epsilon \cdot \nabla) \langle \boldsymbol{u} \rangle_f - \nabla \langle \boldsymbol{u} \rangle_f \cdot \nabla \epsilon + \frac{1}{V} \int_{\mathcal{A}_s} \nabla \tilde{\boldsymbol{u}} \cdot \boldsymbol{n} \mathrm{d}S, \tag{3.13}$$

and because

$$(\nabla \epsilon \cdot \nabla) \langle \boldsymbol{u} \rangle_f = \nabla \epsilon \cdot \left(\nabla \langle \boldsymbol{u} \rangle_f \right)^T = \nabla \langle \boldsymbol{u} \rangle_f \cdot \nabla \epsilon$$

we finally obtain for the Laplacian

$$\left\langle \nabla^2 \boldsymbol{u} \right\rangle_v = \epsilon \nabla^2 \langle \boldsymbol{u} \rangle_f + \langle \boldsymbol{u} \rangle_f \nabla^2 \epsilon + \nabla \langle \boldsymbol{u} \rangle_f \cdot \nabla \epsilon + \frac{1}{V} \int_{\mathcal{A}_s} \nabla \tilde{\boldsymbol{u}} \cdot \boldsymbol{n} \mathrm{d}S. \tag{3.14}$$

To sum up, we can express the averaged momentum balance equation (3.2) by using Eqs. (3.11) and (3.14)

$$\varrho \boldsymbol{g} - \nabla \langle p \rangle_f + \frac{\mu}{\epsilon} \left(\epsilon \nabla^2 \langle \boldsymbol{u} \rangle_f + \langle \boldsymbol{u} \rangle_f \nabla^2 \epsilon + \nabla \langle \boldsymbol{u} \rangle_f \cdot \nabla \epsilon \right) + \boldsymbol{f} = 0, \tag{3.15}$$

where $\boldsymbol{f}$ is the drag force density:

$$\boldsymbol{f} = \frac{1}{V_f} \int_{\mathcal{A}_s} (\mu \nabla \tilde{\boldsymbol{u}} - \tilde{p} \boldsymbol{I}) \cdot \boldsymbol{n} \mathrm{d}S, \tag{3.16}$$

where $\boldsymbol{I}$ is the identity tensor. The first term $\mu_{eff} \nabla^2 \langle \boldsymbol{u} \rangle_v$ (with $\mu_{eff} = \mu/\epsilon$) in the bracketed term in equation (3.15) is the usual Brinkman correction, while the term $-\nabla \langle \boldsymbol{u} \rangle_f \cdot \nabla \epsilon$ is called the second Brinkman correction. The latter reflects the effect of porosity changes (Whitaker 1996).

3.4. *Note: errors in the Whitaker's paper*

There are two small errors in equations (2.22) and (2.29) in (Whitaker 1986), which led Whitaker to end up with $\nabla \epsilon \cdot \nabla \langle \boldsymbol{u} \rangle_f$ rather than $\nabla \langle \boldsymbol{u} \rangle_f \cdot \nabla \epsilon$. The first error is due to an incorrect application of relation (2.9) to $\boldsymbol{\psi} = \nabla \boldsymbol{u}$. Whitaker (1986) posed

$$\langle \nabla \cdot \nabla \boldsymbol{u} \rangle_v = \nabla \cdot \langle \nabla \boldsymbol{u} \rangle_v + \frac{1}{V} \int_{\mathcal{A}_s} \boldsymbol{n} \cdot \nabla \boldsymbol{u} \mathrm{d}S,$$

where the correct form is

$$\langle \nabla \cdot \nabla \boldsymbol{u} \rangle_v = \nabla \cdot \langle \nabla \boldsymbol{u} \rangle_v + \frac{1}{V} \int_{\mathcal{A}_s} \nabla \boldsymbol{u} \cdot \boldsymbol{n} \mathrm{d}S.$$

The second error is that when expanding the Laplacian of a scalar function and a vector, we have

$$\nabla^2 (\epsilon \boldsymbol{u}) = \nabla^2 \epsilon \boldsymbol{u} + \epsilon \nabla^2 \boldsymbol{u} + 2(\nabla \epsilon \cdot \nabla) \boldsymbol{u} = \nabla^2 \epsilon \boldsymbol{u} + \epsilon \nabla^2 \boldsymbol{u} + 2 \nabla \boldsymbol{u} \cdot \nabla \epsilon,$$

whereas Whitaker implicity wrote

$$\nabla^2 (\epsilon \boldsymbol{u}) = \nabla^2 \epsilon \boldsymbol{u} + \epsilon \nabla^2 \boldsymbol{u} + 2 \nabla \epsilon \cdot \nabla \boldsymbol{u}.$$

4. Averaging the Navier-Stokes equations

Averaging the Navier-Stokes equations involves two further steps: adding inertial terms to the momentum balance equations, and considering temporal fluctuations in the velocity and pressure decompositions. The Navier-Stokes equations can be double-averaged in two different ways: first, averaging them over time (we obtain the Reynolds-averaged Navier-Stokes equations), then averaging over space; or first averaging over space, then over time. The first procedure is more common (Nikora *et al.* 2007), and we use it here. As will be seen below with the drag force parametrisation, the order in which time and space averaging is done may lead to differences with what other authors obtained.

Applying equation (2.9) to the velocity tensor $\boldsymbol{uu}$ provides

$$\langle \nabla \cdot \boldsymbol{uu} \rangle_v = \nabla \cdot \langle \boldsymbol{uu} \rangle_v + \frac{1}{V} \int_{\mathcal{A}_s} \boldsymbol{uu} \cdot \boldsymbol{n} \mathrm{d}S = \nabla \cdot \langle \boldsymbol{uu} \rangle_v \tag{4.1}$$

because of the no-slip condition over $\mathcal{A}_s$. Similarly, applying equation (2.7) to the velocity's time derivative $\partial_t \boldsymbol{u}$

$$\left\langle \frac{\partial \boldsymbol{u}}{\partial t} \right\rangle_v = \frac{\partial}{\partial t} \langle \boldsymbol{u} \rangle_v . \tag{4.2}$$

We define the time average of ψ as

$$\bar{\psi}(\boldsymbol{x},t) = \frac{1}{T}\int_T \psi(\boldsymbol{x},t)\mathrm{d}t, \tag{4.3}$$

where T denotes the observation time, which is assumed to be much longer than the characteristic time of velocity fluctuations.

Averaging the material derivative over time and using the Reynolds decomposition $\boldsymbol{u} = \bar{\boldsymbol{u}} + \boldsymbol{u}'$ leads to

$$\overline{\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}} = \frac{\partial}{\partial t}\bar{\boldsymbol{u}} + \nabla\cdot\overline{\boldsymbol{u}\boldsymbol{u}} = \frac{\partial}{\partial t}\bar{\boldsymbol{u}} + \nabla\cdot\bar{\boldsymbol{u}}\bar{\boldsymbol{u}} + \nabla\cdot\overline{\boldsymbol{u}'\boldsymbol{u}'}. \tag{4.4}$$

For the Navier-Stokes equations, Gray's decomposition is made with respect to the space-time average:

$$\bar{\psi} = \langle\bar{\psi}\rangle_f + \tilde{\psi}. \tag{4.5}$$

Using this decomposition, we can expand the tensor $\langle\bar{\boldsymbol{u}}\bar{\boldsymbol{u}}\rangle_v$:

$$\begin{aligned}\langle\bar{\boldsymbol{u}}\bar{\boldsymbol{u}}\rangle_v &= \epsilon\langle\bar{\boldsymbol{u}}\bar{\boldsymbol{u}}\rangle_f,\\ &= \epsilon\langle\langle\bar{\boldsymbol{u}}\rangle_f\langle\bar{\boldsymbol{u}}\rangle_f\rangle_f + \epsilon\langle\langle\bar{\boldsymbol{u}}\rangle_f\tilde{\boldsymbol{u}}\rangle_f + \epsilon\langle\tilde{\boldsymbol{u}}\langle\bar{\boldsymbol{u}}\rangle_f\rangle_f + \epsilon\langle\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}\rangle_f.\end{aligned}$$

By using equation (2.4), we obtain

$$\begin{aligned}\langle\bar{\boldsymbol{u}}\bar{\boldsymbol{u}}\rangle_v &= \epsilon\langle\bar{\boldsymbol{u}}\rangle_f\langle\bar{\boldsymbol{u}}\rangle_f + \epsilon\langle\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}\rangle_f,\\ &= \epsilon^{-1}\langle\bar{\boldsymbol{u}}\rangle_v\langle\bar{\boldsymbol{u}}\rangle_v + \langle\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}\rangle_v.\end{aligned}$$

We average equation (4.4) over the fluid phase:

$$\left\langle\overline{\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}}\right\rangle_v = \frac{\partial}{\partial t}\langle\bar{\boldsymbol{u}}\rangle_v + \nabla\cdot(\epsilon^{-1}\langle\bar{\boldsymbol{u}}\rangle_v\langle\bar{\boldsymbol{u}}\rangle_v) + \nabla\cdot\langle\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}\rangle_v + \nabla\cdot\langle\overline{\boldsymbol{u}'\boldsymbol{u}'}\rangle_v.$$

If we define the turbulent stress and dispersive stress tensors

$$\boldsymbol{\tau}_t = -\varrho\langle\overline{\boldsymbol{u}'\boldsymbol{u}'}\rangle_v \text{ and } \boldsymbol{\tau}_d = -\varrho\langle\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}\rangle_v \tag{4.6}$$

then the double-averaged momentum variation can be expressed as

$$\left\langle\varrho\overline{\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}}\right\rangle_v = \varrho\frac{\partial}{\partial t}\langle\bar{\boldsymbol{u}}\rangle_v + \varrho\epsilon^{-1}(\langle\bar{\boldsymbol{u}}\rangle_v\cdot\nabla)\langle\bar{\boldsymbol{u}}\rangle_v - \nabla\cdot\boldsymbol{\tau}_d - \nabla\cdot\boldsymbol{\tau}_t. \tag{4.7}$$

We now double-average the Navier-Stokes momentum balance equation

$$\left\langle\varrho\overline{\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}}\right\rangle_v = \langle\varrho\boldsymbol{g}\rangle_v - \langle\nabla\bar{p}\rangle_v + \mu\langle\nabla^2\bar{\boldsymbol{u}}\rangle_v \tag{4.8}$$

which gives

$$\varrho\frac{\partial}{\partial t}\langle\bar{\boldsymbol{u}}\rangle_v + \varrho\epsilon^{-1}(\langle\bar{\boldsymbol{u}}\rangle\cdot\nabla)\langle\bar{\boldsymbol{u}}\rangle - \nabla\cdot\boldsymbol{\tau}_d - \nabla\cdot\boldsymbol{\tau}_t = \langle\varrho\boldsymbol{g}\rangle_v - \langle\nabla\bar{p}\rangle_v + \mu\langle\nabla^2\bar{\boldsymbol{u}}\rangle_v. \tag{4.9}$$

We pose $\bar{p} = \langle\bar{p}\rangle_f + \tilde{p}$ and generalize the pressure gradient in equation (3.11)

$$\langle\nabla\bar{p}\rangle_v = \epsilon\langle\nabla\bar{p}\rangle_f = \epsilon\nabla\langle\bar{p}\rangle_f + \frac{1}{V}\int_{\mathcal{A}_s}\tilde{p}\boldsymbol{n}\mathrm{d}S. \tag{4.10}$$

The next step is to generalize the Laplacian in equation (3.14)

$$\left\langle\nabla^2\bar{\boldsymbol{u}}\right\rangle_v = \epsilon\nabla^2\left\langle\bar{\boldsymbol{u}}\right\rangle_f + \left\langle\bar{\boldsymbol{u}}\right\rangle_f\nabla^2\epsilon + \nabla\left\langle\bar{\boldsymbol{u}}\right\rangle_f\cdot\nabla\epsilon + \frac{1}{V}\int_{\mathcal{A}_s}\nabla\tilde{\boldsymbol{u}}\cdot\boldsymbol{n}\mathrm{d}S. \tag{4.11}$$

We define the (time-averaged) drag force density $\bar{\boldsymbol{f}}$

$$\bar{\boldsymbol{f}} = \frac{1}{\epsilon V}\int_{\mathcal{A}_s} (\mu\nabla\tilde{\boldsymbol{u}} - \tilde{p}\boldsymbol{I})\cdot\boldsymbol{n}\mathrm{d}S. \tag{4.12}$$

Substituting equations (4.10), (4.11), and (3.16) into the Navier-Stokes equation (4.9) provides

$$\frac{\partial}{\partial t}\langle\bar{\boldsymbol{u}}\rangle_v + (\langle\bar{\boldsymbol{u}}\rangle_v\cdot\nabla)\langle\bar{\boldsymbol{u}}\rangle_f = \epsilon\boldsymbol{g} - \frac{\epsilon}{\varrho}\nabla\langle\bar{p}\rangle_f + \nu\left(\nabla^2\langle\bar{\boldsymbol{u}}\rangle_v - \nabla\langle\bar{\boldsymbol{u}}\rangle_f\cdot\nabla\epsilon\right) + \epsilon\frac{\bar{\boldsymbol{f}}}{\varrho} + \frac{1}{\varrho}\nabla\cdot(\boldsymbol{\tau}_d+\boldsymbol{\tau}_t) \tag{4.13}$$

with $\langle\bar{\boldsymbol{u}}\rangle_f = \epsilon^{-1}\langle\bar{\boldsymbol{u}}\rangle_v$, $\langle p\rangle_f = \epsilon^{-1}\langle p\rangle_v$, and $\boldsymbol{\tau}_t$ and $\boldsymbol{\tau}_d$ the turbulent and dispersive stress tensors defined by equation (4.6).

When using phase average variables, we obtain the following double-averaged momentum balance equation:

$$\frac{\partial}{\partial t}\langle\bar{\boldsymbol{u}}\rangle_f + \epsilon^{-1}\nabla\cdot(\epsilon\langle\bar{\boldsymbol{u}}\rangle_f\langle\bar{\boldsymbol{u}}\rangle_f) = \boldsymbol{g} - \frac{1}{\varrho}\nabla\langle\bar{p}\rangle_f + \frac{\nu}{\epsilon}\left(\nabla^2(\epsilon\langle\bar{\boldsymbol{u}}\rangle_f) - \nabla\langle\bar{\boldsymbol{u}}\rangle_f\cdot\nabla\epsilon\right) + \frac{\bar{\boldsymbol{f}}}{\varrho} + \frac{1}{\epsilon\varrho}\nabla\cdot(\boldsymbol{\tau}_d+\boldsymbol{\tau}_t) \tag{4.14}$$

where $\boldsymbol{\tau}_t = -\epsilon\varrho\langle\overline{\boldsymbol{u}'\boldsymbol{u}'}\rangle_f$ and $\boldsymbol{\tau}_d = -\epsilon\varrho\langle\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}\rangle_f$. This equation is the same as equation (2.8) in the body of the article.

We can arrange the convective acceleration term differently by using the continuity equation (3.8)

$$\nabla\cdot(\epsilon\langle\bar{\boldsymbol{u}}\rangle_f\langle\bar{\boldsymbol{u}}\rangle_f) = \nabla\langle\bar{\boldsymbol{u}}\rangle_f\cdot(\epsilon\langle\bar{\boldsymbol{u}}\rangle_f) + \langle\bar{\boldsymbol{u}}\rangle_f\nabla\cdot(\epsilon\langle\bar{\boldsymbol{u}}\rangle_f) = \nabla\langle\bar{\boldsymbol{u}}\rangle_f\cdot(\epsilon\langle\bar{\boldsymbol{u}}\rangle_f) = (\epsilon\langle\bar{\boldsymbol{u}}\rangle_f\cdot\nabla)\langle\bar{\boldsymbol{u}}\rangle_f.$$

We can also rearrange the Laplacian term

$$\nabla^2(\epsilon\langle\bar{\boldsymbol{u}}\rangle_f) = \langle\bar{\boldsymbol{u}}\rangle_f\nabla^2\epsilon + 2\nabla\langle\bar{\boldsymbol{u}}\rangle_f\cdot\nabla\epsilon + \epsilon\nabla^2\langle\bar{\boldsymbol{u}}\rangle_f.$$

Substituting these two equations into equation (4.14) leads to the following form

$$\begin{aligned}\frac{\partial}{\partial t}\langle\bar{\boldsymbol{u}}\rangle_f + (\langle\bar{\boldsymbol{u}}\rangle_f\cdot\nabla)\langle\bar{\boldsymbol{u}}\rangle_f = \boldsymbol{g} - \frac{1}{\varrho}\nabla\langle\bar{p}\rangle_f + \nu\nabla^2\langle\bar{\boldsymbol{u}}\rangle_f + \\ \frac{\nu}{\epsilon}\left(\langle\bar{\boldsymbol{u}}\rangle_f\nabla^2\epsilon + \nabla\langle\bar{\boldsymbol{u}}\rangle_f\cdot\nabla\epsilon\right) + \\ \frac{\bar{\boldsymbol{f}}}{\varrho} + \frac{1}{\epsilon\varrho}\nabla\cdot(\boldsymbol{\tau}_d+\boldsymbol{\tau}_t).\end{aligned} \tag{4.15}$$

The first three contributions to the right-hand side of equation (4.15) resemble the usual terms (weight per unit volume, pressure gradient, divergence of the viscous stresses) found in the momentum balance equation in the Navier-Stokes equation. Three additional contributions are included: (i) a Brinkmann correction depending on porosity:

$$\boldsymbol{B} = \frac{\nu}{\epsilon}\left(\langle\bar{\boldsymbol{u}}\rangle_f\nabla^2\epsilon + \nabla\langle\bar{\boldsymbol{u}}\rangle_f\cdot\nabla\epsilon\right),$$

(ii) the drag force density $\bar{\boldsymbol{f}}$, and (iii) the divergence of the dispersive and turbulent stress tensors $\boldsymbol{\tau}_d$ and $\boldsymbol{\tau}_t$.

Following Ochoa-Tapia & Whitaker (1995), we can express the drag force density using Darcy's permeability tensor $\boldsymbol{K}$

$$\bar{\boldsymbol{f}} = -\mu\boldsymbol{K}^{-1}\cdot\langle\bar{\boldsymbol{u}}\rangle_v = -\mu\epsilon\boldsymbol{K}^{-1}\cdot\langle\bar{\boldsymbol{u}}\rangle_f = \frac{1}{V_f}\int_{\mathcal{A}_s}(\mu\nabla\tilde{\boldsymbol{u}} - \tilde{p}\boldsymbol{I})\cdot\boldsymbol{n}\mathrm{d}S$$

Using this parametrisation of the drag force density, we can cast this equation in the

following form to obtain a more compact form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\bar{\boldsymbol{u}}\rangle_f = \boldsymbol{g} - \frac{1}{\varrho}\nabla\langle\bar{p}\rangle_f + \nu\nabla^2\langle\bar{\boldsymbol{u}}\rangle_f + \boldsymbol{B} - \nu\epsilon\boldsymbol{K}^{-1}\cdot\langle\bar{\boldsymbol{u}}\rangle_f + \frac{1}{\epsilon\varrho}\nabla\cdot(\boldsymbol{\tau}_d + \boldsymbol{\tau}_t), \tag{4.16}$$

which is close to the form used by Ochoa-Tapia & Whitaker (1995), Whitaker (1996), Breugem *et al.* (2006), and Kuwata & Kawaguchi (2019). Nikora and his coworkers used a slightly different form (Nikora *et al.* 2007, 2013, 2019)

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\boldsymbol{u}\rangle_f = \boldsymbol{g} - \frac{1}{\varrho}\nabla(\epsilon\langle\bar{p}\rangle_f) + \frac{\nu}{\epsilon}\nabla^2(\epsilon\,\langle\bar{\boldsymbol{u}}\rangle_f) + \frac{\boldsymbol{\Phi}_v + \boldsymbol{\Phi}_p}{\varrho} + \frac{1}{\epsilon\varrho}\nabla\cdot(\boldsymbol{\tau}_d + \boldsymbol{\tau}_t), \tag{4.17}$$

with the viscous drag per unit volume and the from (pressure) drag per unit volume:

$$\boldsymbol{\Phi}_v = \frac{1}{V_f}\int_{\mathcal{A}_s}\mu\nabla\bar{\boldsymbol{u}}\cdot\boldsymbol{n}\mathrm{d}S \text{ and } \boldsymbol{\Phi}_p = -\frac{1}{V_f}\int_{\mathcal{A}_s}\bar{p}\boldsymbol{I}\cdot\boldsymbol{n}\mathrm{d}S. \tag{4.18}$$

The main difference between equations (4.14) and (4.17) lies in the definition of the drag force density. Similarly to what Whitaker (1996) and Breugem *et al.* (2006) did, we decomposed the surface integral $\int_{\mathcal{A}_s}(\mu\nabla\bar{\boldsymbol{u}} - \bar{p}\boldsymbol{I})\cdot\boldsymbol{n}\mathrm{d}S$ into average and fluctuating components:

$$\begin{aligned}
\epsilon(\boldsymbol{\Phi}_v + \boldsymbol{\Phi}_p) &= \frac{1}{V}\int_{\mathcal{A}_s}(\mu\nabla\bar{\boldsymbol{u}} - \bar{p}\boldsymbol{I})\cdot\boldsymbol{n}\mathrm{d}S,\\
&= \underbrace{-\mu\nabla\langle\bar{\boldsymbol{u}}\rangle_f\cdot\nabla\epsilon + \langle\bar{p}\rangle_f\nabla\epsilon}_{\text{mean behaviour}} + \underbrace{\frac{1}{V}\int_{\mathcal{A}_s}(\mu\nabla\tilde{\boldsymbol{u}} - \tilde{p}\boldsymbol{I})\cdot\boldsymbol{n}\mathrm{d}S}_{\text{fluctuating behaviour}},\\
&= -\mu\nabla\langle\bar{\boldsymbol{u}}\rangle_f\cdot\nabla\epsilon + \langle\bar{p}\rangle_f\nabla\epsilon + \epsilon\bar{\boldsymbol{f}}.
\end{aligned}$$

In equation (4.12) defining the drag force density $\bar{\boldsymbol{f}}$, we grouped together the contributions created by velocity and pressure fluctuations. By doing so, we were consistent with the definition of Stokes' drag force (Batchelor 1967). Nikora and his coworkers have considered that the drag force is the total force exerted by the fluid on the bed particles.

Note also that different parametrisations of the drag force density $\boldsymbol{f}$ have been used. For instance, Kuwata & Suga (2013) used the space-time double-averaged Navier–Stokes equation: they first took the space-average of the Navier–Stokes. The drag force density is the same as equation (3.16) above:

$$\boldsymbol{f} = \frac{1}{V_f}\int_{\mathcal{A}_s}(\mu\nabla\tilde{\boldsymbol{u}} - \tilde{p}\boldsymbol{I})\cdot\boldsymbol{n}\mathrm{d}S,$$

Kuwata & Suga (2013) used the Forchheimer parametrisation in the form

$$\boldsymbol{f} = -\epsilon\frac{\mu}{K}\langle\boldsymbol{u}\rangle_f - \varrho\epsilon^2\frac{c}{\sqrt{K}}\langle\boldsymbol{u}\rangle_f|\langle\boldsymbol{u}\rangle_f| \tag{4.19}$$

where K is permeability and c is a material constant. Kuwata & Suga (2013) then took the time-average of the space-averaged Navier–Stokes equation. In the resulting time-averaged drag force density

$$\bar{\boldsymbol{f}} = -\epsilon\frac{\mu}{K}\langle\bar{\boldsymbol{u}}\rangle_f - \varrho\epsilon^2\frac{c}{\sqrt{K}}\overline{\langle\boldsymbol{u}\rangle_f|\langle\boldsymbol{u}\rangle_f|}, \tag{4.20}$$

the time-averaged quadratic term $\overline{\langle\boldsymbol{u}\rangle_f|\langle\boldsymbol{u}\rangle_f|}$ was simplified by using the Reynolds decomposition $\boldsymbol{u} = \bar{\boldsymbol{u}} + \boldsymbol{u}'$, expanding the product, and keeping the zero- and first-order

terms

$$\bar{\boldsymbol{f}} = -\epsilon \frac{\mu}{K} \langle \bar{\boldsymbol{u}} \rangle_f - \varrho \epsilon^2 \frac{c}{\sqrt{K}} \left(\langle \bar{\boldsymbol{u}} \rangle_f |\langle \bar{\boldsymbol{u}} \rangle_f| + \langle \overline{\boldsymbol{u}'\boldsymbol{u}'} \rangle_f | \frac{\langle \bar{\boldsymbol{u}} \rangle_f}{|\langle \bar{\boldsymbol{u}} \rangle_f|} \right). \tag{4.21}$$

Compared to equation (2.8), this equation includes an additional term (proportional to the Reynolds stress tensor $\boldsymbol{\tau}_t = -\epsilon \varrho \langle \overline{\boldsymbol{u}'\boldsymbol{u}'} \rangle_f$). This difference in the definition of the drag force density shows that the order in which the averaging has been done affects the final form of the governing equations.

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