SUPPLEMENTARY MATERIAL for

Thermal Fluctuations in the Dissipation Range

of Homogeneous Isotropic Turbulence

Nonlinear Fluctuation-Dissipation Relation Proof

Here, we present a detailed proof of the nonlinear fluctuation-dissipation relation (FDR) for a finite-volume discretization of incompressible Navier-Stokes in a periodic space domain. The proof parallels the proof for the truncated "continuum" fluctuating incompressible hydrodynamics using the space Fourier transform to diagonalize the Leray-Hodge projection given by Eyink *et al.* [2]. We first briefly review the continuum case. We then discuss the spatial discretization and extend the FDR to the space-discretized model. The elements needed to show the discrete nonlinear FDR are that the linearized systems satisfy a discrete FDR, that the inviscid dynamics conserves kinetic energy, and that the inviscid dynamics satisfies a Liouville theorem. The first two requirements are well known; the key issue is showing the Liouville theorem for the discrete dynamics.

Truncated Continuum Fluctuation-Dissipation Theorem

The incompressible fluctuating Navier-Stokes equation in the torus domain $\Omega = \mathbb{T}^d$

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \nu\Delta\mathbf{u} - \boldsymbol{\nabla}\boldsymbol{\cdot}\tilde{\boldsymbol{\tau}}, \quad \boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{u} = 0. \tag{1}$$

can be written as a system of stochastic ODE's for the Fourier modes

$$\hat{\mathbf{u}}_{\mathbf{k}} = \int_\Omega d^3x \; e^{-i\mathbf{k}\boldsymbol{\cdot}\mathbf{x}}\mathbf{u}(\mathbf{x}) \tag{2}$$

of the form

$$\partial_t \hat{u}_{\mathbf{k},m} + ik_n \left(\delta_{mp} - \frac{k_m k_p}{k^2}\right) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \hat{u}_{\mathbf{p},n}\hat{u}_{\mathbf{q},p} + \nu k^2 \hat{u}_{\mathbf{k},m} = \left(\frac{2\nu k_B T}{\rho}\right)^{1/2} ik\, \eta_{\mathbf{k},m}(t) \tag{3}$$

where $\eta_{\mathbf{k},m}(t)$ for each wavevector $\mathbf{k}$ and space component m are complex white-noise with covariances

$$\langle \eta^*_{\mathbf{k},m}(t)\eta_{\mathbf{k}',m'}(t')\rangle = V\left(\delta_{mm'} - \frac{k_m k_{m'}}{k^2}\right)\delta_{\mathbf{k},\mathbf{k}'}\delta(t-t'). \tag{4}$$

Physically this is a "quasi-continuum" mesoscopic description valid only at wavenumbers $|\mathbf{k}| < \Lambda$, some cutoff wavenumber $\ll \lambda_{mfp}^{-1}$ (inverse mean-free path length). For this reason, and also to give a precise mathematical meaning to the dynamics, all wavevectors in the above dynamical equations are restricted to have magnitudes less than Λ.

The resulting stochastic dynamics satisfies an exact nonlinear fluctuation-dissipation relation, according to which the long-time invariant measure is the Gaussian thermal equilibrium distribution

$$P[\mathbf{u}] = \frac{1}{Z}\exp\left(-\frac{1}{2k_B T}\sum_{|\mathbf{k}|<\Lambda} |\hat{\mathbf{u}}(\mathbf{k})|^2\right). \tag{5}$$

This is a well-known "folklore" result, a careful proof of which can be found in Eyink *et al.* [2]. In fact, this invariant measure is unique because of energy bounds and non-degeneracy of the noise and is in "detailed balance" or time-reversible for the dynamics. The proof in [2] based on the Fokker-Planck equation for the Fourier modes of velocity rests on two key results for the inviscid deterministic dynamics given by the truncated Euler equations: (i) exact conservation of kinetic energy, and (ii) a Liouville Theorem on conservation of phase-volume. Conservation of kinetic energy is a consequence of the "detailed energy conservation" for individual triads of Fourier modes, first noted by Onsager [6]. We comment here briefly on the conservation of phase-volume.

The Liouville Theorem for truncated Euler was derived by T. D. Lee [4], who employed the Fourier representation of the dynamics. The statement of this result involves the term

$$B_{\mathbf{k},m}(\hat{\mathbf{u}},\hat{\mathbf{u}}^*) = -ik_n\left(\delta_{mp} - \frac{k_m k_p}{k^2}\right)\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \hat{u}_{\mathbf{p},n}\hat{u}_{\mathbf{q},p} \tag{6}$$

in Eq.(3). A significant complication, however, is that not all of the Fourier modes are independent, because of the reality condition under complex conjugation

$$\hat{\mathbf{u}}^*_{\mathbf{k}} = \hat{\mathbf{u}}_{-\mathbf{k}}. \tag{7}$$

In his original proof, T.D. Lee used real and imaginary parts of these modes for a subset of wavevectors. The proof in [2], Appendix A, instead considered the modes whose wavevector lies in the half-set

$$K^+ = \left\{ \mathbf{k} : \begin{array}{ll} k_x > 0, & \text{or} \\ k_y > 0 & \text{if } k_x = 0, \text{ or} \\ k_z \geq 0 & \text{if } k_x = k_y = 0 \end{array} \right\} \tag{8}$$

and chose $\hat{\mathbf{u}}_{\mathbf{k},m}$ for $\mathbf{k} \in K^+$ as the independent complex modes. This proof used the standard device of treating $\hat{\mathbf{u}}_{\mathbf{k},m}$ and its complex conjugate $\hat{\mathbf{u}}^*_{\mathbf{k},m}$ as formally independent variables in the calculus of Wirtinger derivatives $\frac{\partial}{\partial \hat{u}_{\mathbf{k},m}}$, $\frac{\partial}{\partial \hat{u}^*_{\mathbf{k},m}}$, simplifying the original calculations of Lee. Here we note that the wavenumbers $\mathbf{p}$, $\mathbf{q}$ which are summed over in Eq.(6) may lie in the complementary set $K^- = -K^+$ and when $\mathbf{p} \in K^-$, then $\hat{\mathbf{u}}_{\mathbf{p}}$ should be interpreted instead as $\hat{\mathbf{u}}^*_{-\mathbf{p}}$. There is a corresponding equation of motion for the complex-conjugate variables

$$\partial_t \hat{u}^*_{\mathbf{k},m} = B^*_{\mathbf{k},m}[\hat{\mathbf{u}}, \hat{\mathbf{u}}^*] - \nu k^2 \hat{u}^*_{\mathbf{k},m} - \left(\frac{2\nu k_B T}{\rho}\right)^{1/2} ik\, \eta^*_{\mathbf{k},m}(t) \tag{9}$$

with $B^*_{\mathbf{k},m}[\hat{\mathbf{u}}, \hat{\mathbf{u}}^*] := B_{\mathbf{k},m}[\hat{\mathbf{u}}, \hat{\mathbf{u}}^*]^*$ when $\mathbf{k} \in K^+$ and $|\mathbf{k}| < \Lambda$. The statement of the Liouville Theorem follows from the easily verified results

$$\frac{\partial}{\partial \hat{\mathbf{u}}_{\mathbf{k}}} \boldsymbol{\cdot} \mathbf{B}_{\mathbf{k}}[\hat{\mathbf{u}}, \hat{\mathbf{u}}^*] = -(d-1) i \mathbf{k} \boldsymbol{\cdot} \hat{\mathbf{u}}(\mathbf{0}), \quad \frac{\partial}{\partial \hat{\mathbf{u}}^*_{\mathbf{k}}} \boldsymbol{\cdot} \mathbf{B}^*_{\mathbf{k}}[\hat{\mathbf{u}}, \hat{\mathbf{u}}^*] = (d-1) i \mathbf{k} \boldsymbol{\cdot} \hat{\mathbf{u}}(\mathbf{0}) \tag{10}$$

in space dimension $d > 1$. In fact, summing over all independent modes then gives

$$\sum_{\mathbf{k} \in K^+, |\mathbf{k}| < \Lambda} \left(\frac{\partial}{\partial \hat{\mathbf{u}}_{\mathbf{k}}} \boldsymbol{\cdot} \mathbf{B}_{\mathbf{k}}[\hat{\mathbf{u}}, \hat{\mathbf{u}}^*] + \frac{\partial}{\partial \hat{\mathbf{u}}^*_{\mathbf{k}}} \boldsymbol{\cdot} \mathbf{B}^*_{\mathbf{k}}[\hat{\mathbf{u}}, \hat{\mathbf{u}}^*] \right) = 0. \tag{11}$$

Centered Finite-Volume Space Discretization

We now describe the finite-volume space-discretization for fluctuating incompressible Navier-Stokes discussed in Usabiaga *et al.* [7], Delong *et al.* [1] and Nonaka *et al.* [5]. We note that the discretization is based on incorporating fluctuations into a classical discretization of Navier-Stokes originally introduced by Harlow and Welch [3]. For simplicity we consider only $d = 2$, since that suffices to illustrate the basic ideas. We shall also consider only the periodic domain $\mathbb{T}^2 := \mathbb{R}^2/(L_x\mathbb{Z} \times L_y\mathbb{Z})$ and consider a spatial discretization $(x_i, y_j) = (i\Delta x, j\Delta y)$ with $0 \leq i < N_x$, $0 \leq j < N_y$ and $L_x = N_x\Delta x$, $L_y = N_y\Delta y$. In this scheme, scalar fields like pressure p live on cell centers at lattice sites (x_i, y_j), denoted $p_{i,j}$. Vector components live on cell faces displaced in the corresponding directions, so that x-component of velocity is $u_{i+\frac{1}{2},j}$ and y-component of velocity is $v_{i,j+\frac{1}{2}}$. The spatially-discretized equations of motion (but continuous in time) have the form, ignoring for the moment stochastic terms:

$$\begin{aligned}\dot{u}_{i+\frac{1}{2},j} &= -\boldsymbol{\nabla}\boldsymbol{\cdot}(\mathbf{v}u)_{i+\frac{1}{2},j} - (\nabla_x p)_{i+\frac{1}{2},j} - \nu(\Delta u)_{i+\frac{1}{2},j} \\ \dot{v}_{i,j+\frac{1}{2}} &= -\boldsymbol{\nabla}\boldsymbol{\cdot}(\mathbf{v}v)_{i,j+\frac{1}{2}} - (\nabla_y p)_{i,j+\frac{1}{2}} - \nu(\Delta v)_{i,j+\frac{1}{2}}\end{aligned} \tag{12}$$

where all gradients denote centered-differences, for example,

$$(\nabla_x p)_{i+\frac{1}{2},j} = \frac{p_{i+1,j} - p_{i,j}}{\Delta x}, \quad (\nabla_y p)_{i,j+\frac{1}{2}} = \frac{p_{i,j+1} - p_{i,j}}{\Delta y} \tag{13}$$

and Δ is the standard 5-point laplacian. The nonlinear terms are calculated on interpolated lattice sites by averaging adjacent values. Thus,

$$\begin{aligned}\boldsymbol{\nabla}\boldsymbol{\cdot}(\mathbf{v}u)_{i+\frac{1}{2},j} &= \frac{1}{\Delta x}\left[\overline{u}^2_{i+1,j} - \overline{u}^2_{i,j}\right] + \frac{1}{\Delta y}\left[\overline{u}_{i+\frac{1}{2},j+\frac{1}{2}}\overline{v}_{i+\frac{1}{2},j+\frac{1}{2}} - \overline{u}_{i+\frac{1}{2},j-\frac{1}{2}}\overline{v}_{i+\frac{1}{2},j-\frac{1}{2}}\right] \\ \boldsymbol{\nabla}\boldsymbol{\cdot}(\mathbf{v}v)_{i,j+\frac{1}{2}} &= \frac{1}{\Delta x}\left[\overline{u}_{i+\frac{1}{2},j+\frac{1}{2}}\overline{v}_{i+\frac{1}{2},j+\frac{1}{2}} - \overline{u}_{i-\frac{1}{2},j+\frac{1}{2}}\overline{v}_{i-\frac{1}{2},j+\frac{1}{2}}\right] + \frac{1}{\Delta y}\left[\overline{v}^2_{i,j+1} - \overline{v}^2_{i,j}\right]\end{aligned} \tag{14}$$

where, for example,

$$\begin{aligned}&\overline{u}_{i,j} = \frac{u_{i+\frac{1}{2},j} + u_{i-\frac{1}{2},j}}{2}, \quad \overline{v}_{i,j} = \frac{v_{i,j+\frac{1}{2}} + v_{i,j-\frac{1}{2}}}{2}, \\ &\overline{u}_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{u_{i+\frac{1}{2},j+1} + u_{i+\frac{1}{2},j}}{2}, \quad \overline{v}_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{v_{i+1,j+\frac{1}{2}} + v_{i,j+\frac{1}{2}}}{2}, \quad \text{etc.}\end{aligned} \tag{15}$$

The velocity field satisfies the discrete incompressibility condition

$$(\nabla_x u)_{ij} + (\nabla_y v)_{ij} = \frac{u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}}{\Delta x} + \frac{v_{i,j+\frac{1}{2}} - v_{i,j-\frac{1}{2}}}{\Delta y} = 0, \tag{16}$$

which implies a Poisson equation to determine the pressure:

$$(-\nabla^2 p)_{ij} = \frac{1}{\Delta x}\left[\boldsymbol{\nabla}\boldsymbol{\cdot}(\mathbf{v}u)_{i+\frac{1}{2},j} - \boldsymbol{\nabla}\boldsymbol{\cdot}(\mathbf{v}u)_{i-\frac{1}{2},j}\right] + \frac{1}{\Delta x}\left[\boldsymbol{\nabla}\boldsymbol{\cdot}(\mathbf{v}v)_{i,j+\frac{1}{2}} - \boldsymbol{\nabla}\boldsymbol{\cdot}(\mathbf{v}v)_{i,j-\frac{1}{2}}\right] \tag{17}$$

In order to prove the Liouville theorem for the space-discretized dynamics, it is convenient, just as for the continuum case, to use Fourier modes. We introduce the discrete Fourier transforms

$$a_{k,\ell} = \frac{1}{N}\sum_{i,j} e^{-i(ki\Delta x + \ell j\Delta y)} u_{i+\frac{1}{2},j}, \quad b_{k,\ell} = \frac{1}{N}\sum_{i,j} e^{-i(ki\Delta x + \ell j\Delta y)} v_{i,j+\frac{1}{2}}, \tag{18}$$

$$q_{k,\ell} = \frac{1}{N}\sum_{i,j} e^{-i(ki\Delta x + \ell j\Delta y)} p_{i,j}, \tag{19}$$

with $N = N_x N_y$ and with $k \in 2\pi\mathbb{Z}_{N_x}/L_x$, $\ell \in 3\pi\mathbb{Z}_{N_y}/L_y$ In that case, reality of the basic variables $u_{i+\frac{1}{2},j}$, $v_{i,j+\frac{1}{2}}$, implies that relations $a^*_{k,\ell} = a_{-k,-\ell}$, $b^*_{k,\ell} = b_{-k,-\ell}$ hold. The consequence is that not all of these variables are independent and, in the proof of the Liouville Theorem, we must select an independent subset.

It is convenient here for bookkeeping purposes to label these modes as $a_{\alpha,\beta}$, $b_{\alpha,\beta}$ using the integers $\alpha \in \mathbb{Z}_{N_x}$, $\beta \in \mathbb{Z}_{N_y}$, where $k_\alpha = 2\pi\alpha/L_x$, $\ell_\beta = 2\pi\beta/L_y$. In that case we may choose $0 \le \alpha < N_x$, $0 \le \beta < N_y$ as representative values. Consider first the case with N_x, N_y both odd. In this case the reality conditions become

$$\begin{aligned} \alpha \neq 0\colon a^*_{\alpha,\beta} = a_{N_x-\alpha,N_y-\beta} &\implies \alpha \text{ can be restricted to } 1 \le \alpha \le \tfrac{N_x-1}{2} \\ \beta \neq 0\colon a^*_{0,\beta} = a_{0,N_y-\beta} &\implies \beta \text{ can be restricted to } 1 \le \beta \le \tfrac{N_y-1}{2} \\ a^*_{0,0} = a_{0,0} &\implies a_{0,0} \text{ is real} \end{aligned} \tag{20}$$

and similarly for $b_{\alpha,\beta}$. We may thus take as independent variables the complex quantities $a_{\alpha,\beta}$ for $1 \le \alpha \le \frac{N_x-1}{2}$, $0 \le \beta < N_y$ and $a_{0,\beta}$ for $1 \le \beta \le \frac{N_y-1}{2}$ and the single real variable $a_{0,0}$, and similarly for $b_{\alpha,\beta}$. On the other hand, with N_x, N_y both even, the reality conditions become

$$\begin{aligned} \alpha \neq 0, \tfrac{N_x}{2}\colon a^*_{\alpha,\beta} = a_{N_x-\alpha,N_y-\beta} &\implies \alpha \text{ can be restricted to } 1 \le \alpha \le \tfrac{N_x-2}{2} \\ \beta \neq 0, \tfrac{N_y}{2}\colon a^*_{0,\beta} = a_{0,N_y-\beta},\ a^*_{\frac{N_x}{2},\beta} = a_{\frac{N_x}{2},N_y-\beta} &\implies \beta \text{ can be restricted to } 1 \le \beta \le \tfrac{N_y-1}{2} \\ a^*_{0,0} = a_{0,0}, a^*_{0,\frac{N_y}{2}} = a_{0,\frac{N_y}{2}}, a^*_{\frac{N_x}{2},0} = a_{\frac{N_x}{2},0}, a^*_{\frac{N_x}{2},\frac{N_y}{2}} = a_{\frac{N_x}{2},\frac{N_y}{2}} &\implies a_{0,0}, a_{0,\frac{N_y}{2}}, a_{\frac{N_x}{2},0}, a_{\frac{N_x}{2},\frac{N_y}{2}} \text{ are real} \end{aligned} \tag{21}$$

We may thus take as independent variables the complex quantities $a_{\alpha,\beta}$ for $1 \le \alpha \le \frac{N_x-2}{2}$, $0 \le \beta < N_y$ and $a_{0,\beta}$, $a_{\frac{N_x}{2},\beta}$ for $1 \le \beta \le \frac{N_y-2}{2}$, and the four real variables $a_{0,0}, a_{0,\frac{N_y}{2}}, a_{\frac{N_x}{2},0}, a_{\frac{N_x}{2},\frac{N_y}{2}}$, and similarly for $b_{\alpha,\beta}$. The cases with one of N_x, N_y odd and the other even can be treated likewise.

The inverse relations hold

$$u_{i+\frac{1}{2},j} = \sum_{k,\ell} e^{i(ki\Delta x+\ell j\Delta y)} a_{k,\ell}, \quad v_{i,j+\frac{1}{2}} = \sum_{k,\ell} e^{i(ki\Delta x+\ell j\Delta y)} b_{k,\ell} \tag{22}$$

$$p_{i,j} = \sum_{i,j} e^{i(ki\Delta x+\ell j\Delta y)} q_{k,\ell}. \tag{23}$$

We then find that spatial derivatives are given by

$$(\nabla_x u)_{i,j} = \sum_{i,j} e^{i(ki\Delta x+\ell j\Delta y)} ik^- a_{k,\ell}, \quad (\nabla_y v)_{i,j} = \sum_{i,j} e^{i(ki\Delta x+\ell j\Delta y)} i\ell^- b_{k,\ell} \tag{24}$$

with

$$k^- := \frac{1}{i\Delta x}(1 - e^{-ik\Delta x}), \quad \ell^- := \frac{1}{i\Delta y}(1 - e^{-i\ell\Delta y}) \tag{25}$$

and the complex conjugates $k^+ = (k^-)^*$, $k^+ = (k^-)^*$ given by

$$k^+ := \frac{1}{i\Delta x}(e^{ik\Delta x} - 1), \quad \ell^+ := \frac{1}{i\Delta y}(e^{i\ell\Delta y} - 1) \tag{26}$$

Similarly, the discrete Laplacian Fourier transforms as

$$-\widehat{(\nabla^2 p)}_{k,\ell} = (|k^+|^2 + |\ell^+|^2) q_{k,\ell} = \left[\frac{4}{(\Delta x)^2} \sin^2\left(\frac{k\Delta x}{2}\right) + \frac{4}{(\Delta y)^2} \sin^2\left(\frac{\ell\Delta y}{2}\right) \right] q_{k,\ell} \tag{27}$$

Lastly, we note that averaged fields can be Fourier analyzed as well, for example

$$\overline{u}_{i+\frac{1}{2},j+\frac{1}{2}} = \sum_{k,\ell} e^{i(ki\Delta x+\ell j\Delta y)} \overline{a}^{(+y)}_{k,\ell}, \quad \overline{v}_{i+\frac{1}{2},j+\frac{1}{2}} = \sum_{k,\ell} e^{i(ki\Delta x+\ell j\Delta y)} \overline{b}^{(+x)}_{k,\ell} \tag{28}$$

with

$$\overline{a}^{(\pm y)}_{k,\ell} := \frac{1}{2}(1+e^{\pm ik\Delta y}) a_{k,\ell}, \quad \overline{b}^{(\pm x)}_{k,\ell} := \frac{1}{2}(1+e^{\pm ik\Delta x}) b_{k,\ell} \tag{29}$$

and similarly for other fields.

With these definitions we note that a straightforward but tedious calculation gives the deterministic dynamics of Fourier modes as:

$$\begin{aligned} \dot{a}_{k,\ell} &= -ik^+ \sum_{k',\ell'} \overline{a}^{(-x)}_{k',\ell'} \overline{a}^{(-x)}_{k-k',\ell-\ell'} - i\ell^- \sum_{k',\ell'} \overline{a}^{(+y)}_{k',\ell'} \overline{b}^{(+x)}_{k-k',\ell-\ell'} - ik^+ q_{k,\ell} - \nu(|k^+|^2 + |\ell^+|^2) a_{k,\ell} \\ \dot{b}_{k,\ell} &= -ik^- \sum_{k',\ell'} \overline{a}^{(+y)}_{k',\ell'} \overline{b}^{(+x)}_{k-k',\ell-\ell'} - i\ell^+ \sum_{k',\ell'} \overline{b}^{(-y)}_{k',\ell'} \overline{b}^{(-y)}_{k-k',\ell-\ell'} - i\ell^+ q_{k,\ell} - \nu(|k^+|^2 + |\ell^+|^2) b_{k,\ell} \end{aligned} \tag{30}$$

We next prove for this dynamics the two essential ingredients needed for the nonlinear FDR, namely: (i) exact conservation of kinetic energy and (ii) the Liouville Theorem on conservation of phase volume. We begin with the latter.

Discrete Liouville Theorem

We introduce the following notation for the inviscid part of the dynamics

$$\begin{aligned} A_{k,\ell} &= -ik^+ \sum_{k',\ell'} \overline{a}^{(-x)}_{k',\ell'} \overline{a}^{(-x)}_{k-k',\ell-\ell'} - i\ell^- \sum_{k',\ell'} \overline{a}^{(+y)}_{k',\ell'} \overline{b}^{(+x)}_{k-k',\ell-\ell'} - ik^+ q_{k,\ell} \\ B_{k,\ell} &= -ik^- \sum_{k',\ell'} \overline{a}^{(+y)}_{k',\ell'} \overline{b}^{(+x)}_{k-k',\ell-\ell'} - i\ell^+ \sum_{k',\ell'} \overline{b}^{(-y)}_{k',\ell'} \overline{b}^{(-y)}_{k-k',\ell-\ell'} - i\ell^+ q_{k,\ell} \end{aligned} \tag{31}$$

and state our main result:

Proposition 1 *The formula holds*

$$\frac{\partial A_{k,\ell}}{\partial a_{k,\ell}} + \frac{\partial B_{k,\ell}}{\partial b_{k,\ell}} = -i\frac{\sin(k\Delta x)}{\Delta x} a_{0,0} - i\frac{\sin(\ell\Delta y)}{\Delta y} b_{0,0} \tag{32}$$

and thus

$$\sum_{\textit{complex modes } (k,\ell)} \left(\frac{\partial A_{k,\ell}}{\partial a_{k,\ell}} + \frac{\partial B_{k,\ell}}{\partial b_{k,\ell}} + \frac{\partial A^*_{k,\ell}}{\partial a^*_{k,\ell}} + \frac{\partial B^*_{k,\ell}}{\partial b^*_{k,\ell}} \right) + \sum_{\textit{real modes } (k,\ell)} \left(\frac{\partial A_{k,\ell}}{\partial a_{k,\ell}} + \frac{\partial B_{k,\ell}}{\partial b_{k,\ell}} \right) = 0 \tag{33}$$

Note that these are the discrete analogues of the continuum results (10),(11) for $d = 2$.

Proof: Note that the advective part of the dynamics is represented by

$$\begin{aligned} A^{adv}_{k,\ell} &= -ik^{+}\sum_{k',\ell'}\overline{a}^{(-x)}_{k',\ell'}\overline{a}^{(-x)}_{k-k',\ell-\ell'} - i\ell^{-}\sum_{k',\ell'}\overline{a}^{(+y)}_{k',\ell'}\overline{b}^{(+x)}_{k-k',\ell-\ell'} \\ B^{adv}_{k,\ell} &= -ik^{-}\sum_{k',\ell'}\overline{a}^{(+y)}_{k',\ell'}\overline{b}^{(+x)}_{k-k',\ell-\ell'} - i\ell^{+}\sum_{k',\ell'}\overline{b}^{(-y)}_{k',\ell'}\overline{b}^{(-y)}_{k-k',\ell-\ell'} \end{aligned} \tag{34}$$

and the Poisson equation for the pressure in Fourier representation becomes

$$q_{k,\ell} = -\frac{ik^{-}A^{adv}_{k,\ell} + i\ell^{-}B^{adv}_{k,\ell}}{|k^{+}|^2 + |\ell^{+}|^2}. \tag{35}$$

Thus, the inviscid dynamics can be represented via a discrete Leray projection as

$$\begin{aligned} A_{k,\ell} &= \frac{|\ell^{+}|^2}{|k^{+}|^2 + |\ell^{+}|^2}A^{adv}_{k,\ell} - \frac{k^{+}\ell^{-}}{|k^{+}|^2 + |\ell^{+}|^2}B^{adv}_{k,\ell} \\ B_{k,\ell} &= -\frac{\ell^{+}k^{-}}{|k^{+}|^2 + |\ell^{+}|^2}A^{adv}_{k,\ell} + \frac{|k^{+}|^2}{|k^{+}|^2 + |\ell^{+}|^2}B^{adv}_{k,\ell} \end{aligned} \tag{36}$$

The following straightforward derivatives

$$\begin{aligned} \frac{\partial A^{adv}_{k,\ell}}{\partial a_{k,\ell}} &= -2i\frac{\sin(k\Delta x)}{\Delta x}a_{0,0} - i\frac{\sin(\ell\Delta y)}{\Delta y}b_{0,0}, \quad & \frac{\partial A^{adv}_{k,\ell}}{\partial b_{k,\ell}} &= -i\ell^{-}\frac{1}{2}(1+e^{ik\Delta x})a_{0,0} \\ \frac{\partial B^{adv}_{k,\ell}}{\partial b_{k,\ell}} &= -i\frac{\sin(k\Delta x)}{\Delta x}a_{0,0} - 2i\frac{\sin(\ell\Delta y)}{\Delta y}b_{0,0}, \quad & \frac{\partial B^{adv}_{k,\ell}}{\partial a_{k,\ell}} &= -ik^{-}\frac{1}{2}(1+e^{i\ell\Delta y})b_{0,0} \end{aligned} \tag{37}$$

together with (36) yields the result (32).

Finally, we note that the expression in (32) for complex modes is pure imaginary and thus cancels in (33) with the contribution from the complex conjugate. On the other hand, for real modes the expressions in (32) vanish individually because $k\Delta x$, $\ell\Delta y$ are equal either to 0 or π. $\square$

Discrete Energy Conservation

It is well know that the discretization Eqs. (12)-(17) for the inviscid case $\nu = 0$ in periodic boundary conditions exactly conserves the discrete kinetic energy (per mass):

$$H = \frac{1}{2}\sum_{ij}(u^2_{i+\frac{1}{2},j} + v^2_{i,j+\frac{1}{2}}). \tag{38}$$

This result follows from the skew-adjoint property of the advection discretization applied to discretely divergence free fields and orthogonality of discrete gradients with discretely divergence free fields. See, for example, Delong *et al.* [1], Usabiaga *et al.* [7]. We summarize the argument below for completeness.

To prove the first statement, we note using $(u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j})\overline{u}_{i,j} = \frac{1}{2}(u^2_{i+\frac{1}{2},j} - u^2_{i-\frac{1}{2},j})$ that

$$\frac{1}{\Delta x}\sum_{ij}u_{i+\frac{1}{2},j}(\overline{u}^2_{i+1,j} - \overline{u}^2_{i,j}) = -\frac{1}{\Delta x}\sum_{ij}(u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j})\overline{u}^2_{i,j}$$

$$
\begin{aligned}
&= -\frac{1}{\Delta x}\sum_{ij}\frac{1}{2}(u^2_{i+\frac{1}{2},j} - u^2_{i-\frac{1}{2},j})\overline{u}_{i,j} \\
&= \frac{1}{\Delta x}\sum_{ij}\frac{1}{2}u^2_{i+\frac{1}{2},j}(\overline{u}_{i+1,j} - \overline{u}_{i,j})
\end{aligned}
\tag{39}
$$

An exactly analogous calculation shows that

$$
\frac{1}{\Delta y}\sum_{ij} u_{i+\frac{1}{2},j}(\overline{u}_{i+\frac{1}{2},j+\frac{1}{2}}\overline{v}_{i+\frac{1}{2},j+\frac{1}{2}} - \overline{u}_{i+\frac{1}{2},j-\frac{1}{2}}\overline{v}_{i+\frac{1}{2},j-\frac{1}{2}}) = \frac{1}{\Delta y}\sum_{ij}\frac{1}{2}u^2_{i+\frac{1}{2},j}(\overline{v}_{i+\frac{1}{2},j+\frac{1}{2}} - \overline{v}_{i+\frac{1}{2},j-\frac{1}{2}}). \tag{40}
$$

Adding these results gives

$$
\sum_{ij} u_{i+\frac{1}{2},j}[\boldsymbol{\nabla}\boldsymbol{\cdot}(\mathbf{u}u)]_{i+\frac{1}{2},j} = 0 \tag{41}
$$

since

$$
\frac{1}{\Delta x}(\overline{u}_{i+1,j} - \overline{u}_{i,j}) + \frac{1}{\Delta y}(\overline{v}_{i+\frac{1}{2},j+\frac{1}{2}} - \overline{v}_{i+\frac{1}{2},j-\frac{1}{2}}) = 0 \tag{42}
$$

is implied by discrete incompressibility. This shows conservation of $\frac{1}{2}\sum_{ij} u^2_{i+\frac{1}{2},j}$ by discretized advection and conservation of $\frac{1}{2}\sum_{ij} v^2_{i,j+\frac{1}{2}}$ follows by an identical argument.

The conservation of total energy by the pressure gradient is more direct, and follows from discrete incompressibility and the fact that the finite-volume discretizations of the gradient operator $\mathbf{G}$ and divergence operator $\mathbf{D}$ satisfy $\mathbf{G}^* = -\mathbf{D}$.

Proof of the Nonlinear FDR

To complete the proof of the nonlinear FDR, we note that Usabiaga *et al.* [7] added noise to the discretized Stokes equation so that $P = (1/Z)\exp(-H/k_BT)$ is the exact stationary measure of this linear stochastic dynamics. This was guaranteed by adding the noise in the form

$$
\partial_t \mathbf{v} = -\mathbf{G}p + \nu\mathbf{L}\mathbf{v} + \tilde{\mathbf{f}} \tag{43}
$$

where $\mathbf{L} = \mathbf{D}\mathbf{G}$ is the discrete Laplacian, where for $\Delta V = \Delta x \Delta y$

$$
\tilde{\mathbf{f}} = \mathbf{D}\left(\sqrt{\frac{2\nu k_B T}{\rho \Delta V}}\mathbf{W}\right) \tag{44}
$$

and where $\mathbf{W}$ is a space-discretized set of temporal white noises living on the faces of the shifted velocity grids, e.g. in 2D consisting of two independent white noises $W^{(x)}_{i,j}$, $W^{(y)}_{i,j}$, at the cell centers and another two $W^{(x)}_{i+\frac{1}{2},j+\frac{1}{2}}$, $W^{(y)}_{i+\frac{1}{2},j+\frac{1}{2}}$ at the corner points/nodes. This result is easily verified by taking discrete Fourier transforms. Adding this same noise into the discretized nonlinear equations (12), the invariant measure is preserved. Indeed, because of energy conservation and Liouville Theorem, the gaussian Gibbs measure is also invariant for the inviscid deterministic dynamics. See Eyink *et al.* (2021) for more details.

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