Supplementary Material: The onset of zonal modes in two-dimensional Rayleigh–Bénard convection

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DNS and linear stability analysis comparison

In this section we check the results obtained in the linear stability analysis (SA) against those obtained from the DNS. From the DNS, we determine the growth rate (σ) of the $\widehat{\psi}_{0,1}$ mode, the dominant mode in the odd perturbation, by the method of least squares, and the frequency (f) by computing the FFT of the time series. In Table 1 we see that there is excellent agreement between the DNS and SA for Rayleigh numbers close to the stability boundary where the SCRS becomes unstable to odd perturbations and $Pr \in \{10^{-2}, 10^{-1}, 1, 10\}$.

To further illustrate this point, in Figure 1 we plot the time series of $\widehat{\psi}_{0,1}$ from the DNS (red full) alongside the behaviour predicted by the linear stability analysis (black dashed). In Figure 2 we plot the FFT of $\widehat{\psi}_{0,1}$ from the DNS (red full) with the predicted frequency from the linear stability analysis (black dashed).

In Figure 3, we show an extension of the times series presented in Figure 6 of the main article to clearly illustrate the exponential growth and decay of $\widehat{\psi}_{0,1}$ close to the point q_3.

Further details of SRCS computations

In Tables 2–4 we list the Nusselt number (Nu) of the SCRS for parameter values in the range $(Pr, Ra) \in [10^{-6}, 10^{6}] \times (8\pi^4, 10^8]$. The Nusselt number is defined as the ratio of total mean heat flux in the vertical direction to the flux from conduction alone:

$$Nu = 1 + \langle \psi_x \theta \rangle, \tag{1}$$

where ψ and θ are dimensionless solutions to equations (2.1) in the main text and $\langle \cdot \rangle$ indicates an average over space. Table 5 lists the Nusselt number from the results presented in [1]. Note that there is excellent agreement with those presented by us in Table 3.

In Figure 4 we show visualisations of the SCRS at all the points p_i and q_i highlighted in Figure 2 of the main text.

Movies

Bellow are details of the movies found in the Supplementary Material.

- **Movie 1**: $Ra = 3.35 \times 10^5$, $Pr = 0.2$. Close to q_1 as annotated in figure 1 of the main article, illustrating when the pitchfork bifurcation has transitioned into the Hopf bifurcation.
- **Movies 2 and 3**: $Ra = 3.949 \times 10^6$. $Pr = 4.97$ and 4.99, respectively. These are movies close to q_2, showing the qualitative change in ψ^O and θ^O.
- **Movies 4 and 5**: $Pr = 8.58$. $Ra = 1.29 \times 10^6$ and 1.27×10^6. These are movies close to q_3, showing the qualitative change in ψ^O and θ^O.
- **Movie 6**: $Ra = 1.06 \times 10^5$, $Pr = 100$. This is an instance where the presence of zonal mode in the symmetry breaking eigenfunction (E_{SB}) is clear.
- **Movies 7 and 8**: $Ra = 2.54 \times 10^7$. $Pr = 6.7 \times 10^5$ and 7×10^5, respectively. Instances of the symmetry breaking (E_{SB}) and period doubling (E_{PD}) eigenfunctions close to q_4.

Pr	Ra	σ_{DNS}	f_{DNS}	σ_{SA}	f_{SA}
10^{-2}	7.85×10^2	-1.21×10^{-3}	-	-1.21×10^{-3}	-
	7.86×10^2	1.62×10^{-3}	-	1.62×10^{-3}	-
10^{-1}	3.7×10^4	-1.10×10^{-2}	-	-1.10×10^{-2}	-
	3.8×10^4	2.43×10^{-3}	-	2.43×10^{-3}	-
1	1.1×10^6	-1.19	8.18	-1.23	8.17
	1.2×10^6	1.42	8.39	1.37	8.33
10	3.2×10^4	-3.41×10^{-1}	24.5	-3.37×10^{-1}	24.5
	3.3×10^4	4.95×10^{-1}	24.9	4.99×10^{-1}	24.9

Table 1: The growth rate (σ) and frequency (f) of an odd perturbation added to the SCRS as calculated from the DNS and the linear stability analysis (SA). Both σ and f are non-dimensionalised with respect to the thermal diffusive time-scale (d^2/κ).

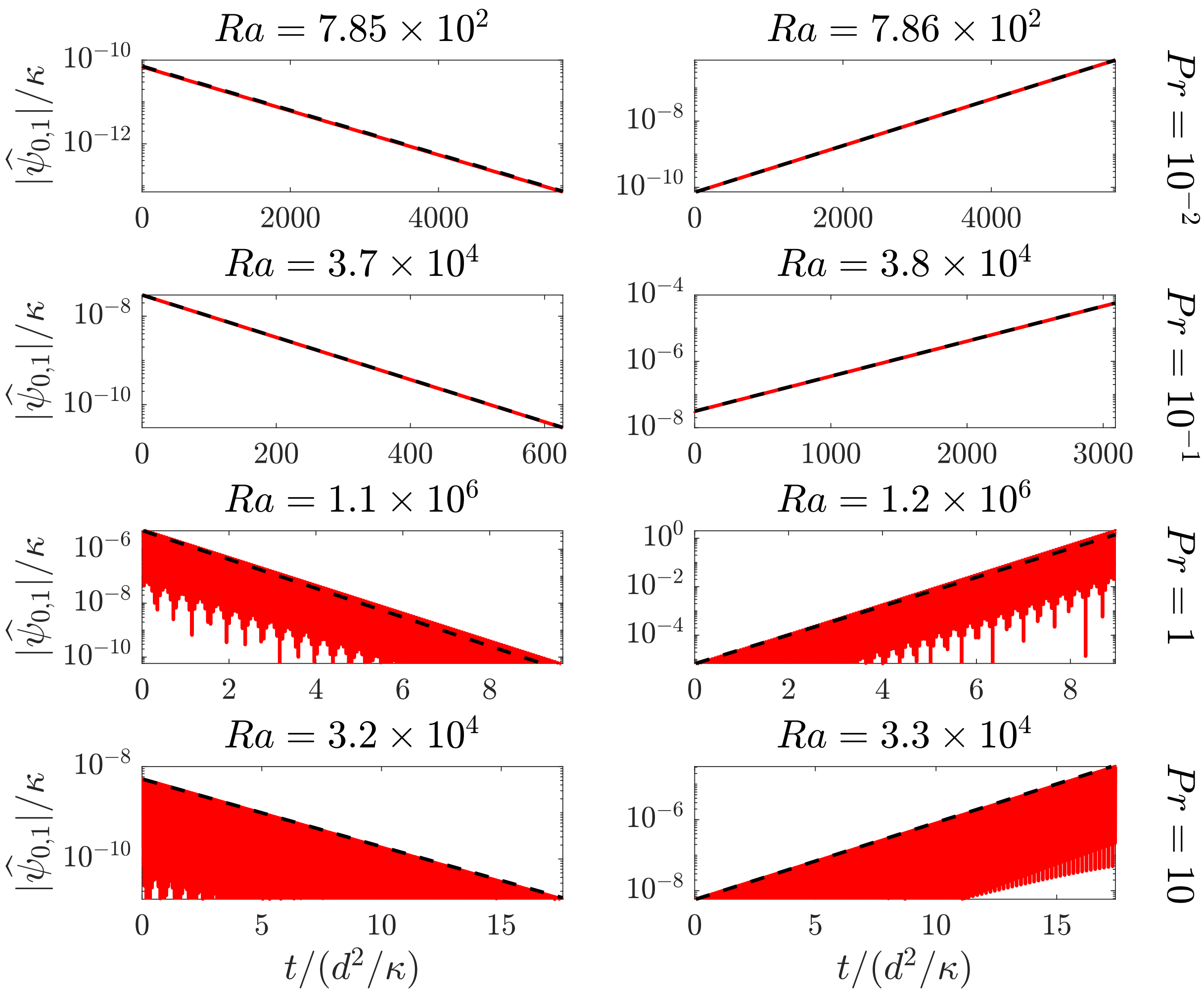

Figure 1: Time series of $\widehat{\psi}_{0,1}$ from DNS (red full) compared with the predicted growth rate from the linear stability analysis (black dashed).

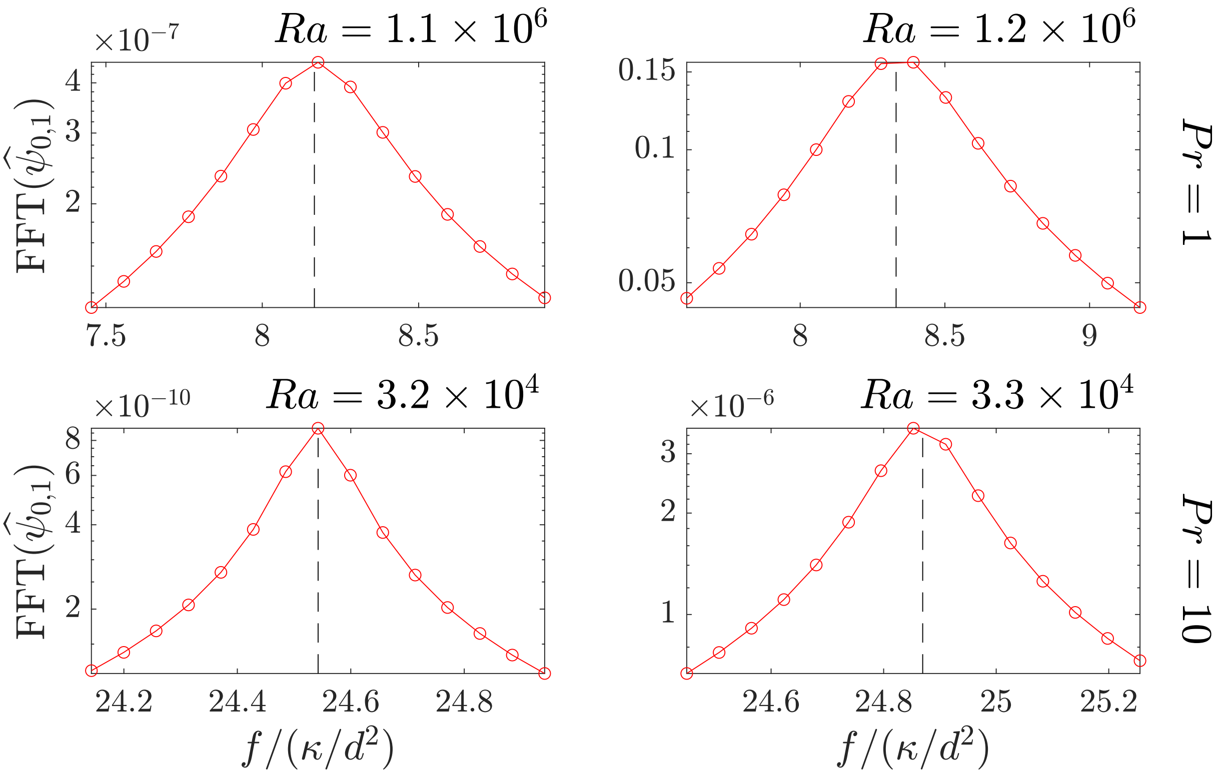

Figure 2: The FFT of $\widehat{\psi}_{0,1}$ from the DNS (red full) with the predicted frequency from the linear stability analysis (black dashed).

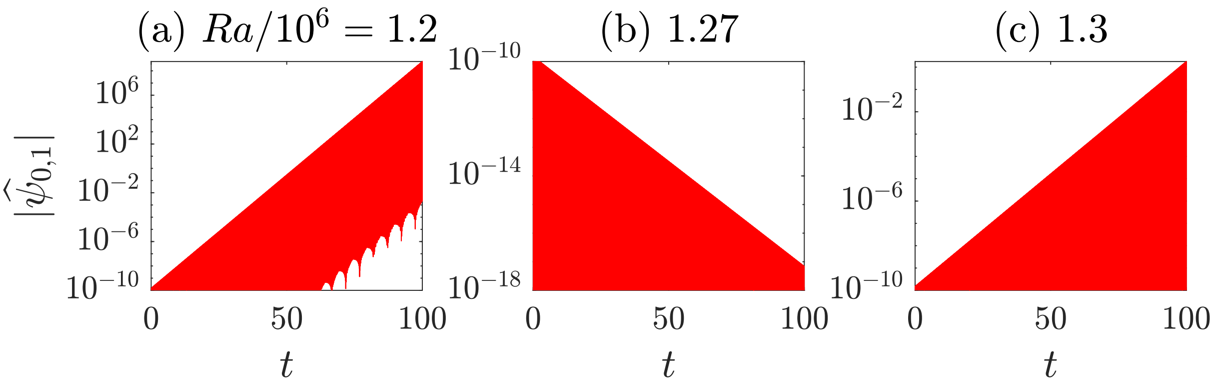

Figure 3: An extension of the time series presented in Figure 6 of the main article. We have fixed $Pr = 8.57$.

Ra	N	Nu $Pr = 10^{-6}$	$Pr = 10^{-5}$	$Pr = 10^{-4}$	$Pr = 10^{-3}$
10^3	64	1.46633	1.46633	1.46633	1.46633
2×10^3	64	2.54127	2.54127	2.54127	2.54124
3×10^3	64	3.11158	3.11158	3.11158	3.11153
6×10^3	152	4.15903	4.15903	4.15902	4.15895
10^4	152	5.08008	5.08008	5.08007	5.07999
2×10^4	152	6.61094	6.61094	6.61093	6.61084
3×10^4	152	7.68231	7.68231	7.68230	7.68220
6×10^4	152	9.88104	9.88104	9.88103	9.88093
10^5	152	11.8579	11.8579	11.8579	11.8578
2×10^5	152	15.1387	15.1387	15.1386	15.1385
3×10^5	152	17.4397	17.4397	17.4397	17.4396
6×10^5	152	22.1706	22.1706	22.1706	22.1705
10^6	256	26.4286	26.4286	26.4286	26.4285
2×10^6	256	33.4971	33.4971	33.4971	33.4970
3×10^6	256	38.4560	38.4560	38.4560	38.4559
6×10^6	400	48.6515	48.6515	48.6515	48.6514
10^7	400	57.8266	57.8266	57.8265	57.8264
2×10^7	400	73.0618	73.0618	73.0617	73.0616
3×10^7	400	83.7543	83.7543	83.7543	83.7542
6×10^7	400	105.699	105.699	105.699	105.699
10^8	400	125.102	125.102	125.102	125.102

Table 2: The value of the Nusselt number (Nu) for the SCRS with $Ra \in [10^3, 10^8]$, $Pr \in [10^{-6}, 10^{-3}]$ and $\Gamma = 2$. The resolutions ($N_x = N_y = N$) are also given.

Ra	N	Nu $Pr = 10^{-2}$	$Pr = 10^{-1}$	$Pr = 1$	$Pr = 10$	$Pr = 10^2$
10^3	64	1.46630	1.46614	1.46687	1.46716	1.46718
2×10^3	64	2.54093	2.53805	2.53065	2.53948	2.54099
3×10^3	64	3.11103	3.10613	3.08042	3.09180	3.09613
6×10^3	152	4.15820	4.15066	4.08758	4.05554	4.06805
10^4	152	5.07914	5.07051	4.98831	4.86435	4.88129
2×10^4	152	6.60992	6.60055	6.50471	6.18817	6.19053
3×10^4	152	7.68126	7.67164	7.57095	7.12370	7.09066
6×10^4	152	9.87995	9.87001	9.76305	9.10021	8.91216
10^5	152	11.8568	11.8467	11.7360	10.9457	10.5267
2×10^5	152	15.1375	15.1271	15.0121	14.0979	13.1717
3×10^5	152	17.4386	17.4280	17.3108	16.3403	15.0125
6×10^5	152	22.1694	22.1584	22.0380	20.9875	18.8001
10^6	256	26.4275	26.4165	26.2929	25.1935	22.2598
2×10^6	256	33.4959	33.4849	33.3580	32.2031	28.2230
3×10^6	256	38.4548	38.4437	38.3150	37.1323	32.6025
6×10^6	400	48.6503	48.6388	48.5076	47.2839	42.0018
10^7	400	57.8253	57.8138	57.6805	56.4309	50.7119
2×10^7	400	73.0605	73.0491	72.9134	71.6332	65.4185
3×10^7	400	83.7531	83.7419	83.6060	82.3098	75.8429
6×10^7	400	105.698	105.688	105.560	104.250	97.3668
10^8	400	125.101	125.093	124.983	123.711	116.541

Table 3: The value of the Nusselt number (Nu) for the SCRS with $Ra \in [10^3, 10^8]$, $Pr \in [10^{-2}, 10^2]$ and $\Gamma = 2$. The resolutions ($N_x = N_y = N$) are also given.

Ra	N	Nu			
		$Pr = 10^3$	$Pr = 10^4$	$Pr = 10^5$	$Pr = 10^6$
10^3	64	1.46719	1.46719	1.46719	1.46719
2×10^3	64	2.54114	2.54116	2.54116	2.54116
3×10^3	64	3.09659	3.09664	3.09664	3.09664
6×10^3	152	4.06959	4.06975	4.06976	4.06977
10^4	152	4.88409	4.88438	4.88441	4.88441
2×10^4	152	6.19538	6.19591	6.19596	6.19597
3×10^4	152	7.09670	7.09740	7.09748	7.09748
6×10^4	152	8.91937	8.92044	8.92055	8.92056
10^5	152	10.5325	10.5339	10.5341	10.5341
2×10^5	152	13.1660	13.1680	13.1683	13.1683
3×10^5	152	14.9872	14.9898	14.9901	14.9901
6×10^5	152	18.6827	18.6863	18.6868	18.6869
10^6	256	21.9674	21.9716	21.9724	21.9725
2×10^6	256	27.3643	27.3671	27.3685	27.3687
3×10^6	256	31.1230	31.1211	31.1231	31.1233
6×10^6	400	38.8158	38.7834	38.7868	38.7873
10^7	400	45.7362	45.6338	45.6387	45.6393
2×10^7	400	57.3332	56.9488	56.9556	56.9567
3×10^7	400	65.6360	64.8617	64.8686	64.8702
6×10^7	400	83.3475	81.0633	81.0635	81.0664
10^8	400	99.9470	95.4377	95.4182	95.4226

Table 4: The value of the Nusselt number (Nu) for the SCRS with $Ra \in [10^3, 10^8]$, $Pr \in [10^3, 10^6]$ and $\Gamma = 2$. The resolutions ($N_x = N_y = N$) are also given.

Ra	$N_x \times N_y$	Nu				
		$Pr = 10^{-2}$	$Pr = 10^{-1}$	$Pr = 1$	$Pr = 10$	$Pr = 10^2$
10^3	128×65	1.46630	1.46614	1.46687	1.46716	1.46718
10^4	128×65	5.07914	5.07051	4.98831	4.86435	4.88129
10^5	128×65	11.8568	11.8467	11.7360	10.9457	10.5267
10^6	256×97	26.4274	26.4164	26.2929	25.1935	22.2598
10^7	512×129	57.8250	57.8132	57.6802	56.4307	50.7115
10^8	512×129	125.484	125.472	125.330	123.991	116.909

Table 5: These results are taken from [1] and show the value of the Nusselt number (Nu) for the SCRS with $Ra \in [10^3, 10^8]$, $Pr \in [10^{-2}, 10^2]$ and $\Gamma = 2$. The resolution of Fourier modes (N_x) and Chebyshev collocation points (N_y) is given also. There is excellent agreement with the results calculated by us displayed in Table 3.

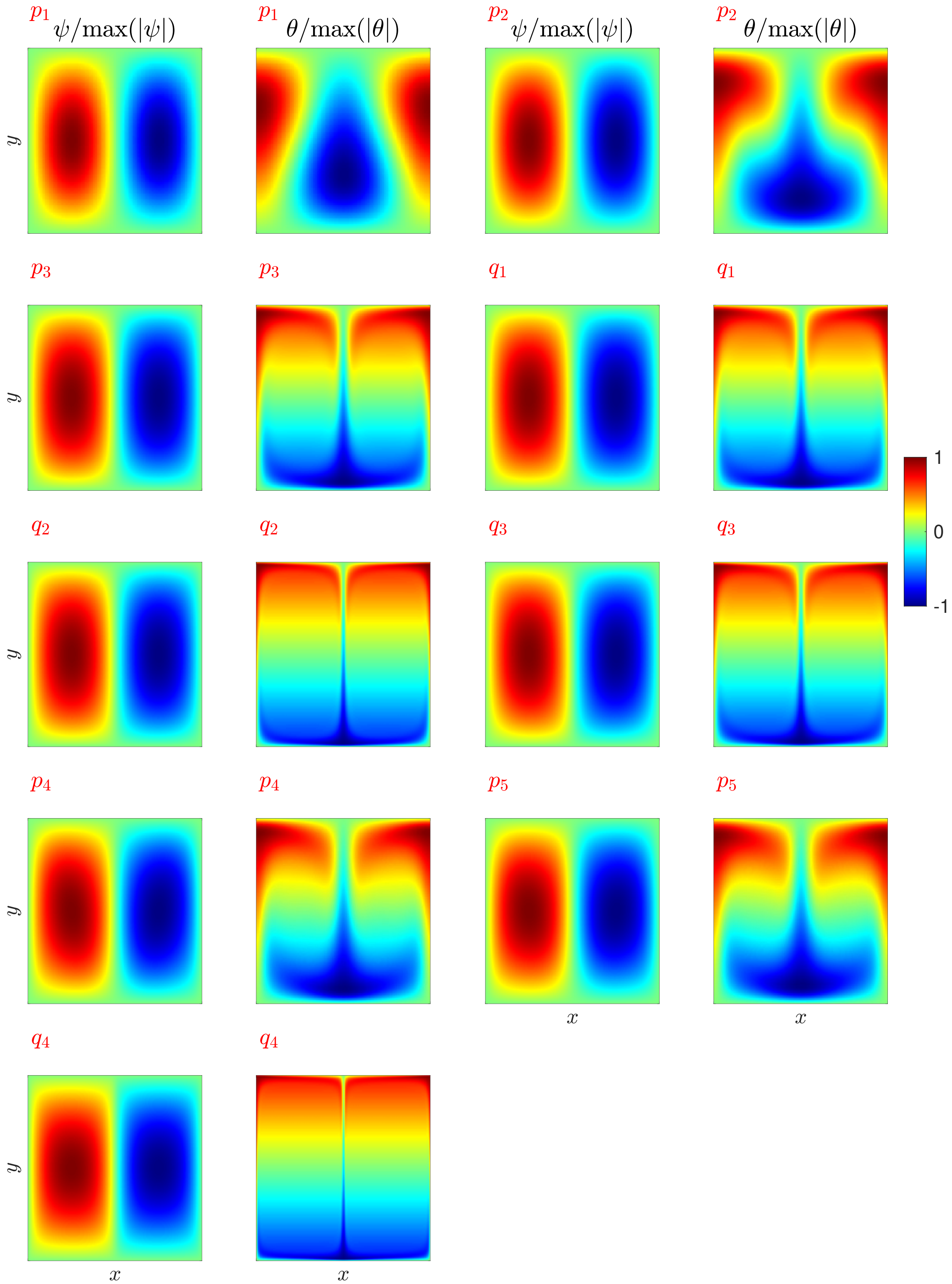

Figure 4: Visualisations of the SCRS at the points $p_1 = (Pr, Ra) = (5.48 \times 10^{-2}, 1.16 \times 10^3)$, $p_2 = (1.81 \times 10^{-2}, 4.12 \times 10^3)$, $p_3 = (0.2175, 2.59 \times 10^5)$, $q_1 = (0.2, 3.32 \times 10^5)$, $q_2 = (4.97, 3.949 \times 10^6)$, $q_3 = (8.58, 1.27 \times 10^6)$, $p_4 = (6.16, 5.43 \times 10^4)$, $p_5 = (9.53, 3.24 \times 10^4)$ and $q_4 = (7 \times 10^5, 2.54 \times 10^7)$. The domain is spanned by $x \in [0, \Gamma]$, $y \in [0, 1]$ and $\Gamma = 2$.

References

[1] B. Wen, D. Goluskin, M. LeDuc, G. P. Chini, and C. R. Doering. Steady Rayleigh–Bénard convection between stress-free boundaries. *Journal of Fluid Mechanics*, 905:R4, 2020.