Supplementary material for 'The instability of a helical vortex filament under a free surface'

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For a single horizontal helical vortex of infinite extent in the unbounded ideal fluid, the translation speed U_0 and rotational speed Ω_0 due to self-induction are given by

$$U_0 = \int_0^\infty a^2(1-\cos\lambda)J_0^{-3/2}\mathrm{d}\lambda, \quad \Omega_0 = \int_0^\infty (1-\cos\lambda-\lambda\sin\lambda)J_0^{-3/2}\mathrm{d}\lambda, \tag{1}$$

where $J_0 = \lambda^2 + 2a^2(1-\cos\lambda) + (ae_r)^2$ and $e_r = e_0 e^{-3/4}$. The corresponding components of the self-induced velocity in FrenetSerret frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ take the form

$$U_T = \frac{U_0 + a^2\Omega_0}{\sqrt{1+a^2}}, \quad U_N = 0, \quad U_B = \frac{a}{\sqrt{1+a^2}}(U_0 - \Omega_0), \tag{2}$$

indicating that the single helical vortex does not possess any self-induced motion in the normal direction $\boldsymbol{N}$. In following contexts, we first drive the asymptotic behaviors of U_0 and Ω_0 for both small and large a cases, and then show the inequality $U_B > 0$.

1 Asymptotic analysis of U_0 and Ω_0

First, we analyze the behavior of integral $f(a, e_r) = \int_0^\infty (1-\cos\lambda)J_0^{-3/2}\mathrm{d}\lambda$ when $e_r \to 0$. For any fixed $\delta \in (0,1)$, we split f into two parts $f_1 = \int_0^\delta$ and $f_2 = \int_\delta^\infty$. For small λ, we have following expansions

$$J_0 = \lambda^2 + a^2\lambda^2 + a^2e_r^2 + O(\lambda^4), \quad 1-\cos\lambda = \lambda^2/2 + O(\lambda^4). \tag{3}$$

Using these small λ expansions, we obtain

$$\begin{aligned} f_1 &= \frac{1}{2}\int_0^\delta \left[\lambda^2 + O(\lambda^4)\right]\left[(1+a^2)\lambda^2 + a^2e_r^2\right]^{-3/2}\mathrm{d}\lambda \\ &= \frac{1}{2(1+a^2)^{3/2}}\int_0^\delta \lambda^2(\lambda^2+\nu^2)^{-3/2}\mathrm{d}\lambda + \int_0^\delta O(\lambda)\mathrm{d}\lambda \\ &= \frac{1}{2(1+a^2)^{3/2}}\left[\ln\left(\sqrt{\frac{\delta^2}{\nu^2}+1} + \frac{\delta}{\nu}\right) - \frac{\delta}{\sqrt{\delta^2+\nu^2}}\right] + O(\delta^2) \\ &= \frac{1}{2(1+a^2)^{3/2}}\left[\ln\left(\frac{2\delta}{e_r}\right) + \frac{1}{2}\ln(1+a^{-2}) - 1\right] + O(\delta^2) + o(1), \end{aligned} \tag{4}$$

where $\nu = e_r(1+a^{-2})^{-1/2} \ll 1$, and $o(1)$ represents the terms that tend to 0 as $e_r \to 0$.

As for f_2, since the integrand is not singular when $e_r \to 0$, we have

$$f_2 = \int_\delta^\infty (1-\cos\lambda)\left[\lambda^2 + 2a^2(1-\cos\lambda)\right]^{-3/2} \mathrm{d}\lambda + o(1). \tag{5}$$

When $\lambda \to 0$, we have $(1-\cos\lambda)\left[\lambda^2 + 2a^2(1-\cos\lambda)\right]^{-3/2} = (1+a^2)^{-3/2}/(2\lambda) + O(\lambda)$. Using this small λ expansion, we have

$$\begin{aligned}
f_2 =& \int_\delta^\infty \left\{ \frac{1-\cos\lambda}{\left[\lambda^2 + 2a^2(1-\cos\lambda)\right]^{3/2}} - \frac{1}{2(1+a^2)^{3/2}} \frac{H(1-\lambda)}{\lambda} \right\} \mathrm{d}\lambda \\
&+ \frac{1}{2(1+a^2)^{3/2}} \int_\delta^1 \frac{1}{\lambda} \mathrm{d}\lambda + o(1) \\
=& \int_0^\infty \left\{ \frac{1-\cos\lambda}{\left[\lambda^2 + 2a^2(1-\cos\lambda)\right]^{3/2}} - \frac{1}{2(1+a^2)^{3/2}} \frac{H(1-\lambda)}{\lambda} \right\} \mathrm{d}\lambda + O(\delta^2) \\
&- \frac{1}{2(1+a^2)^{3/2}} \ln\delta + o(1) \\
=& \frac{1}{2a^3} W(a^{-1}) - \frac{1}{2(1+a^2)^{3/2}} \ln\delta + O(\delta^2) + o(1),
\end{aligned} \tag{6}$$

in which $H(x)$ denotes the unit step function, and function $W(x)$ is defined as

$$W(x) = \int_0^\infty \left\{ \frac{\sin^2 t}{\left(x^2 t^2 + \sin^2 t\right)^{3/2}} - \frac{1}{(1+x^2)^{3/2}} \frac{H(1/2 - t)}{t} \right\} \mathrm{d}t. \tag{7}$$

Combining the results of f_1, f_2 and letting $\delta \to 0$, we obtain

$$f = \frac{1}{2(1+a^2)^{3/2}} \left[\ln\left(\frac{2}{e_r}\right) + \frac{1}{2}\ln(1+a^{-2}) - 1 \right] + \frac{1}{2a^3} W(a^{-1}) + o(1), \quad e_r \to 0. \tag{8}$$

The asymptotic behaviors of $W(x)$ for small and large x have been studied in [1]. They are expressed in the form

$$W(x) = (3/2 - \gamma)x^{-3} + O(x^{-5}), \qquad x \to \infty, \tag{9}$$

$$W(x) = x^{-1} + (1+x^2)^{-3/2} \ln(x/2) + 1 + O(x^2), \qquad x \to 0, \tag{10}$$

where $\gamma \approx 0.577$ is the Euler constant. Using above formulae, after some algebraic manipulations, the asymptotic solutions of $f(a, e_r)$ are obtained to be given by

$$f = -\frac{1}{2(1+a^2)^{3/2}} \ln(ae_r) + \frac{1}{4}(1 - 2\gamma + \ln 4) + O(a^2), \qquad a \to 0, \tag{11}$$

$$f = -\frac{1}{2(1+a^2)^{3/2}} \ln(ae_r) + \frac{1}{2a^2} + O(a^{-5}), \qquad a \to \infty. \tag{12}$$

The asymptotic solutions of translation velocity $U_0 = a^2 f$ take the form

$$U_0 = -\frac{a^2}{2(1+a^2)^{3/2}} \ln(ae_r) + O(a^2), \qquad a \to 0, \tag{13}$$

$$U_0 = \frac{1}{2} - \frac{a^2}{2(1+a^2)^{3/2}} \ln(ae_r) + O(a^{-3}), \qquad a \to \infty. \tag{14}$$

For rotational velocity Ω_0, since $2(1-\cos\lambda)-\lambda\sin\lambda = O(\lambda^4)$ as $\lambda \to 0$, there is no singularity in $f + \Omega_0$ when $e_r = 0$. Specifically, we have

$$\begin{aligned} f + \Omega_0 &= \int_0^\infty \frac{2(1-\cos\lambda)-\lambda\sin\lambda}{[\lambda^2 + 2a^2(1-\cos\lambda)]^{3/2}} \mathrm{d}\lambda + o(1) \\ &= \int_0^\infty a^{-2} \mathrm{d}\left\{\lambda\left[\lambda^2 + 2a^2(1-\cos\lambda)\right]^{-1/2}\right\} + o(1) \\ &= a^{-2}\left[1-(1+a^2)^{-1/2}\right] + o(1). \end{aligned} \tag{15}$$

Therefore, the asymptotic solutions of Ω_0 are

$$\Omega_0 = \frac{1}{2(1+a^2)^{3/2}} \ln(ae_r) + \frac{1}{4}(1+2\gamma-\ln 4) + O(a^2), \qquad a \to 0, \tag{16}$$

$$\Omega_0 = \frac{1}{2a^2} + \frac{1}{2(1+a^2)^{3/2}} \ln(ae_r) + O(a^{-3}), \qquad a \to \infty. \tag{17}$$

2 Proof of inequality $U_B > 0$

In this section, we introduce β for simplicity in expressions by letting $a = \tan\beta$, where $\beta \in (0, \pi/2)$.

Lemma 1. *For $0 \le e_r < 1$, we have $U_T \le U_T(e_r = 0) = \cos\beta(1-\cos\beta)$*

Proof. When $e_r = 0$ we obtain from formula (15) that $U_0 + a^2\Omega_0 = 1-(1+a^2)^{-1/2} = 1-\cos\beta$, and then $U_T(e_r = 0) = \cos\beta(1-\cos\beta)$. When $e_r > 0$, considering that

$$J_0 \le (1+a^2)\lambda^2 + a^2 e_r^2, \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\lambda J_0^{-1/2}\right] = a^2\left[2(1-\cos\lambda) - \lambda\sin\lambda + e_r^2\right] J_0^{-3/2}, \tag{18}$$

we have

$$\begin{aligned} U_0 + a^2\Omega_0 &= \int_0^\infty \mathrm{d}\left[\lambda J_0^{-1/2}\right] - a^2 e_r^2 \int_0^\infty J_0^{-3/2} \mathrm{d}\lambda = 1 - a^2 e_r^2 \int_0^\infty J_0^{-3/2} \mathrm{d}\lambda \\ &\le 1 - a^2 e_r^2 \int_0^\infty \left[(1+a^2)\lambda^2 + a^2 e_r^2\right]^{-3/2} \mathrm{d}\lambda = 1 - \cos\beta. \end{aligned} \tag{19}$$

Therefore, the inequality $U_T \le U_T(e_r = 0) = \cos\beta(1-\cos\beta)$ must hold. □

Lemma 2. $1 - \sin\beta \mathrm{K}_1(\sin\beta) > \cos\beta\,(1-\cos\beta)$ *holds for all* $\beta \in (0, \pi/2)$, *where* $\mathrm{K}_1(x)$ *is the modified Bessel function.*

Proof. For arbitrary $z \in (0,1)$, we have [2]:

$$\mathrm{I}_1(z) = \frac{z}{2}\sum_{k=0}^{\infty}\frac{(\frac{1}{4}z^2)^k}{k!(k+1)!} > \frac{z}{2}, \tag{20}$$

$$\mathrm{K}_1(z) = \frac{1}{z} + \ln\left(\frac{z}{2}\right)\mathrm{I}_1(z) - \frac{z}{4}\sum_{k=0}^{\infty}\left[\psi\left(k+1\right)+\psi\left(k+2\right)\right]\frac{(\frac{1}{4}z^2)^k}{k!(k+1)!}, \tag{21}$$

where $\psi(z)$ is the digamma function that satisfies $\psi(n+1) = \sum_{k=1}^{n} k^{-1} - \gamma$ for $n \in \mathbb{N}$. Then we have:

$$\mathrm{K}_1(z) < \frac{1}{z} + \frac{z}{2}\ln\left(\frac{z}{2}\right) - \frac{z}{4}\left[\psi(1)+\psi(2)\right] = \frac{1}{z} + \frac{z}{2}\ln\left(\frac{z}{2}\right) - \frac{z}{4}(1-2\gamma), \tag{22}$$

$$1 - z\mathrm{K}_1(z) > -\frac{1}{2}z^2\ln\left(\frac{z}{2}\right) + \frac{1}{4}(1-2\gamma)z^2 = \frac{1}{4}\left[1-2\gamma+\ln 4-\ln(z^2)\right]z^2. \tag{23}$$

Therefore, by inequality (23), we only need to prove that

$$\begin{aligned}&\frac{1}{4}\left[1-2\gamma+\ln 4-\ln(\sin^2\beta)\right]\sin^2\beta > \cos\beta\,(1-\cos\beta)\\ \Longleftrightarrow\, &1-2\gamma+\ln 4 > \ln(\sin^2\beta) + \frac{4\cos\beta}{1+\cos\beta}.\end{aligned} \tag{24}$$

For function $f(x) = \ln(1-x^2) + 4x/(1+x)$, its maximum on interval $(0,1)$ is

$$f_{\max} = f\left(\frac{\sqrt{17}-3}{2}\right) \approx 1.06 < 1.23 \approx 1 - 2\gamma + \ln 4. \tag{25}$$

Therefore, inequality (24) holds for all $\beta \in (0, \pi/2)$ and the lemma is then proved. □

Theorem 1. *Bi-normal velocity $U_B > 0$ and $U_0 > \Omega_0$.*

Proof. By Basset's integral [2]

$$\mathrm{K}_1(z) = z\int_0^{\infty}\frac{\cos t}{(t^2+z^2)^{3/2}}\mathrm{d}t. \tag{26}$$

Using this and the fact $J_0 < (1+a^2)\lambda^2 + a^2$, we have

$$\begin{aligned}U_0 &= \int_0^{\infty} a^2(1-\cos\lambda)J_0^{-3/2}\mathrm{d}\lambda > \int_0^{\infty} a^2(1-\cos\lambda)\left[(1+a^2)\lambda^2+a^2\right]^{-3/2}\mathrm{d}\lambda\\ &= \cos\beta\left[1-\sin\beta\cdot\mathrm{K}_1(\sin\beta)\right] > \cos^2\beta(1-\cos\beta),\end{aligned} \tag{27}$$

where the last step is based on lemma 2. Also, by inequality (19), we have

$$\Omega_0 \le a^{-2}\left(1-\cos\beta-U_0\right) < \tan^{-2}\beta(1-\cos\beta)(1-\cos^2\beta) = (1-\cos\beta)\cos^2\beta. \tag{28}$$

Therefore, the result of $U_0 > (1-\cos\beta)\cos^2\beta > \Omega_0$ holds for any helical vortex filament. □

References

[1] J Boersma and DH Wood. On the self-induced motion of a helical vortex. *Journal of Fluid Mechanics*, 384:263–279, 1999.

[2] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.1.1 of 2021-03-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.