Two-dimensional response of a floating ice plate to a line load moving at variable speed

Roger J. Hosking, School of Mathematical Sciences, University of Adelaide, Australia

Fausto Milinazzo, Department of Mathematics, Royal Roads Military College, Canada*

7 Appendix

This material is provided to supplement the material found in the Appendix found in the above article. More precisely, the details of the contour integration required to evaluate the steady state solutions are given here. The methods used follow closely the work in Davys *et al.* and Schulkes & Sneyd in Chapter 5 of Squire et al. (1996).

7.3 The initial condition

The results in Miles and Sneyd (2003) envisage loads started impulsively at time $t = 0$, but for loads initially moving at constant speed V the Fourier transform of the deflection becomes

$$-\frac{\tanh kH}{k}\left[\frac{\cos kct - i\frac{V}{c}\sin kct}{c^2 - V^2} + \frac{k}{c}\int_0^t e^{-ikX(\tau)}\sin kc(t-\tau)\,d\tau\right]. \qquad \text{(A.1)}$$

The first term is the Fourier transform at time t of the component of the deflection due to the evolution of the initial condition and the second term is the Fourier transform of the component of the deflection at time t due to the moving load at the position $X(t)$. Since the second term has no poles on the real k axes for the class of load positions $X(t)$ considered in this article, the discrete inverse Fourier transform can be applied directly to obtain an approximation to its inverse Fourier transform. The first term has poles on the real k axes at points where $c^2 - V^2 = 0$ and consequently for $V > c_{\min}$ the discrete Fourier transform can not be applied directly. However, this term can be re-written in the form

$$\begin{aligned}
-\frac{\tanh kH}{k}\left[\frac{\cos kct - i\frac{V}{c}\sin kct}{c^2 - V^2}\right] = \frac{\tanh kH}{c+V}t&\left[\sin\frac{k(c+V)t}{2}\mathrm{sinc}\frac{k(c-V)t}{2}\right.\\
&\left.+ i\frac{V}{c}\left[\cos\frac{k(c+V)t}{2}\mathrm{sinc}\frac{k(c-V)t}{2} - \mathrm{sinc}(kVt)\right]\right]\\
&- \frac{\tanh kH}{k}\frac{e^{-ikVt}}{c^2 - V^2}\,. \qquad \text{(A.2)}
\end{aligned}$$

The term within the outside square brackets is regular on the real k axis so that the inverse Fourier transform of the corresponding expression can be approximated using the discrete inverse Fourier transform. The inverse Fourier transform of the last term (the steadily moving load contribution) can be evaluated analytically using the method of residues.

The contour of integration is taken to be the real axis from $-R$ to R, where $R > 0$ and sufficiently large, together with the semi-circle $\Gamma_R : z = Re^{i\theta}$ where $0 \leq \theta \leq \pi$ if $x - Vt > 0$ and $0 \geq \theta \geq -\pi$ if $x - Vt < 0$. The poles are computed numerically and can be obtained by noting that

$$\frac{\tanh kH}{k}\frac{e^{-ikVt}}{c^2 - V^2} = \frac{e^{-ikVt}}{(c^2 - V^2)\frac{k}{\tanh kH}}$$

and writing the denominator in the form

$$(c^2 - V^2)\frac{k}{\tanh kH} = g\left(1 + \frac{Dk^4}{\rho g}\right) - V^2\frac{k}{\tanh kH}$$

There are two sets of poles. The first set is made up of poles that are purely imaginary and can be obtained by setting $k = i\kappa$ and writing

$$\left(1 + \frac{D}{\rho g}\kappa^4\right) - \left(\frac{V}{\sqrt{gH}}\right)^2 \frac{\kappa H}{\tan \kappa H} = 0$$

where it can be seen that the poles (the zeros of the above equation) depend on the Froude number $Fr = V/\sqrt{gH}$. Since the expression

$$\frac{\tan \kappa H}{\kappa H} = \frac{Fr^2}{1 + \frac{D}{\rho g}\kappa^4}$$

is even in κ, it is sufficient to consider $\kappa \geq 0$. For $\kappa \geq 0$ the function $Fr^2/(1 + \frac{D}{\rho g}\kappa^4)$ decreases from Fr^2 to zero as κ increases to infinity and the function $\tan \kappa H/\kappa H$ increases from 1 to infinity on the interval $J_0 = (\kappa H : 0 \leq \kappa H < \frac{\pi}{2})$ and increases from minus infinity to infinity on the the intervals $J_n = (\kappa H : \frac{\pi}{2}(2n-1) < \kappa H < \frac{\pi}{2}(2n+1))$ for $n = 1, 2, \dots$. Consequently, for $Fr^2 > 1$ there is a simple pole κ_n in each of the intervals J_n. For $Fr^2 < 1$, there are no poles in the interval J_0. For $Fr^2 = 1$ (i.e. when $V = \sqrt{gH}$), there is a double pole at $k = \kappa_0 = 0$. The poles in the second set are complex for $V < c_{\min}$. Two of the poles k_L and $-\bar{k}_L$ are in the lower half plane and two of the poles k_U and $-\bar{k}_U$ are in the upper half plane. As $V \to c_{\min}$, the poles approach the real k axis until at $V = c_{\min}$ the imaginary parts of k_L and k_U become zero and k_L=k_U. For $V = c_{\min}$ they become double poles, so the deflection for the constant speed $V = c_{\min}$ (the critical speed) is predicted to grow without bound as t increases. For $c_{\min} < V < \sqrt{gH}$, the poles $\pm k_L$ and $\pm k_U$ are simple poles on the real k axis. For $V > \sqrt{gH}$, there are only two real poles $\pm k_U$. The artifact attributed to Lighthill (cf. Lighthill (1957, 1978)) moves the real poles off the real axis and so attributes their contribution to the appropriate half plane by introducing an artificial time dependence, where the load is zero in the distant past and grows to p_0 at time t. For $V > c_{\min}$, this artifact treats the poles $\pm k_U$ as if they were in the upper half plane (see also Davys *et al.* and Schulkes & Sneyd discussed in Chapter 5 of Squire et al. (1996)); and for $c_{\min} < V < \sqrt{gH}$ it treats the poles $\pm k_L$ as if they were in the lower half plane.

Noting that the contribution to the contour integration along the path Γ_R goes to zero as $R \to \infty$, the steady deflection moving with speed V is given by

$$\eta(x - Vt) = -\frac{p_0}{2\pi\rho}\int_{-\infty}^{\infty} \frac{\tanh kH}{k}\frac{e^{ik(x-Vt)}}{c^2 - V^2}\,dk = \pm i\frac{p_0}{\rho}\sum_{\kappa\in\Pi} \text{residue}(\kappa, \frac{\tanh \kappa H}{\kappa}\frac{e^{i\kappa(x-Vt)}}{c^2 - V^2})\ ,$$

where Π is the set of poles within the contour which in this case is either the lower or upper half plane.

Written in the frame moving with the load, we have

$$\eta(x) = -\frac{p_0}{\rho}\begin{cases} \operatorname{Im}\left[\dfrac{\tanh k_L H}{k_L}\dfrac{e^{ik_L x}}{c(k_L)\dot{c}(k_L)} + \displaystyle\sum_{n=1}^{\infty}\dfrac{\tan\kappa_n H}{2\kappa_n}\dfrac{e^{\kappa_n x}}{c(-i\kappa_n)\dot{c}(-i\kappa_n)}\right] \\ \hfill \text{for } V < c_{\min}\ , \\ \dfrac{\tanh k_L H}{k_L}\dfrac{\sin k_L x}{c(k_L)\dot{c}(k_L)} + \operatorname{Im}\left[\displaystyle\sum_{n=1}^{\infty}\dfrac{\tan\kappa_n H}{2\kappa_n}\dfrac{e^{\kappa_n x}}{c(-i\kappa_n)\dot{c}(-i\kappa_n)}\right] \\ \hfill \text{for } c_{\min} < V < \sqrt{gH}\ , \\ \operatorname{Im}\left[\displaystyle\sum_{n=0}^{\infty}\dfrac{\tan\kappa_n H}{2\kappa_n}\dfrac{e^{\kappa_n x}}{c(-i\kappa_n)\dot{c}(-i\kappa_n)}\right] \text{ for } V > \sqrt{gH}\ , \end{cases}$$

for $x < 0$, and

$$\eta(x) = -\frac{p_0}{\rho}\begin{cases} \operatorname{Im}\left[\dfrac{\tanh k_U H}{k_U}\dfrac{e^{ik_U x}}{c(k_U)\dot{c}(k_U)} + \displaystyle\sum_{n=1}^{\infty}\dfrac{\tan\kappa_n H}{2\kappa_n}\dfrac{e^{-\kappa_n x}}{c(i\kappa_n)\dot{c}(i\kappa_n)}\right] \\ \hfill \text{for } V < c_{\min}\ , \\ \dfrac{\tanh k_U H}{k_U}\dfrac{\sin k_U x}{c(k_U)\dot{c}(k_U)} + \operatorname{Im}\left[\displaystyle\sum_{n=1}^{\infty}\dfrac{\tan\kappa_n H}{2\kappa_n}\dfrac{e^{-\kappa_n x}}{c(i\kappa_n)\dot{c}(i\kappa_n)}\right] \\ \hfill \text{for } c_{\min} < V < \sqrt{gH}\ , \\ \dfrac{\tanh k_U H}{k_U}\dfrac{\sin k_U x}{c(k_U)\dot{c}(k_U)} + \operatorname{Im}\left[\displaystyle\sum_{n=0}^{\infty}\dfrac{\tan\kappa_n H}{2\kappa_n}\dfrac{e^{-\kappa_n x}}{c(i\kappa_n)\dot{c}(i\kappa_n)}\right] \\ \hfill \text{for } V > \sqrt{gH}\ , \end{cases}$$

for $x > 0$.

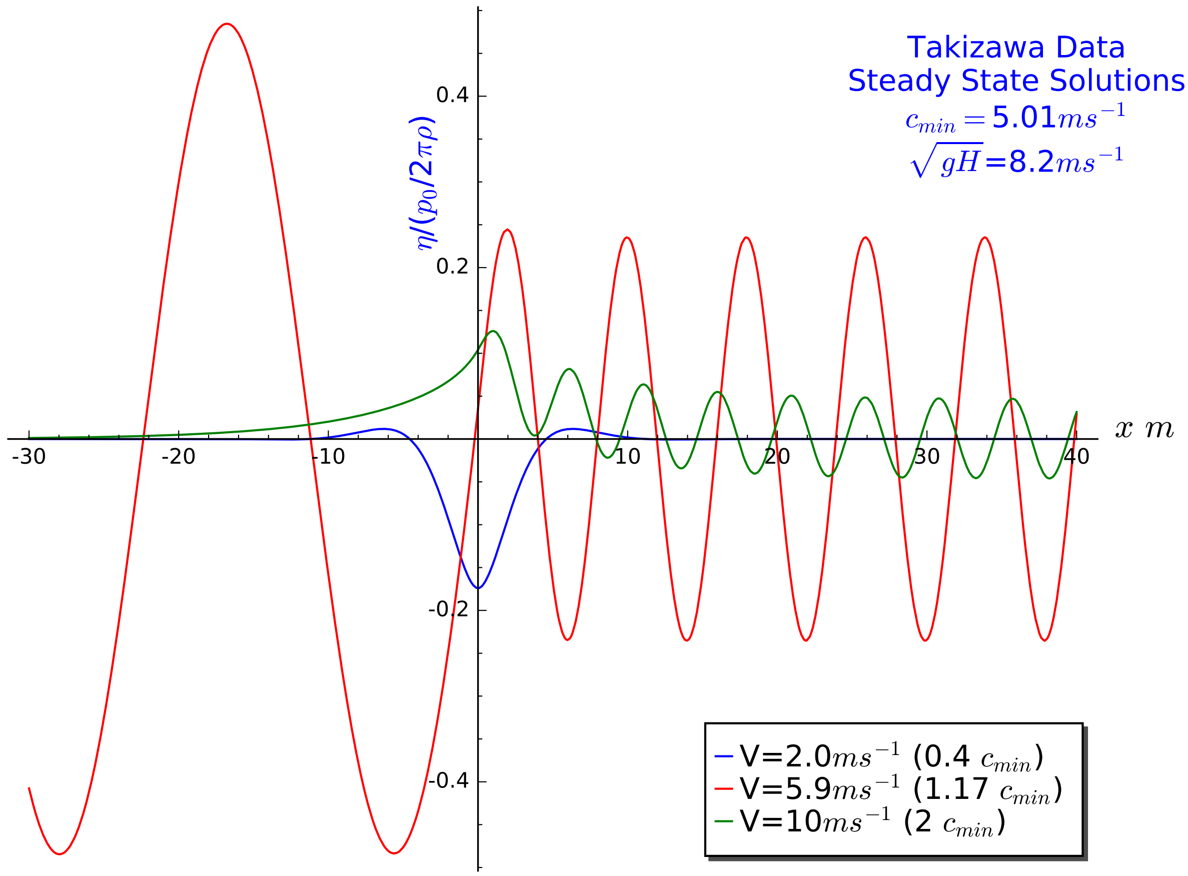

Figure 1: Steady state deflections at the constant subcritical and supercritical speeds consistent with the three limiting results produced in Section 3 for the Takizawa data.

References

Lighthill, M. (1957), *River waves. Naval hydrodynamics publication 515*, National Academy of Sciences - National Research Council.

Lighthill, M. (1978), *Waves in Fluids*, Cambridge University Press, Cambridge, England.

Miles, J. and Sneyd, A. (2003), 'The response of a floating ice sheet to an accelerating line load', *J. Fluid Mech.* **497**, 435–439.

Squire, V., Hosking, R., Kerr, A. and Langhorne, P. (1996), *Moving Loads on Ice Plates*, Kluwer Academic Publishers, Dordrecht. Also Arkiv, Springer (2012).