Supplementary Materials A: shift of streamwise root-mean-square velocity near the leading edge of the isothermal wall

As seen in Section 3 of the article, an upward shift of streamwise root-mean-square (r.m.s.) velocity u_{rms} is observed near the leading edge of the isothermal wall, *i.e.*, for $x/\delta \approx 0$. This perturbation vanishes as x/δ increases, and we have attributed this anomaly to the recycling method described in §2.2. In fact, between the inlet (at $x/\delta = -4\pi$) and the recycling plane (at $x/\delta = -2\pi$), there is not a real periodicity in the sense of absence of boundary conditions, like in a bi-periodic channel flow or like along the spanwise direction in our simulation. In our case, we extract the fields at $x/\delta = -2\pi$ and re-inject them at the inlet using relaxed characteristic boundary conditions, and although we use an extremely high relaxation coefficient ($\sim 10^{+5}$), at $x/\delta = -4\pi$ we do not have the exact velocity values of $x/\delta = -2\pi$.

Other aspects might contribute to the shift we observe on u_{rms}, namely the discontinuity of surface temperature at x/δ, and potential auto-correlation issues between the inlet and the recycling plane. The objective of this document is to provide further elements to support the explanation given in Section 3.

We have performed two additional direct numerical simulations. The first simulation's domain is $6\pi\delta$ long and the recycling is prescribed at $x = 2\pi\delta$ (*i.e.*, exactly like the non-equilibrium simulation of our paper); the second simulation's domain is still $6\pi\delta$ long, yet the recycling is prescribed at $x = 4\pi\delta$, so that we are sure to avoid any auto-correlation problem. In both cases, the walls are adiabatic and the point distribution is uniform along the streamwise direction, thus avoiding any discontinuity in the streamwise direction. Table 1 summarises the main numerical parameters of the two simulations (S1 and S2).

In the following, $(\cdot)^+$ denotes classic wall-scaling, as in the paper, and x/δ denotes the non-dimensional distance from the inlet (therefore, $x/\delta \in [0, 6\pi]$); results are compared to the equilibrium adiabatic channel flow of Section 3.1. Let us commence by analysing the evolution of the mean streamwise velocity. Figure 1 shows the profiles obtained at $x/\delta = 0.5\pi$ (a), $x/\delta = 2.5\pi$ (b), $x/\delta = 4.5\pi$ (c) and $x/\delta = 5.5\pi$ (d) with both S1 and S2. First of all, notice that no appreciable difference can be observed between S1 and S2 at any x/δ, indicating that the recycling location has no influence on the mean velocity. As can be seen, the profiles at $x/\delta = 0.5\pi$ are strongly perturbed, with an upward shift of around 10%; however, the perturbation quickly decreases and, even if it is still noticeable at $x/\delta = 2.5\pi$, it appears to have fully disappeared by $x/\delta = 4.5\pi$. Note that, in the paper, the leading edge of the isothermal wall is at a distance of $x/\delta = 4\pi$ from the inlet, which explains why no perturbation of the mean streamwise velocity has been reported in Section 3.1.

Now, let us focus on the r.m.s. velocity profiles. Figure 2 shows the streamwise, wall-normal and spanwise r.m.s. velocity profiles obtained at $x/\delta = 0.5\pi$ (a), $x/\delta = 2.5\pi$ (b), $x/\delta = 4.5\pi$ (c) and $x/\delta = 5.5\pi$ (d) with both S1 and S2. In this case, larger differences between S1 and S2 are observed. However, these deviations become remarkably smaller as x/δ increases. Therefore, also in this case we can conclude that the location of the

	Size	Number of nodes	Recycling location
S1	$6\pi\delta, 2\delta, \pi\delta$	598, 179, 200	$x = 2\pi\delta$
S2	$6\pi\delta, 2\delta, \pi\delta$	598, 179, 200	$x = 4\pi\delta$

Table 1: Size, number of nodes and resolutions of S1 and S2.

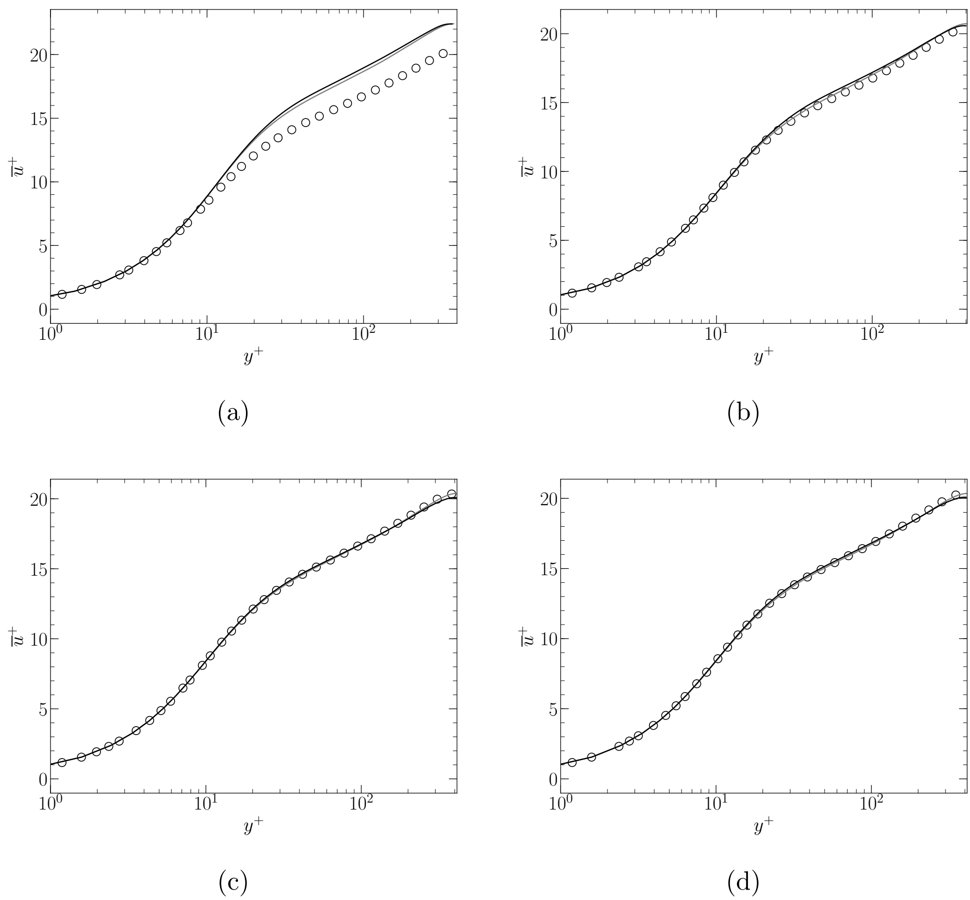

Figure 1: Mean streamwise velocity profiles at $x/\delta = 0.5\pi$ (a), $x/\delta = 2.5\pi$ (b), $x/\delta = 4.5\pi$ (c) and $x/\delta = 5.5\pi$ (d). Black solid line, S1; Gray solid line, S2; $\bigcirc$ bi-periodic adiabatic channel flow from Section 3.1.

recycling plane does not seem to have a strong influence on the flow statistics. The most perturbed profiles appear to be the ones at $x/\delta = 0.5\pi$, *i.e.*, the closest to the inlet, where none of the profiles agree with the equilibrium ones; this is true for the streamwise component (as observed in the paper), yet also for the wall-normal and spanwise components. As x/δ increases, v_{rms} and w_{rms} quickly return to equilibrium, which explains why no shifts have been reported for these components in Section 3.1. Concerning u_{rms}, instead, notice how the perturbation persists even until $x/\delta = 5.5\pi$, consistently with what has been observed in Section 3.1 of the paper. In this case,

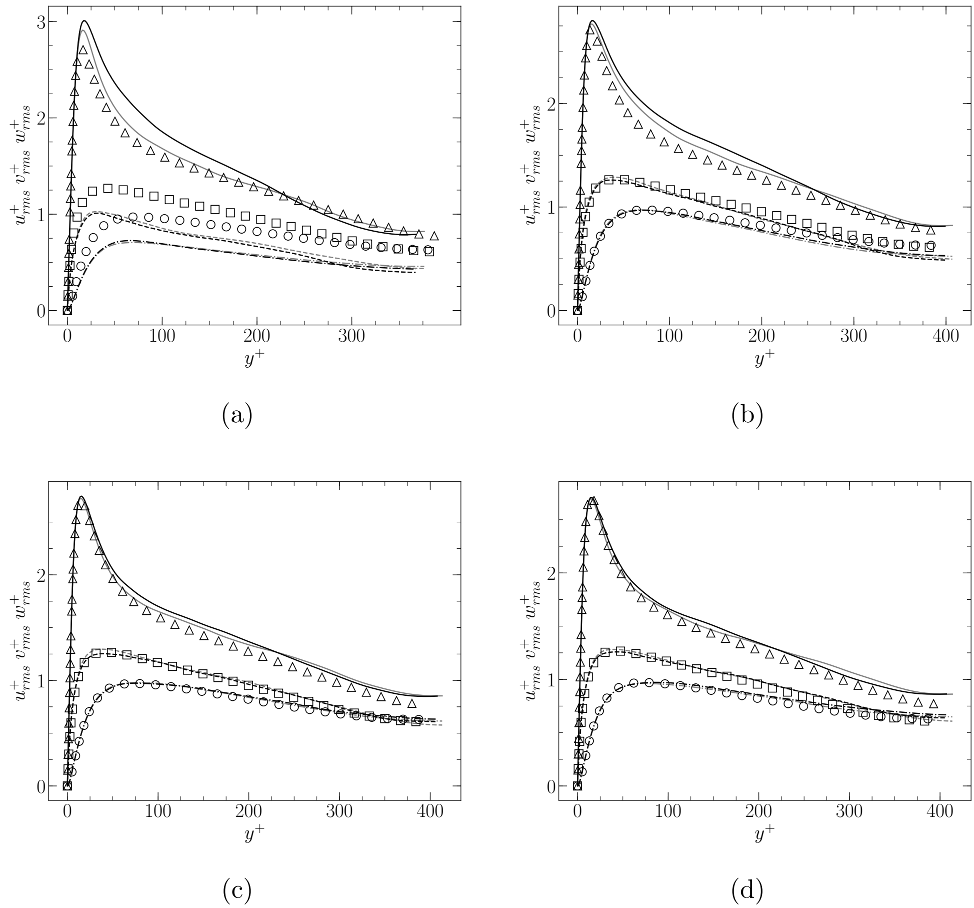

Figure 2: Profiles of r.m.s. streamwise, wall-normal and spanwise velocity, respectively, at $x/\delta = 0.5\pi$ (a), $x/\delta = 2.5\pi$ (b), $x/\delta = 4.5\pi$ (c) and $x/\delta = 5.5\pi$ (d): ——, − · − · − and - - - - present results (black lines for S1, gray lines for S2); △, ○ and □ results from bi-periodic adiabatic channel flow of Section 3.1.

however, the channel flow's length is limited, and, therefore, the return to equilibrium of u_{rms} does not occur before the outlet.

Our results can be summarised as follows:

— The mean and r.m.s. velocity profiles are perturbed by the recycling method;

— The perturbation appears near the inlet and vanishes as x/δ increases;

— The most persisting impact, as observed in the non-equilibrium simulation of the paper, seems to be the one on the streamwise r.m.s. velocity, which exhibits a shift even at $x/\delta = 5.5$.

These results allow us to conclude that:

— The perturbation cannot be attributed to the discontinuity of wall temperature since, in this case, the walls of the channel flow are entirely adiabatic;

— The perturbation cannot be ascribed to potential auto-correlation problems since the results of S1 and S2, obtained with two different recycling plane locations, are extremely similar;

— The perturbation is due to the recycling method itself and by the use of relaxed characteristic boundary conditions at the inlet towards target values determined at the recycling plane. Despite the very short response time, this introduces a non-ideal recycling, all the more so as negative streamwise velocity can occasionally be encountered, which the inlet characteristic boundary cannot handle properly. This conclusion, in particular, is corroborated by the fact that the most affected profiles are the ones at $x/\delta = 0.5\pi$ and the deviation decreases as the distance from the inlet is increased.