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Supplementary material

[redacted]

In this supplementary material are the appendices with more details.

Appendix A. About the vector $\boldsymbol{B}$

Like the other quantities, the vector $\boldsymbol{B}$ in (2.14) can be decomposed as

$$\boldsymbol{B}_i = S_i(r)\boldsymbol{e}_r P_l^m(\cos\theta)\exp(\mathrm{j}m\varphi - \omega t), \;\; i = 1,\; 2,\; 3, \tag{A 1}$$

where $S(r)$ is the initial amplitude of $\boldsymbol{B}$.

Substituting the decomposition (A 1) into (2.14) and then (2.14) into (2.13), we get the following governing equation for $S(r)$,

$$\frac{\mathrm{d}^2 S_i}{\mathrm{d}r^2} - \frac{l(l+1)}{r^2}S_i + \frac{\rho_i\omega}{\mu_i}S_i = 0, \quad i = 1,\; 2,\; 3. \tag{A 2}$$

Considering the boundedness of $S(r)$ at the origin $r = 0$ and at infinity, the solution to (A 2) is

$$S_1 = C_1 r^{\frac{1}{2}} J_{l+\frac{1}{2}}(\vartheta_1 r), \tag{A 3}$$

$$S_2 = C_2 r^{\frac{1}{2}} J_{l+\frac{1}{2}}(\vartheta_2 r) + C_3 r^{\frac{1}{2}} Y_{l+\frac{1}{2}}(\vartheta_2 r), \tag{A 4}$$

$$S_3 = C_4 r^{\frac{1}{2}} H^{(1)}_{l+\frac{1}{2}}(\vartheta_3 r), \;\; \text{if} \;\; \mathrm{Im}(\omega) \neq 0 \tag{A 5a}$$

$$S_3 = C_5 r^{\frac{1}{2}} J_{l+\frac{1}{2}}(\vartheta_3 r) + C_6 r^{\frac{1}{2}} Y_{l+\frac{1}{2}}(\vartheta_3 r), \;\; \text{if} \;\; \mathrm{Im}(\omega) = 0 \tag{A 5b}$$

where

$$\vartheta_i = \sqrt{\frac{\rho_i\omega}{\mu_i}}, \quad i = 1,\; 2,\; 3, \tag{A 6}$$

$J_{l+\frac{1}{2}}(\cdot)$ and $Y_{l+\frac{1}{2}}(\cdot)$ are, respectively, the Bessel functions of the first and second kinds, $H^{(1)}_{l+\frac{1}{2}}(\cdot)$ is the Hankel function of the first kind with order $l + \frac{1}{2}$, and C_1-C_6 are the coefficients to be determined. Considering that the Bessel/Hankel functions of fractional order $l + \frac{1}{2}$ relate to the spherical Bessel functions of integer order l, the solution to (A 2) can also be expressed in terms of the latter.

Now substitute the decompositions (2.14), (2.16) and (A 1) into the boundary conditions at the inner and outer interfaces (2.3)-(2.10). While doing this, one can easily find that neither the vector $\boldsymbol{B}$ nor its initial amplitude $S(r)$ plays a role in the kinematic boundary condition, the continuity of velocity in the r direction or the normal force balance at an interface. They only need to satisfy the continuity of velocity in the θ and φ directions and the tangential force balance.

The continuity of velocity in either θ or φ direction in (2.4) yields

$$S_1 = S_2 \quad \text{at} \quad r = R_1. \tag{A 7}$$

The equality of either $r\theta$- or $r\varphi$-component of the deviatoric stress tensors in (2.5) yields

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mu_2 S_2 - \mu_1 S_1}{r^2}\right) \quad \text{at} \quad r = R_1. \tag{A 8}$$

Similarly, (2.8) yields

$$S_2 = S_3 \quad \text{at} \quad r = R_2, \tag{A 9}$$

and (2.9) yields

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mu_3 S_3 - \mu_2 S_2}{r^2}\right) \quad \text{at} \quad r = R_2. \tag{A 10}$$

Note that in the bulk equation (A 2) and boundary conditions (A 7)-(A 10) only the eigenfunction $S(r)$ is involved.

Substitution of (A 3)-(A 5b) into (A 7)-(A 10) yields

$$C_1 J_{l+\frac{1}{2}}(\vartheta_1 R_1) = C_2 J_{l+\frac{1}{2}}(\vartheta_2 R_1) + C_3 Y_{l+\frac{1}{2}}(\vartheta_2 R_1), \tag{A 11}$$

$$\begin{aligned}\mu_1 C_1 \left[(l-1) J_{l+\frac{1}{2}}(\vartheta_1 R_1) - \vartheta_1 R_1 J_{l+\frac{3}{2}}(\vartheta_1 R_1)\right] = \mu_2 C_2 \left[(l-1) J_{l+\frac{1}{2}}(\vartheta_2 R_1)\right.\\ \left. - \vartheta_2 R_1 J_{l+\frac{3}{2}}(\vartheta_2 R_1)\right] + \mu_2 C_3 \left[(l-1) Y_{l+\frac{1}{2}}(\vartheta_2 R_1) - \vartheta_2 R_1 Y_{l+\frac{3}{2}}(\vartheta_2 R_1)\right],\end{aligned} \tag{A 12}$$

$$C_4 H^{(1)}_{l+\frac{1}{2}}(\vartheta_3 R_2) = C_2 J_{l+\frac{1}{2}}(\vartheta_2 R_2) + C_3 Y_{l+\frac{1}{2}}(\vartheta_2 R_2), \quad \text{if} \quad \mathrm{Im}(\omega) \neq 0 \tag{A 13}$$

$$C_5 J_{l+\frac{1}{2}}(\vartheta_3 R_2) + C_6 Y_{l+\frac{1}{2}}(\vartheta_3 R_2) = C_2 J_{l+\frac{1}{2}}(\vartheta_2 R_2) + C_3 Y_{l+\frac{1}{2}}(\vartheta_2 R_2), \quad \text{if} \quad \mathrm{Im}(\omega) = 0 \tag{A 14}$$

$$\begin{aligned}\mu_3 C_4 \left[(l-1) H^{(1)}_{l+\frac{1}{2}}(\vartheta_3 R_2) - \vartheta_3 R_2 H^{(1)}_{l+\frac{3}{2}}(\vartheta_3 R_2)\right] = \mu_2 C_2 \left[(l-1) J_{l+\frac{1}{2}}(\vartheta_2 R_2)\right.\\ \left. - \vartheta_2 R_2 J_{l+\frac{3}{2}}(\vartheta_2 R_2)\right] + \mu_2 C_3 \left[(l-1) Y_{l+\frac{1}{2}}(\vartheta_2 R_2) - \vartheta_2 R_2 Y_{l+\frac{3}{2}}(\vartheta_2 R_2)\right], \quad \text{if } \mathrm{Im}(\omega) \neq 0\end{aligned} \tag{A 15}$$

$$\begin{aligned}\mu_3 C_5 \left[(l-1) J_{l+\frac{1}{2}}(\vartheta_3 R_2) - \vartheta_3 R_2 J_{l+\frac{3}{2}}(\vartheta_3 R_2)\right] + \mu_3 C_6 \left[(l-1) Y_{l+\frac{1}{2}}(\vartheta_3 R_2)\right.\\ \left. - \vartheta_3 R_2 Y_{l+\frac{3}{2}}(\vartheta_3 R_2)\right] = \mu_2 C_2 \left[(l-1) J_{l+\frac{1}{2}}(\vartheta_2 R_2) - \vartheta_2 R_2 J_{l+\frac{3}{2}}(\vartheta_2 R_2)\right] +\\ \mu_2 C_3 \left[(l-1) Y_{l+\frac{1}{2}}(\vartheta_2 R_2) - \vartheta_2 R_2 Y_{l+\frac{3}{2}}(\vartheta_2 R_2)\right], \quad \text{if} \quad \mathrm{Im}(\omega) = 0.\end{aligned} \tag{A 16}$$

In the case $\mathrm{Im}(\omega) \neq 0$, (A 11)-(A 13) and (A 15) are four linear homogenous equations in the four coefficients C_1-C_4. Hence nontrivial solutions exist only if the determinant of coefficients is zero, which gives the following characteristic equation

$$\begin{vmatrix} 1 & \kappa_1 & \upsilon_1 & 0 \\ 0 & \kappa_2 & \upsilon_2 & 1 \\ \mu_1\left[(l-1)-\vartheta_1 R_1 \varUpsilon_1\right] & \mu_2 H_5 & \mu_2 \varPi_5 & 0 \\ 0 & \mu_2 H_6 & \mu_2 \varPi_6 & \mu_3\left[(l-1)-\vartheta_3 R_2 \varUpsilon_2\right] \end{vmatrix} = 0, \tag{A 17}$$

where

$$\varUpsilon_1 = \frac{J_{l+\frac{3}{2}}(\vartheta_1 R_1)}{J_{l+\frac{1}{2}}(\vartheta_1 R_1)},\ \varUpsilon_2 = \frac{H^{(1)}_{l+\frac{3}{2}}(\vartheta_3 R_2)}{H^{(1)}_{l+\frac{1}{2}}(\vartheta_3 R_2)}, \tag{A 18}$$

$$H_5 = (l-1)\kappa_1 - \vartheta_2 R_1 \kappa_3,\ H_6 = (l-1)\kappa_2 - \vartheta_2 R_2 \kappa_4, \tag{A 19}$$

$$\varPi_5 = (l-1)\upsilon_1 - \vartheta_2 R_1 \upsilon_3,\ \varPi_6 = (l-1)\upsilon_2 - \vartheta_2 R_2 \upsilon_4, \tag{A 20}$$

$$\kappa_1 = J_{l+\frac{1}{2}}(\vartheta_2 R_1),\ \ \kappa_2 = J_{l+\frac{1}{2}}(\vartheta_2 R_2),\ \kappa_3 = J_{l+\frac{3}{2}}(\vartheta_2 R_1),\ \ \kappa_4 = J_{l+\frac{3}{2}}(\vartheta_2 R_2), \tag{A 21}$$

$$\upsilon_1 = Y_{l+\frac{1}{2}}(\vartheta_2 R_1),\ \ \upsilon_2 = Y_{l+\frac{1}{2}}(\vartheta_2 R_2),\ \upsilon_3 = Y_{l+\frac{3}{2}}(\vartheta_2 R_1),\ \ \upsilon_4 = Y_{l+\frac{3}{2}}(\vartheta_2 R_2). \tag{A 22}$$

All the roots of the transcendental equation (A 17) are purely real, i.e. $\mathrm{Im}(\omega) = 0$, as in the case of a viscous droplet suspended in a viscous host fluid (Miller & Scriven 1968; Prosperetti 1980b). This result is against the hypothesis $\mathrm{Im}(\omega) \neq 0$ based on which (A 17) is derived. Accordingly, the solution of S_3 expressed by the Hankel function in (A 5*a*) is incorrect.

Under the hypothesis $\mathrm{Im}(\omega) = 0$, the four equations (A 11), (A 12), (A 14) and (A 16) constitute an under-determined system for the five unknowns (C_1-C_3, C_5, C_6). In such a case, the eigenvalue ω can be any real, non-negative number, which forms a continuous spectrum occupying the entire positive real semi-axis in the complex frequency plane. Physically, the continuous spectrum corresponds to purely rotational waves or shear waves (Miller & Scriven 1968; Prosperetti 1980b). Waves of this type cause no interface displacement and are irrelevant to shape oscillations of the droplet. Moreover, due to the lack of restoring force, these waves are always damped without oscillation.

Appendix B. Derivation of the characteristic equation for small-amplitude shape oscillations of a viscous compound droplet suspended in a viscous host fluid

Taking the divergence of (2.2) and then combing it with (2.1), we get

$$\nabla^2 p_i = 0, \quad i = 1,\ 2,\ 3. \tag{B 1}$$

The pressure perturbation p can be decomposed as

$$p_i = \hat{p}_i(r) P_l^m(\cos\theta)\exp(\mathrm{j}m\varphi - \omega t),\ i = 1,\ 2,\ 3, \tag{B 2}$$

where $\hat{p}(r)$ is the initial amplitude of p and also its eigenfunction.

Substitution of (B 2) into (B 1) yields

$$\frac{\mathrm{d}^2\hat{p}_i}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\hat{p}_i}{\mathrm{d}r} - \frac{l(l+1)}{r^2}\hat{p}_i = 0, \quad i = 1,\ 2\ ,3. \tag{B 3}$$

Considering the boundedness of the pressure at the origin $r = 0$ and at infinity, the solution to (B 3) is

$$\hat{p}_1 = A_1 r^l, \tag{B 4}$$

$$\hat{p}_2 = A_2 r^l + A_3 r^{-(l+1)}, \tag{B 5}$$

$$\hat{p}_3 = A_4 r^{-(l+1)}, \tag{B 6}$$

where A_1-A_4 are the coefficients to be determined.

For each phase, the velocity field $\boldsymbol{v}$ is purely poloidal, whose three components can be expressed in terms of a scalar defining function $U(r)$ (Chandrasekhar 1959), i.e.

$$v_r = \frac{l(l+1)}{r^2} U P_l^m(\cos\theta) \exp(\mathrm{j}m\varphi - \omega t), \tag{B 7}$$

$$v_\theta = \frac{1}{r}\frac{\mathrm{d}U}{\mathrm{d}r}\frac{\partial P_l^m(\cos\theta)}{\partial\theta} \exp(\mathrm{j}m\varphi - \omega t), \tag{B 8}$$

$$v_\varphi = \frac{1}{r\sin\theta}\frac{\mathrm{d}U}{\mathrm{d}r} \mathrm{j}m P_l^m(\cos\theta) \exp(\mathrm{j}m\varphi - \omega t). \tag{B 9}$$

Further, the momentum equation (2.2) can be replaced by the following ordinary differential equation of $U(r)$ (Chandrasekhar 1959)

$$\frac{\mathrm{d}^2 U_1}{\mathrm{d}r^2} - \frac{l(l+1)}{r^2} U_1 + \frac{\rho_1\omega}{\mu_1} U_1 = \frac{A_1}{\mu_1}\frac{r^{l+1}}{l+1}, \tag{B 10}$$

$$\frac{\mathrm{d}^2 U_2}{\mathrm{d}r^2} - \frac{l(l+1)}{r^2} U_2 + \frac{\rho_2\omega}{\mu_2} U_2 = \frac{A_2}{\mu_2}\frac{r^{l+1}}{l+1} - \frac{A_3}{\mu_2}\frac{r^{-l}}{l}, \tag{B 11}$$

$$\frac{\mathrm{d}^2 U_3}{\mathrm{d}r^2} - \frac{l(l+1)}{r^2} U_3 + \frac{\rho_3\omega}{\mu_3} U_3 = -\frac{A_4}{\mu_3}\frac{r^{-l}}{l}. \tag{B 12}$$

The solutions to (B 10)-(B 12) are

$$U_1 = \frac{A_1}{\rho_1\omega}\frac{r^{l+1}}{l+1} + A_5 r^{\frac{1}{2}} J_{l+\frac{1}{2}}(\vartheta_1 r), \tag{B 13}$$

$$U_2 = \frac{A_2}{\rho_2\omega}\frac{r^{l+1}}{l+1} - \frac{A_3}{\rho_2\omega}\frac{r^{-l}}{l} + A_6 r^{\frac{1}{2}} J_{l+\frac{1}{2}}(\vartheta_2 r) + A_7 r^{\frac{1}{2}} Y_{l+\frac{1}{2}}(\vartheta_2 r), \tag{B 14}$$

$$U_3 = -\frac{A_4}{\rho_3\omega}\frac{r^{-l}}{l} + A_8 r^{\frac{1}{2}} H^{(1)}_{l+\frac{1}{2}}(\vartheta_3 r), \tag{B 15}$$

where A_5-A_8 are the coefficients to be determined and ϑ_i is defined in (A 6). By writing the solution of U_3 in the form of (B 15), we hypothesize that $\mathrm{Im}(\omega)$ is not equal to zero. Similar to the solution of S_3 discussed in appendix A, the other hypothesis $\mathrm{Im}(\omega) = 0$ results in a continuous spectrum consisting of the entire positive semi-axis in the complex frequency plane, which corresponds to overdamped modes and has no influence on shape oscillations of the droplet (Miller & Scriven 1968; Prosperetti 1980b).

Now express the boundary conditions (2.3)-(2.10) in terms of the scalar defining function $U(r)$. From (2.3),

$$\frac{l(l+1)}{R_1^2} U_1|_{r=R_1} = -\omega\hat{\xi}_1, \tag{B 16}$$

$$U_1|_{r=R_1} = U_2|_{r=R_1}. \tag{B 17}$$

Both the conditions in (2.4) yield

$$\left.\frac{\mathrm{d}U_1}{\mathrm{d}r}\right|_{r=R_1} = \left.\frac{\mathrm{d}U_2}{\mathrm{d}r}\right|_{r=R_1}. \tag{B 18}$$

Both the conditions in (2.5) yield

$$\mu_1\left[\frac{l(l+1)}{R_1^2}U_1|_{r=R_1} - \frac{2}{R_1}\left.\frac{\mathrm{d}U_1}{\mathrm{d}r}\right|_{r=R_1} + \left.\frac{\mathrm{d}^2U_1}{\mathrm{d}r^2}\right|_{r=R_1}\right] = \mu_2\left[\frac{l(l+1)}{R_1^2}U_2|_{r=R_1} - \frac{2}{R_1}\left.\frac{\mathrm{d}U_2}{\mathrm{d}r}\right|_{r=R_1} + \left.\frac{\mathrm{d}^2U_2}{\mathrm{d}r^2}\right|_{r=R_1}\right]. \tag{B 19}$$

From (2.6),

$$-A_2R_1^l - A_3R_1^{-(l+1)} + 2\mu_2\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{l(l+1)}{r^2}U_2\right]\Bigg|_{r=R_1} + A_1R_1^l - 2\mu_1\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{l(l+1)}{r^2}U_1\right]\Bigg|_{r=R_1} = \frac{\gamma_1(l-1)(l+2)}{R_1^2}\hat{\xi}_1. \tag{B 20}$$

From (2.7),

$$\frac{l(l+1)}{R_2^2}U_3|_{r=R_2} = -\omega\hat{\xi}_2, \tag{B 21}$$

$$U_2|_{r=R_2} = U_3|_{r=R_2}. \tag{B 22}$$

Both the conditions in (2.8) yield

$$\left.\frac{\mathrm{d}U_2}{\mathrm{d}r}\right|_{r=R_2} = \left.\frac{\mathrm{d}U_3}{\mathrm{d}r}\right|_{r=R_2}. \tag{B 23}$$

Both the conditions in (2.9) yield

$$\mu_3\left[\frac{l(l+1)}{R_2^2}U_3|_{r=R_2} - \frac{2}{R_2}\left.\frac{\mathrm{d}U_3}{\mathrm{d}r}\right|_{r=R_2} + \left.\frac{\mathrm{d}^2U_3}{\mathrm{d}r^2}\right|_{r=R_2}\right] = \mu_2\left[\frac{l(l+1)}{R_2^2}U_2|_{r=R_2} - \frac{2}{R_2}\left.\frac{\mathrm{d}U_2}{\mathrm{d}r}\right|_{r=R_2} + \left.\frac{\mathrm{d}^2U_2}{\mathrm{d}r^2}\right|_{r=R_2}\right]. \tag{B 24}$$

From (2.10),

$$A_2R_2^l + A_3R_2^{-(l+1)} - 2\mu_2\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{l(l+1)}{r^2}U_2\right]\Bigg|_{r=R_2} - A_4R_2^{-(l+1)} + 2\mu_3\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{l(l+1)}{r^2}U_3\right]\Bigg|_{r=R_2} = \frac{\gamma_2(l-1)(l+2)}{R_2^2}\hat{\xi}_2. \tag{B 25}$$

Then substituting (B 13)-(B 15) into (B 16)-(B 25) yields

$$l(l+1)\left[\frac{A_1}{\rho_1\omega}\frac{R_1^{l-1}}{l+1} + A_5R_1^{-\frac{3}{2}}J_{l+\frac{1}{2}}(\vartheta_1R_1)\right] = -\omega\hat{\xi}_1, \tag{B 26}$$

$$\frac{A_1}{\rho_1\omega}\frac{R_1^{l-1}}{l+1} + A_5R_1^{-\frac{3}{2}}J_{l+\frac{1}{2}}(\vartheta_1R_1) = \frac{A_2}{\rho_2\omega}\frac{R_1^{l-1}}{l+1} - \frac{A_3}{\rho_2\omega}\frac{R_1^{-(l+2)}}{l}$$

$$+ A_6 R_1^{-\frac{3}{2}} J_{l+\frac{1}{2}}(\vartheta_2 R_1) + A_7 R_1^{-\frac{3}{2}} Y_{l+\frac{1}{2}}(\vartheta_2 R_1), \quad \text{(B 27)}$$

$$\frac{A_1}{\rho_1\omega} R_1^l + A_5 R_1^{-\frac{1}{2}} \left[(l+1) J_{l+\frac{1}{2}}(\vartheta_1 R_1) - \vartheta_1 R_1 J_{l+\frac{3}{2}}(\vartheta_1 R_1)\right]$$
$$= \frac{A_2}{\rho_2\omega} R_1^l + \frac{A_3}{\rho_2\omega} R_1^{-(l+1)} + A_6 R_1^{-\frac{1}{2}} \left[(l+1) J_{l+\frac{1}{2}}(\vartheta_2 R_1) - \vartheta_2 R_1 J_{l+\frac{3}{2}}(\vartheta_2 R_1)\right]$$
$$+ A_7 R_1^{-\frac{1}{2}} \left[(l+1) Y_{l+\frac{1}{2}}(\vartheta_2 R_1) - \vartheta_2 R_1 Y_{l+\frac{3}{2}}(\vartheta_2 R_1)\right], \quad \text{(B 28)}$$

$$\mu_1 \left\{ \frac{2(l-1)}{\rho_1\omega} A_1 R_1^{l-1} + A_5 R_1^{-\frac{3}{2}} \left[(2l^2 - 2 - \vartheta_1^2 R_1^2) J_{l+\frac{1}{2}}(\vartheta_1 R_1) + 2\vartheta_1 R_1 J_{l+\frac{3}{2}}(\vartheta_1 R_1)\right] \right\}$$
$$= \mu_2 \left\{ \frac{2(l-1)}{\rho_2\omega} A_2 R_1^{l-1} - \frac{2(l+2)}{\rho_2\omega} A_3 R_1^{-(l+2)} + A_6 R_1^{-\frac{3}{2}} \left[(2l^2 - 2 - \vartheta_2^2 R_1^2) J_{l+\frac{1}{2}}(\vartheta_2 R_1) \right.\right.$$
$$\left.\left. + 2\vartheta_2 R_1 J_{l+\frac{3}{2}}(\vartheta_2 R_1)\right] + A_7 R_1^{-\frac{3}{2}} \left[(2l^2 - 2 - \vartheta_2^2 R_1^2) Y_{l+\frac{1}{2}}(\vartheta_2 R_1) + 2\vartheta_2 R_1 Y_{l+\frac{3}{2}}(\vartheta_2 R_1)\right] \right\}, \quad \text{(B 29)}$$

$$\left[1 - \frac{2\mu_1}{\rho_1\omega} \frac{l(l-1)}{R_1^2}\right] A_1 R_1^l - 2\mu_1 l(l+1) R_1^{-\frac{5}{2}} A_5 \left[(l-1) J_{l+\frac{1}{2}}(\vartheta_1 R_1) - \vartheta_1 R_1 J_{l+\frac{3}{2}}(\vartheta_1 R_1)\right]$$
$$- \left[1 - \frac{2\mu_2}{\rho_2\omega} \frac{l(l-1)}{R_1^2}\right] A_2 R_1^l - \left[1 - \frac{2\mu_2}{\rho_2\omega} \frac{(l+1)(l+2)}{R_1^2}\right] A_3 R_1^{-(l+1)}$$
$$+ 2\mu_2 l(l+1) R_1^{-\frac{5}{2}} A_6 \left[(l-1) J_{l+\frac{1}{2}}(\vartheta_2 R_1) - \vartheta_2 R_1 J_{l+\frac{3}{2}}(\vartheta_2 R_1)\right]$$
$$+ 2\mu_2 l(l+1) R_1^{-\frac{5}{2}} A_7 \left[(l-1) Y_{l+\frac{1}{2}}(\vartheta_2 R_1) - \vartheta_2 R_1 Y_{l+\frac{3}{2}}(\vartheta_2 R_1)\right] = \frac{\gamma_1 (l-1)(l+2)}{R_1^2} \hat{\xi}_1, \quad \text{(B 30)}$$

$$l(l+1) \left[-\frac{A_4}{\rho_3\omega} \frac{R_2^{-(l+2)}}{l} + A_8 R_2^{-\frac{3}{2}} H^{(1)}_{l+\frac{1}{2}}(\vartheta_3 R_2) \right] = -\omega \hat{\xi}_2, \quad \text{(B 31)}$$

$$-\frac{A_4}{\rho_3\omega} \frac{R_2^{-(l+2)}}{l} + A_8 R_2^{-\frac{3}{2}} H^{(1)}_{l+\frac{1}{2}}(\vartheta_3 R_2) = \frac{A_2}{\rho_2\omega} \frac{R_2^{l-1}}{l+1} - \frac{A_3}{\rho_2\omega} \frac{R_2^{-(l+2)}}{l}$$
$$+ A_6 R_2^{-\frac{3}{2}} J_{l+\frac{1}{2}}(\vartheta_2 R_2) + A_7 R_2^{-\frac{3}{2}} Y_{l+\frac{1}{2}}(\vartheta_2 R_2), \quad \text{(B 32)}$$

$$\frac{A_4}{\rho_3\omega} R_2^{-(l+1)} + A_8 R_2^{-\frac{1}{2}} \left[(l+1) H^{(1)}_{l+\frac{1}{2}}(\vartheta_3 R_2) - \vartheta_3 R_2 H^{(1)}_{l+\frac{3}{2}}(\vartheta_3 R_2)\right]$$
$$= \frac{A_2}{\rho_2\omega} R_2^l + \frac{A_3}{\rho_2\omega} R_2^{-(l+1)} + A_6 R_2^{-\frac{1}{2}} \left[(l+1) J_{l+\frac{1}{2}}(\vartheta_2 R_2) - \vartheta_2 R_2 J_{l+\frac{3}{2}}(\vartheta_2 R_2)\right]$$
$$+ A_7 R_2^{-\frac{1}{2}} \left[(l+1) Y_{l+\frac{1}{2}}(\vartheta_2 R_2) - \vartheta_2 R_2 Y_{l+\frac{3}{2}}(\vartheta_2 R_2)\right], \quad \text{(B 33)}$$

$$\mu_3 \left\{ -\frac{2(l+2)}{\rho_3\omega} A_4 R_2^{-(l+2)} + A_8 R_2^{-\frac{3}{2}} \left[(2l^2 - 2 - \vartheta_3^2 R_2^2) H^{(1)}_{l+\frac{1}{2}}(\vartheta_3 R_2) + 2\vartheta_3 R_2 H^{(1)}_{l+\frac{3}{2}}(\vartheta_3 R_2)\right] \right\}$$

$$
= \mu_2 \left\{ \frac{2(l-1)}{\rho_2 \omega} A_2 R_2^{l-1} - \frac{2(l+2)}{\rho_2 \omega} A_3 R_2^{-(l+2)} + A_6 R_2^{-\frac{3}{2}} \left[(2l^2 - 2 - \vartheta_2^2 R_2^2) J_{l+\frac{1}{2}}(\vartheta_2 R_2) \right.\right.
$$

$$
\left.\left. + 2\vartheta_2 R_2 J_{l+\frac{3}{2}}(\vartheta_2 R_2) \right] + A_7 R_2^{-\frac{3}{2}} \left[(2l^2 - 2 - \vartheta_2^2 R_2^2) Y_{l+\frac{1}{2}}(\vartheta_2 R_2) + 2\vartheta_2 R_2 Y_{l+\frac{3}{2}}(\vartheta_2 R_2) \right] \right\}, \tag{B 34}
$$

$$
- \left[1 - \frac{2\mu_3}{\rho_3 \omega} \frac{(l+1)(l+2)}{R_2^2} \right] A_4 R_2^{-(l+1)} + 2\mu_3 l(l+1) R_2^{-\frac{5}{2}} A_8 \left[(l-1) H^{(1)}_{l+\frac{1}{2}}(\vartheta_3 R_2) \right.
$$

$$
\left. - \vartheta_3 R_2 H^{(1)}_{l+\frac{3}{2}}(\vartheta_3 R_2) \right] + \left[1 - \frac{2\mu_2}{\rho_2 \omega} \frac{l(l-1)}{R_2^2} \right] A_2 R_2^{l} + \left[1 - \frac{2\mu_2}{\rho_2 \omega} \frac{(l+1)(l+2)}{R_2^2} \right] A_3 R_2^{-(l+1)}
$$

$$
- 2\mu_2 l(l+1) R_2^{-\frac{5}{2}} A_6 \left[(l-1) J_{l+\frac{1}{2}}(\vartheta_2 R_2) - \vartheta_2 R_2 J_{l+\frac{3}{2}}(\vartheta_2 R_2) \right]
$$

$$
-2\mu_2 l(l+1) R_2^{-\frac{5}{2}} A_7 \left[(l-1) Y_{l+\frac{1}{2}}(\vartheta_2 R_2) - \vartheta_2 R_2 Y_{l+\frac{3}{2}}(\vartheta_2 R_2) \right] = \frac{\gamma_2 (l-1)(l+2)}{R_2^2} \hat{\xi}_2. \tag{B 35}
$$

Equations (B 26)-(B 35) constitute a linear homogeneous system for the ten unknowns (A_1-A_8, $\hat{\xi}_1$, $\hat{\xi}_2$). In order that the system has nontrivial solutions, the determinant of coefficients must be zero, which gives the following characteristic equation

$$
\begin{vmatrix}
\omega & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \omega \\
0 & 1 & 0 & R_1^{l-1} & R_1^{-(l+2)} & R_1^{-\frac{3}{2}} \kappa_1 & R_1^{-\frac{3}{2}} \upsilon_1 & 0 & 0 & 0 \\
0 & 0 & 0 & R_2^{l-1} & R_2^{-(l+2)} & R_2^{-\frac{3}{2}} \kappa_2 & R_2^{-\frac{3}{2}} \upsilon_2 & 1 & 0 & 0 \\
0 & l+1 & X_6 & (l+1) R_1^{l-1} & -l R_1^{-(l+2)} & R_1^{-\frac{3}{2}} H_1 & R_1^{-\frac{3}{2}} \Pi_1 & 0 & 0 & 0 \\
0 & 0 & 0 & -(l+1) R_2^{l-1} & l R_2^{-(l+2)} & -R_2^{-\frac{3}{2}} H_2 & -R_2^{-\frac{3}{2}} \Pi_2 & l & Z_6 & 0 \\
0 & X_5 & X_3 & \overline{Y}_5 & \overline{Y}_7 & \mu_2 R_1^{-\frac{3}{2}} H_3 & \mu_2 R_1^{-\frac{3}{2}} \Pi_3 & 0 & 0 & 0 \\
0 & 0 & 0 & \overline{Y}_6 & \overline{Y}_8 & \mu_2 R_2^{-\frac{3}{2}} H_4 & \mu_2 R_2^{-\frac{3}{2}} \Pi_4 & Z_5 & Z_2 & 0 \\
X_1 & -X_2 & X_4 & -\overline{Y}_1 & \overline{Y}_3 & 2\mu_2 R_1^{-\frac{7}{2}} H_5 & 2\mu_2 R_1^{-\frac{7}{2}} \Pi_5 & 0 & 0 & 0 \\
0 & 0 & 0 & \overline{Y}_2 & -\overline{Y}_4 & -2\mu_2 R_2^{-\frac{7}{2}} H_6 & -2\mu_2 R_2^{-\frac{7}{2}} \Pi_6 & -Z_1 & Z_3 & Z_4
\end{vmatrix} = 0, \tag{B 36}
$$

where

$$
H_1 = (l+1)\kappa_1 - \vartheta_2 R_1 \kappa_3, \quad H_2 = (l+1)\kappa_2 - \vartheta_2 R_2 \kappa_4, \tag{B 37}
$$

$$
H_3 = (2l^2 - 2 - \vartheta_2^2 R_1^2)\kappa_1 + 2\vartheta_2 R_1 \kappa_3, \quad H_4 = (2l^2 - 2 - \vartheta_2^2 R_2^2)\kappa_2 + 2\vartheta_2 R_2 \kappa_4, \tag{B 38}
$$

$$
\Pi_1 = (l+1)\upsilon_1 - \vartheta_2 R_1 \upsilon_3, \quad \Pi_2 = (l+1)\upsilon_2 - \vartheta_2 R_2 \upsilon_4, \tag{B 39}
$$

$$
\Pi_3 = (2l^2 - 2 - \vartheta_2^2 R_1^2)\upsilon_1 + 2\vartheta_2 R_1 \upsilon_3, \quad \Pi_4 = (2l^2 - 2 - \vartheta_2^2 R_2^2)\upsilon_2 + 2\vartheta_2 R_2 \upsilon_4, \tag{B 40}
$$

$$
X_1 = \frac{\gamma_1 (l-1)(l+2)}{R_1^3}, \quad X_2 = \frac{\rho_1 \omega}{l} - \frac{2\mu_1 (l-1)}{R_1^2}, \tag{B 41}
$$

$$
X_3 = \mu_1(-\vartheta_1^2 R_1^2 + 2\vartheta_1 R_1 \Upsilon_1), \quad X_4 = \frac{\rho_1 \omega}{l} - \frac{2\mu_1}{R_1^2}(\vartheta_1 R_1 \Upsilon_1), \tag{B 42}
$$

$$
X_5 = 2\mu_1 (l^2 - 1), \quad X_6 = -\vartheta_1 R_1 \Upsilon_1, \tag{B 43}
$$

$$\overline{Y}_1 = \left[\frac{\rho_2\omega}{l} - \frac{2\mu_2(l-1)}{R_1^2}\right] R_1^{l-1}, \;\; \overline{Y}_2 = \left[\frac{\rho_2\omega}{l} - \frac{2\mu_2(l-1)}{R_2^2}\right] R_2^{l-1}, \tag{B 44}$$

$$\overline{Y}_3 = \left[\frac{\rho_2\omega}{l+1} - \frac{2\mu_2(l+2)}{R_1^2}\right] R_1^{-(l+2)}, \;\; \overline{Y}_4 = \left[\frac{\rho_2\omega}{l+1} - \frac{2\mu_2(l+2)}{R_2^2}\right] R_2^{-(l+2)}, \tag{B 45}$$

$$\overline{Y}_5 = 2\mu_2(l^2-1)R_1^{l-1}, \;\; \overline{Y}_6 = 2\mu_2(l^2-1)R_2^{l-1}, \tag{B 46}$$

$$\overline{Y}_7 = 2\mu_2 l(l+2)R_1^{-(l+2)}, \;\; \overline{Y}_8 = 2\mu_2 l(l+2)R_2^{-(l+2)}, \tag{B 47}$$

$$Z_1 = \frac{\rho_3\omega}{l+1} - \frac{2\mu_3(l+2)}{R_2^2}, \;\; Z_2 = \mu_3\left[-2(2l+1) - \vartheta_3^2R_2^2 + 2\vartheta_3R_2\Upsilon_2\right], \tag{B 48}$$

$$Z_3 = \frac{\rho_3\omega}{l+1} - \frac{2\mu_3}{R_2^2}(2l+1-\vartheta_3R_2\Upsilon_2), \;\; Z_4 = \frac{\gamma_2(l-1)(l+2)}{R_2^3}, \tag{B 49}$$

$$Z_5 = 2\mu_3 l(l+2), \;\; Z_6 = -(2l+1-\vartheta_3R_2\Upsilon_2), \tag{B 50}$$

and the other terms are given in (A 6) and (A 18)-(A 22).

Following the method of Miller & Scriven (1968) and introducing the vorticity instead of the scalar defining function of the velocity, Lyell & Wang (1986) derived the characteristic equation in the form of 10×10 matrix as well. Our derivation is different, but the resulting characteristic equation (B 36) is equivalent to the one obtained by Lyell & Wang (1986).

Several limiting cases can be obtained directly from (B 36).

Case 1: a viscous liquid droplet in vacuum

Suppose that only the core fluid exists, and (B 36) reduces to the characteristic equation for a single viscous liquid droplet in vacuum:

$$\begin{vmatrix} \omega & 1 & 0 \\ 0 & X_5 & X_3 \\ X_1 & -X_2 & X_4 \end{vmatrix} = 0. \tag{B 51}$$

After some straightforward manipulations, (B 51) turns into

$$\frac{\omega_{01}^2}{\omega^2} = \frac{2(l^2-1)}{\vartheta_1^2R_1^2 - 2\vartheta_1R_1\Upsilon_1} - 1 + \frac{2l(l-1)}{\vartheta_1^2R_1^2}\left[1 - \frac{2(l+1)\Upsilon_1}{\vartheta_1R_1 - 2\Upsilon_1}\right], \tag{B 52}$$

where ω_{01} is the frequency of oscillation in the inviscid case,

$$\omega_{01}^2 = \frac{\gamma_1 l(l-1)(l+2)}{\rho_1R_1^3}. \tag{B 53}$$

The characteristic equation (B 52) is identical in form to that obtained by Chandrasekhar (1959) and Reid (1960).

Case 2: a gas bubble in a viscous host liquid

In this case, we assume that the core and shell fluids are a gas of negligible hydrodynamic effects. Hence (B 36) reduces to

$$\begin{vmatrix} 1 & 0 & \omega \\ Z_5 & Z_2 & 0 \\ -Z_1 & Z_3 & Z_4 \end{vmatrix} = 0. \tag{B 54}$$

After some manipulations, (B 54) becomes

$$\frac{\omega_{03}^2}{\omega^2} = \frac{2(l+2)}{\vartheta_3^2 R_2^2} \frac{(2l+1)\vartheta_3^2 R_2^2 - 2(l-1)(l+1)(2l+1-\vartheta_3 R_2 \Upsilon_2)}{2(2l+1) - 2\vartheta_3 R_2 \Upsilon_2 + \vartheta_3^2 R_2^2} - 1, \tag{B 55}$$

where ω_{03} is the frequency of oscillation in the inviscid case,

$$\omega_{03}^2 = \frac{\gamma_2 (l-1)(l+1)(l+2)}{\rho_3 R_2^3}. \tag{B 56}$$

The expression (B 55) accords with the characteristic equation presented by Miller & Scriven (1968).

Case 3: a viscous droplet suspended in a viscous host liquid

Suppose that the shell is the same fluid with the core ($\rho_1 = \rho_2$, $\mu_1 = \mu_2$) and remove the inner interface ($\gamma_1 = 0$). Thus (B 36) reduces to the following characteristic equation for a viscous droplet in a viscous host liquid,

$$\begin{vmatrix} \omega & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & l+1 & X_6 & -l & -Z_6 \\ 0 & X_5 & X_3 & Z_5 & Z_2 \\ X_1 & -X_2 & X_4 & Z_1 & -Z_3 \end{vmatrix} = 0. \tag{B 57}$$

Note that all R_1's in (B 57) should be replaced by R_2. The presentation in the form of 5×5 matrix in (B 57) is similar to and also effectively equivalent to the characteristic equation given by Miller & Scriven (1968).

In addition, (B 57) can be reduced to the following form of 3×3 matrix,

$$\begin{vmatrix} X_6 & Z_6 & 2l+1 \\ X_3 & -Z_2 & X_5 - Z_5 \\ -X_4 & -Z_3 & X_2 + Z_1 + \frac{X_1}{\omega} \end{vmatrix} = 0, \tag{B 58}$$

where, again, all R_1's should be replaced by R_2. This equation is identical to that given by Basaran *et al.* (1989).

Appendix C. Checking the eigenvalues with the aid of the characteristic equation

Using the scales chosen in section 2, the characteristic equation (B 36) is nondimensionalized as follows

$\boldsymbol{D}_3 =$

$$
\begin{vmatrix}
\omega & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \omega \\
0 & 1 & 0 & a^{l-1} & a^{-(l+2)} & a^{-\frac{3}{2}}\kappa_1 & a^{-\frac{3}{2}}\upsilon_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & \kappa_2 & \upsilon_2 & 1 & 0 & 0 \\
0 & l+1 & X_6 & (l+1)a^{l-1} & -la^{-(l+2)} & a^{-\frac{3}{2}}H_1 & a^{-\frac{3}{2}}\varPi_1 & 0 & 0 & 0 \\
0 & 0 & 0 & -(l+1) & l & -H_2 & -\varPi_2 & l & Z_6 & 0 \\
0 & X_5 & X_3 & \overline{Y}_5 & \overline{Y}_7 & a^{-\frac{3}{2}}H_3 & a^{-\frac{3}{2}}\varPi_3 & 0 & 0 & 0 \\
0 & 0 & 0 & \overline{Y}_6 & \overline{Y}_8 & H_4 & \varPi_4 & Z_5 & Z_2 & 0 \\
X_1 & -X_2 & X_4 & -\overline{Y}_1 & \overline{Y}_3 & 2Oh_2a^{-\frac{7}{2}}H_5 & 2Oh_2a^{-\frac{7}{2}}\varPi_5 & 0 & 0 & 0 \\
0 & 0 & 0 & \overline{Y}_2 & -\overline{Y}_4 & -2Oh_2H_6 & -2Oh_2\varPi_6 & -Z_1 & Z_3 & Z_4
\end{vmatrix} = 0, \tag{C 1}
$$

where

$$z_1 = \sqrt{\frac{\rho_{r1}\omega}{\mu_{r1}Oh_2}}, \quad z_2 = \sqrt{\frac{\omega}{Oh_2}}, \quad z_3 = \sqrt{\frac{\rho_{r3}\omega}{\mu_{r3}Oh_2}}, \tag{C 2}$$

$$\varUpsilon_1 = \frac{J_{l+\frac{3}{2}}(z_1a)}{J_{l+\frac{1}{2}}(z_1a)}, \quad \varUpsilon_2 = \frac{H^{(1)}_{l+\frac{3}{2}}(z_3)}{H^{(1)}_{l+\frac{1}{2}}(z_3)}, \tag{C 3}$$

$$\kappa_1 = J_{l+\frac{1}{2}}(z_2a), \ \kappa_2 = J_{l+\frac{1}{2}}(z_2), \ \kappa_3 = J_{l+\frac{3}{2}}(z_2a), \ \kappa_4 = J_{l+\frac{3}{2}}(z_2), \tag{C 4}$$

$$\upsilon_1 = Y_{l+\frac{1}{2}}(z_2a), \ \upsilon_2 = Y_{l+\frac{1}{2}}(z_2), \ \upsilon_3 = Y_{l+\frac{3}{2}}(z_2a), \ \upsilon_4 = Y_{l+\frac{3}{2}}(z_2), \tag{C 5}$$

$$H_1 = (l+1)\kappa_1 - z_2a\kappa_3, \ H_2 = (l+1)\kappa_2 - z_2\kappa_4, \tag{C 6}$$

$$H_3 = (2l^2 - 2 - z_2^2a^2)\kappa_1 + 2z_2a\kappa_3, \ H_4 = (2l^2 - 2 - z_2^2)\kappa_2 + 2z_2\kappa_4, \tag{C 7}$$

$$H_5 = (l-1)\kappa_1 - z_2a\kappa_3, \ H_6 = (l-1)\kappa_2 - z_2\kappa_4, \tag{C 8}$$

$$\varPi_1 = (l+1)\upsilon_1 - z_2a\upsilon_3, \ \varPi_2 = (l+1)\upsilon_2 - z_2\upsilon_4, \tag{C 9}$$

$$\varPi_3 = (2l^2 - 2 - z_2^2a^2)\upsilon_1 + 2z_2a\upsilon_3, \ \varPi_4 = (2l^2 - 2 - z_2^2)\upsilon_2 + 2z_2\upsilon_4, \tag{C 10}$$

$$\varPi_5 = (l-1)\upsilon_1 - z_2a\upsilon_3, \ \varPi_6 = (l-1)\upsilon_2 - z_2\upsilon_4, \tag{C 11}$$

$$X_1 = \frac{\gamma_r(l-1)(l+2)}{a^3}, \ X_2 = \frac{\rho_{r1}\omega}{l} - \frac{2\mu_{r1}Oh_2(l-1)}{a^2}, \tag{C 12}$$

$$X_3 = \mu_{r1}(-z_1^2a^2 + 2z_1a\varUpsilon_1), \ X_4 = \frac{\rho_{r1}\omega}{l} - \frac{2\mu_{r1}Oh_2}{a^2}(z_1a\varUpsilon_1), \tag{C 13}$$

$$X_5 = 2\mu_{r1}(l^2-1), \ X_6 = -z_1a\varUpsilon_1, \tag{C 14}$$

$$\overline{Y}_1 = \left[\frac{\omega}{l} - \frac{2Oh_2(l-1)}{a^2}\right]a^{l-1}, \ \overline{Y}_2 = \frac{\omega}{l} - 2Oh_2(l-1), \tag{C 15}$$

$$\overline{Y}_3 = \left[\frac{\omega}{l+1} - \frac{2Oh_2(l+2)}{a^2}\right]a^{-(l+2)}, \ \overline{Y}_4 = \frac{\omega}{l+1} - 2Oh_2(l+2), \tag{C 16}$$

$$\overline{Y}_5 = 2(l^2-1)a^{l-1}, \ \overline{Y}_6 = 2(l^2-1), \tag{C 17}$$

$$\overline{Y}_7 = 2l(l+2)a^{-(l+2)}, \ \overline{Y}_8 = 2l(l+2), \tag{C 18}$$

$$Z_1 = \frac{\rho_{r3}\omega}{l+1} - 2\mu_{r3}Oh_2(l+2), \ Z_2 = \mu_{r3}\left[-2(2l+1) - z_3^2 + 2z_3\varUpsilon_2\right], \tag{C 19}$$

$$Z_3 = \frac{\rho_{r3}\omega}{l+1} - 2\mu_{r3}Oh_2(2l+1-z_3\varUpsilon_2), \ Z_4 = (l-1)(l+2), \tag{C 20}$$

$$Z_5 = 2\mu_{r3}l(l+2), \ Z_6 = -(2l+1-z_3\varUpsilon_2). \tag{C 21}$$

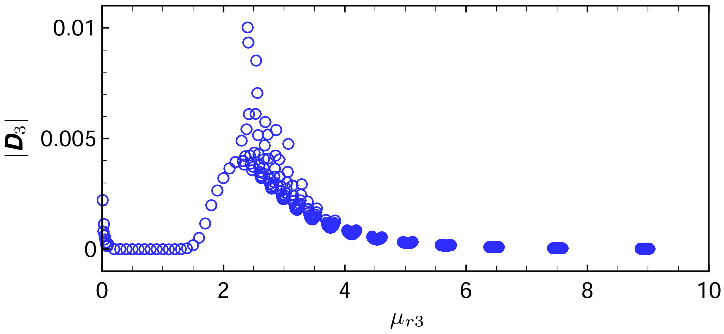

Figure 1: The absolute value of $\boldsymbol{D}_3$, i.e. $|\boldsymbol{D}_3|$, obtained by substituting the eigenvalues in figure 14(c) into the determinant $\boldsymbol{D}_3$ in (C 1), versus the host to shell viscosity ratio μ_{r3}.

Without loss of clarity, the same symbols are used to denote the corresponding nondimensional terms in (C 1). Replacing the element in row 4 and column 10 of the matrix in (C 1) with $-\omega$ and eliminating columns 8 and 9 and rows 2 and 6, (C 1) reduces to the characteristic equation for the case of a viscous compound droplet suspended in vacuum or in a gas of negligible hydrodynamic effects; further, replacing the element in row 3 and column 1 with $-\omega$ and deleting columns 2 and 3 and rows 1 and 5, (C 1) reduces to the characteristic equation for the case of a viscous liquid shell with the core and the host being vacuum or a gas of negligible hydrodynamic effects.

The transcendental equation (C 1) is cumbersome. Instead of solving it to get the eigenvalues, we use it as a tool to check the exactness of the eigenvalues obtained with the aid of the spectral method. The strategy is as follows: We substitute the eigenvalues into the determinant $\boldsymbol{D}_3$ in (C 1) and calculate the corresponding absolute values of $\boldsymbol{D}_3$, denoted by $|\boldsymbol{D}_3|$. If $|\boldsymbol{D}_3| = 0$, the eigenvalues are accurate. However, due to the numerical errors in the use of the spectral method, the values of $|\boldsymbol{D}_3|$ are not exactly zero but remain quite small, as shown in figure 1. In such a case, the eigenvalues obtained by the spectral method are considered to be acceptable in accuracy.

Appendix D. Derivation of the characteristic equation for the thin shell limiting case

In the thin shell limit, the radius ratio $a = 1 - \epsilon$ with $\epsilon \ll 1$. To derive the characteristic equation for this limiting case, we expand the nondimensional characteristic equation (C 1) in a Taylor series in the small parameter ϵ (to save space, only the first two orders of the expansion are explicitly expressed):

$$
\begin{vmatrix}
\omega & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \omega \\
0 & 1 & 0 & D_{34} & D_{35} & D_{36}\kappa_1 & D_{36}\upsilon_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & \kappa_2 & \upsilon_2 & 1 & 0 & 0 \\
0 & l+1 & X_6 & (l+1)D_{34} & -lD_{35} & D_{36}H_1 & D_{36}\varPi_1 & 0 & 0 & 0 \\
0 & 0 & 0 & -(l+1) & l & -H_2 & -\varPi_2 & l & Z_6 & 0 \\
0 & X_5 & X_3 & \overline{Y}_5 & \overline{Y}_7 & D_{36}H_3 & D_{36}\varPi_3 & 0 & 0 & 0 \\
0 & 0 & 0 & \overline{Y}_6 & \overline{Y}_8 & H_4 & \varPi_4 & Z_5 & Z_2 & 0 \\
X_1 & -X_2 & X_4 & -\overline{Y}_1 & \overline{Y}_3 & 2Oh_2D_{86}H_5 & 2Oh_2D_{86}\varPi_5 & 0 & 0 & 0 \\
0 & 0 & 0 & \overline{Y}_2 & -\overline{Y}_4 & -2Oh_2H_6 & -2Oh_2\varPi_6 & -Z_1 & Z_3 & Z_4
\end{vmatrix} = 0, \tag{D 1}
$$

where

$$D_{34} = 1 - \epsilon(l-1) + O(\epsilon^2), \quad D_{35} = 1 + \epsilon(l+2) + O(\epsilon^2),$$

$$D_{36} = 1 + \frac{3}{2}\epsilon + O(\epsilon^2), \quad D_{86} = 1 + \frac{7}{2}\epsilon + O(\epsilon^2)$$

$$\Upsilon_1 = \frac{J_{l+\frac{3}{2}}(z_1)}{J_{l+\frac{1}{2}}(z_1)} - \epsilon\left[\frac{z_1 J_{l+\frac{1}{2}}(z_1) - (l+\frac{3}{2})J_{l+\frac{3}{2}}(z_1)}{J_{l+\frac{1}{2}}(z_1)}\right.$$

$$\left. - \frac{J_{l+\frac{3}{2}}(z_1)\left(z_1 J_{l-\frac{1}{2}}(z_1) - (l+\frac{1}{2})J_{l+\frac{1}{2}}(z_1)\right)}{J^2_{l+\frac{1}{2}}(z_1)}\right] + O(\epsilon^2),$$

$$\kappa_1 = \kappa_2 - \epsilon\left[z_2 J_{l-\frac{1}{2}}(z_2) - \left(l+\frac{1}{2}\right) J_{l+\frac{1}{2}}(z_2)\right] + O(\epsilon^2),$$

$$\kappa_3 = \kappa_4 - \epsilon\left[z_2 J_{l+\frac{1}{2}}(z_2) - \left(l+\frac{3}{2}\right) J_{l+\frac{3}{2}}(z_2)\right] + O(\epsilon^2),$$

$$\upsilon_1 = \upsilon_2 - \epsilon\left[z_2 Y_{l-\frac{1}{2}}(z_2) - \left(l+\frac{1}{2}\right) Y_{l+\frac{1}{2}}(z_2)\right] + O(\epsilon^2),$$

$$\upsilon_3 = \upsilon_4 - \epsilon\left[z_2 Y_{l+\frac{1}{2}}(z_2) - \left(l+\frac{3}{2}\right) Y_{l+\frac{3}{2}}(z_2)\right] + O(\epsilon^2),$$

$$H_1 = H_2 - \epsilon\left[\left(l^2 + \frac{l}{2} - z_2^2\right) J_{l+\frac{1}{2}}(z_2) - \frac{z_2}{2} J_{l-\frac{1}{2}}(z_2)\right] + O(\epsilon^2),$$

$$H_3 = H_4 + \epsilon\left[(2l+1)(l^2 + 2l - \frac{1}{2}z_2^2)J_{l+\frac{1}{2}}(z_2) - z_2(2l^2 + 2l - z_2^2 - 1)J_{l-\frac{1}{2}}(z_2)\right] + O(\epsilon^2),$$

$$H_5 = H_6 + \epsilon\left[\frac{3}{2}z_2 J_{l-\frac{1}{2}}(z_2) - (l^2 + \frac{5}{2}l - z_2^2 + 1)J_{l+\frac{1}{2}}(z_2)\right] + O(\epsilon^2),$$

$$\varPi_1 = \varPi_2 - \epsilon\left[\left(l^2 + \frac{l}{2} - z_2^2\right) Y_{l+\frac{1}{2}}(z_2) - \frac{z_2}{2} Y_{l-\frac{1}{2}}(z_2)\right] + O(\epsilon^2),$$

$$\varPi_3 = \varPi_4 + \epsilon\left[(2l+1)(l^2 + 2l - \frac{1}{2}z_2^2)Y_{l+\frac{1}{2}}(z_2) - z_2(2l^2 + 2l - z_2^2 - 1)Y_{l-\frac{1}{2}}(z_2)\right] + O(\epsilon^2),$$

$$\varPi_5 = \varPi_6 + \epsilon\left[\frac{3}{2}z_2 Y_{l-\frac{1}{2}}(z_2) - (l^2 + \frac{5}{2}l - z_2^2 + 1)Y_{l+\frac{1}{2}}(z_2)\right] + O(\epsilon^2),$$

$$X_1 = \gamma_r(l-1)(l+2) + 3\epsilon\gamma_r(l-1)(l+2) + O(\epsilon^2),$$

$$X_2 = \frac{\rho_{r1}\omega}{l} - 2\mu_{r1}Oh_2(l-1) - 4\epsilon\mu_{r1}Oh_2(l-1) + O(\epsilon^2),$$

$$X_3 = \mu_{r1}\left[-z_1^2 + 2z_1\frac{J_{l+\frac{3}{2}}(z_1)}{J_{l+\frac{1}{2}}(z_1)}\right]$$

$$-2\epsilon\mu_{r1}\left[\frac{z_1^2 J_{l-\frac{1}{2}}^2(z_1) - (2l+1)z_1 J_{l-\frac{1}{2}}(z_1)J_{l+\frac{1}{2}}(z_1)}{J_{l+\frac{1}{2}}^2(z_1)}\right] + O(\epsilon^2),$$

$$X_4 = \frac{\rho_{r1}\omega}{l} - 2\mu_{r1}Oh_2 z_1\frac{J_{l+\frac{3}{2}}(z_1)}{J_{l+\frac{1}{2}}(z_1)} - 2\epsilon\mu_{r1}Oh_2 z_1$$

$$\frac{(4l+2-z_1^2)J_{l+\frac{1}{2}}^2(z_1) - z_1^2 J_{l-\frac{1}{2}}^2(z_1) + (2l-1)z_1 J_{l-\frac{1}{2}}(z_1)J_{l+\frac{1}{2}}(z_1)}{z_1 J_{l+\frac{1}{2}}^2(z_1)} + O(\epsilon^2),$$

$$X_6 = -z_1\frac{J_{l+\frac{3}{2}}(z_1)}{J_{l+\frac{1}{2}}(z_1)} - \epsilon\frac{z_1^2 J_{l-\frac{1}{2}}^2(z_1) + z_1^2 J_{l+\frac{1}{2}}^2(z_1) - (2l+1)z_1 J_{l-\frac{1}{2}}(z_1)J_{l+\frac{1}{2}}(z_1)}{J_{l+\frac{1}{2}}^2(z_1)} + O(\epsilon^2),$$

$$\overline{Y}_1 = \frac{\omega}{l} - 2Oh_2(l-1) + 2\epsilon Oh_2(l-1)(l-3) + O(\epsilon^2),$$

$$\overline{Y}_3 = \frac{\omega}{l+1} - 2Oh_2(l+2) - 2\epsilon Oh_2(l+2)(l+4) + O(\epsilon^2),$$

$$\overline{Y}_5 = 2(l^2-1) - 2\epsilon(l^2-1)(l-1) + O(\epsilon^2),$$

$$\overline{Y}_7 = 2l(l+2) + 2\epsilon l(l+2)^2 + O(\epsilon^2),$$

and the other terms remain unchanged.

The leading order $O(1)$ of (D 1) yields

$$\begin{vmatrix} 1 & 1 & \kappa_2 & \upsilon_2 \\ -(l+1) & l & -H_2 & \mathit{\Pi}_2 \\ 2(l^2-1) & 2l(l+2) & H_4 & \mathit{\Pi}_4 \\ \overline{Y}_2 & -\overline{Y}_4 & -2Oh_2H_6 & -2Oh_2\mathit{\Pi}_6 \end{vmatrix}$$

$$\times\begin{vmatrix} \omega & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \omega \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & l+1 & -\mathit{\Xi}_1 & l & Z_6 & 0 \\ 0 & X_5 & \mu_{r1}(-z_1^2+2\mathit{\Xi}_1) & -Z_5 & -Z_2 & 0 \\ \gamma_r(l-1)(l+2) & -\frac{\rho_{r1}\omega}{l} + 2\mu_{r1}Oh_2(l-1) & \frac{\rho_{r1}\omega}{l} - 2\mu_{r1}Oh_2\mathit{\Xi}_1 & -Z_1 & Z_3 & Z_4 \end{vmatrix} = 0, \tag{D 2}$$

where

$$\mathit{\Xi}_1 = z_1\frac{J_{l+\frac{3}{2}}(z_1)}{J_{l+\frac{1}{2}}(z_1)}. \tag{D 3}$$

At least one of the two determinants in (D 2) equals zero. The eigenvalues given by the first determinant being equal to zero are all purely real, which does not meet the hypothesis $\mathrm{Im}(\omega) \neq 0$. So the only possibility is that the second determinant equals zero. This determinant can be written as the product of two smaller determinants, i.e.

$$\begin{vmatrix} \omega & 0 \\ 0 & \omega \end{vmatrix} \times \begin{vmatrix} -\Xi_1 & 2l+1 & Z_6 \\ \mu_{r1}(-z_1^2+2\Xi_1) & X_5 - Z_5 & -Z_2 \\ -\frac{\rho_{r1}\omega}{l} + 2\mu_{r1} Oh_2 \Xi_1 & X_2 + Z_1 + (\gamma_r+1)\frac{(l-1)(l+2)}{\omega} & -Z_3 \end{vmatrix} = 0. \quad (D\,4)$$

Apparently, the solution to the first determinant in (D 4) being equal to zero is just zero, against the hypothesis $\mathrm{Im}(\omega) \neq 0$. On the other hand, considering that this determinant corresponds to the positions of the interface amplitudes $\hat{\xi}_1$ and $\hat{\xi}_2$, its structure may suggest that $\hat{\xi}_1 = \hat{\xi}_2$. That is, the interfaces oscillate in phase and with equal amplitude.

The second determinant in (D 4) being equal to zero yields the characteristic equation for the thin shell limiting case. It is not surprising to find that the characteristic equation in this limit is identical to (B 58) for the case of a viscous droplet suspended in a viscous host fluid, except that the interfacial tension here is the sum of the inner and outer interfacial tensions $\gamma_r + 1$. It turns out that in the thin shell limiting case the hydrodynamic effects of the shell can be neglected to the leading order.