Compressible potential flows around round bodies: Janzen-Rayleigh expansion inferences - Supplementary material

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(Received xx; revised xx; accepted xx)

Appendix A. Explicit JRE coefficients for low orders

Expressions for the coefficients of the JRE up to order $\mathcal{M}_\infty^6$, valid for any γ, are provided in table 1 for a disk, and in tables 2 and 3 for a sphere. Each table is broken to parts, each corresponding to a different m index of $\mathcal{M}_\infty^{2m}$. Columns correspond to the n appearing in $\cos(n\theta)$ or $P_n(\cos\theta)$ basis functions, and the rows correspond to k of the r^k terms. All coefficients in these tables pertain to non-logarithmic terms, i.e. to $\propto \ln^\ell(r)$ terms with $\ell = 0$, except for the very last term in table 3, which has $\ell = 1$.

Appendix B. hodograph approximation of the solution of a radial flow

In §6, we discuss the hodograph approximation for the flow in front of a sphere, and use the JRE to obtain a more accurate expansion. To relate the radial velocity and the radius, we use Eq. (7) from Keshet & Naor (2016),

$$2\ln(r) = \int_0^{-u_r(r)} \frac{1-\mathcal{M}_0(u')^2/W^2}{1-\mathcal{M}_0(u')^2/S^2}\frac{du'}{q(u')-u'}, \tag{B 1}$$

with $W^2 = 2/(\gamma+1)$, $S^2 = 2/(\gamma-1)$ (notice that $S^2 = w^{-1}$) and

$$\mathcal{M}_0(u) = u/\overline{c} = u/\left(S^{-2}+\mathcal{M}_\infty^{-2}\right)^{1/2},$$

which is the Mach number with respect to the stagnation point. To complete the integral, we calculate the coefficients q_i for $i \in \{0,1,2,3\}$ using the JRE.

We provide explicit expressions for general γ in table 4 which are valid for $0 \leqslant \mathcal{M}_\infty \leqslant \mathcal{M}_c$ (with $q_1 \equiv 1/2$ identically (Keshet & Naor 2016)). These values give a good approximation for the radial velocity near the stagnation point, but deviate far from the sphere. To ensure the correct BC far from the body, we add a fifth term to $q(u)$ such that the denominator in the integral in (B 1) vanishes at $r \to \infty$ i.e. $q_4 = 1-q_0-q_1-q_2-q_3$.

REFERENCES

Keshet, U. & Naor, Y. 2016 Compressible flow in front of an axisymmetric blunt object: analytic approximation and astrophysical implications. *The Astrophysical Journal* **830** (2), 147.

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$a_{m,k,n}$	$m=0$	$m=1$		$m=2$			$m=3$			
	$n=1$	$n=1$	$n=3$	$n=1$	$n=3$	$n=5$	$n=1$	$n=3$	$n=5$	$n=7$
$k=+1$	1									
$k=-1$	1	$\frac{13}{12}$	$-\frac{1}{4}$	$\frac{17\gamma}{60}+\frac{343}{240}$	$-\frac{19}{48}$	$\frac{1}{16}$	$\frac{59\gamma^2}{420}+\frac{41\gamma}{42}+\frac{2273}{1120}$	$-\frac{17\gamma}{240}-\frac{137}{240}$	$\frac{25}{192}$	$-\frac{1}{64}$
$k=-3$		$-\frac{1}{2}$	$\frac{1}{12}$	$-\frac{\gamma}{8}-\frac{29}{24}$	$\frac{3}{20}-\frac{61\gamma}{240}$	$\frac{\gamma}{16}$	$-\frac{163\gamma}{240}-\frac{1759}{720}$	$-\frac{29\gamma^2}{210}-\frac{2117\gamma}{2100}+\frac{583}{14400}$	$\frac{97\gamma}{240}+\frac{157}{960}$	$-\frac{\gamma}{16}-\frac{1}{32}$
$k=-5$		$\frac{1}{12}$		$\frac{\gamma}{12}+\frac{35}{48}$	$\frac{3\gamma}{16}$	$-\frac{3\gamma}{80}-\frac{1}{80}$	$\frac{\gamma^2}{48}+\frac{47\gamma}{64}+\frac{3719}{1440}$	$\frac{\gamma^2}{16}+\frac{387\gamma}{320}+\frac{209}{960}$	$\frac{169\gamma^2}{1680}-\frac{11131\gamma}{33600}-\frac{60911}{302400}$	$-\frac{\gamma^2}{48}+\frac{7\gamma}{192}+\frac{5}{192}$
$k=-7$				$-\frac{\gamma}{16}-\frac{1}{4}$	$-\frac{\gamma}{40}-\frac{1}{60}$		$-\frac{\gamma^2}{16}-\frac{749\gamma}{960}-\frac{5443}{2880}$	$-\frac{\gamma^2}{40}-\frac{1493\gamma}{2400}-\frac{349}{1800}$	$-\frac{\gamma^2}{12}+\frac{\gamma}{16}+\frac{7}{96}$	$\frac{5\gamma^2}{336}+\frac{\gamma}{1344}-\frac{1}{336}$
$k=-9$				$\frac{\gamma}{80}+\frac{1}{30}$	$\frac{1}{144}$		$\frac{3\gamma^2}{80}+\frac{149\gamma}{320}+\frac{2449}{2880}$	$\frac{\gamma^2}{24}+\frac{281\gamma}{1440}+\frac{143}{1080}$	$\frac{\gamma^2}{112}+\frac{13\gamma}{2240}-\frac{13}{840}$	
$k=-11$							$-\frac{\gamma^2}{60}-\frac{17\gamma}{120}-\frac{61}{288}$	$-\frac{\gamma^2}{112}-\frac{83\gamma}{2240}-\frac{5}{96}$	$\frac{1}{1440}-\frac{\gamma}{240}$	
$k=-13$							$\frac{\gamma^2}{336}+\frac{113\gamma}{6720}+\frac{227}{10080}$	$\frac{\gamma}{240}+\frac{11}{1440}$	$\frac{1}{1728}$	

TABLE 1. Coefficients table for the JRE up to third-order ($\mathcal{M}_\infty^6$) in 2D (disk)

$b_{m,k,n,0}$	$m=0$	$m=1$		$m=2$			$m=3$
	$n=1$	$n=1$	$n=3$	$n=1$	$n=3$	$n=5$	$n=1$
$k=+1$	1						
$k=-2$	$\frac{1}{2}$	$\frac{1}{3}$	$-\frac{3}{10}$	$\frac{599\gamma}{16800}+\frac{78733}{271040}$	$-\frac{53}{150}$	$\frac{5}{42}$	$\frac{131\gamma^2}{23520}+\frac{7507037\gamma}{100793000}+\frac{1160213239}{4192988800}$
$k=-4$			$\frac{27}{55}$		$\frac{83133\gamma}{3665200}+\frac{13691721}{19059040}$	$-\frac{6}{11}$	
$k=-5$		$-\frac{1}{5}$	$-\frac{3}{10}$	$-\frac{3\gamma}{70}-\frac{149}{525}$	$-\frac{\gamma}{10}-\frac{23}{75}$	$\frac{\gamma}{7}+\frac{11}{21}$	
$k=-5$							$-\frac{2519\gamma}{21000}-\frac{6013321}{15246000}$
$k=-6$						$\frac{4805247}{12172160}-\frac{50517\gamma}{276640}$	
$k=-7$				$-\frac{243}{1925}$	$-\frac{156}{275}$	$-\frac{60}{77}$	$-\frac{4346757\gamma}{128282000}-\frac{457071093}{3335332000}$
$k=-8$		$\frac{1}{24}$	$\frac{3}{176}$	$\frac{\gamma}{24}+\frac{1381}{4620}$	$\frac{15\gamma}{176}+\frac{373}{880}$	$\frac{3\gamma}{52}+\frac{211}{728}$	$\frac{\gamma^2}{105}+\frac{50143\gamma}{246400}+\frac{26742953}{48787200}$
$k=-9$						$-\frac{3888}{21175}$	
$k=-10$				$\frac{5589}{84700}$	$\frac{1137}{7150}$	$\frac{87}{1540}$	$\frac{573186051\gamma}{5644408000}+\frac{87731975043}{146754608000}$
$k=-11$				$-\frac{13\gamma}{560}-\frac{131}{924}$	$-\frac{57\gamma}{1960}-\frac{1075}{6468}$	$-\frac{9\gamma}{1120}-\frac{1697}{36960}$	$-\frac{17\gamma^2}{840}-\frac{179359\gamma}{646800}-\frac{1289}{1764}$
$k=-12$							$\frac{16281}{125125}$
$k=-13$							$-\frac{81\gamma}{1078}-\frac{355023}{770770}$
$k=-14$				$\frac{13\gamma}{2800}+\frac{6073}{369600}$	$\frac{23\gamma}{6800}+\frac{2957}{224400}$	$\frac{5\gamma}{8512}+\frac{1285}{561792}$	$\frac{43\gamma^2}{2800}+\frac{29644609\gamma}{153938400}+\frac{46207961}{92363040}$
$k=-16$							$\frac{33183\gamma}{1832600}+\frac{7173927}{104824720}$
$k=-17$							$-\frac{\gamma^2}{175}-\frac{7908301\gamma}{128282000}-\frac{3261710029}{25399836000}$
$k=-20$							$\frac{41\gamma^2}{47040}+\frac{2943929\gamma}{439824000}+\frac{2032133}{188496000}$

TABLE 2. Coefficients table for the third-order JRE ($\mathcal{M}^6_\infty$) in 3D (sphere) (For the third-order only $n=1$ terms are present, see table 3 for the remaining terms)

$b_{m,k,n,0}$	$m=3$		
	$n=3$	$n=5$	$n=7$
$k=-2$	$-\frac{599\gamma}{28000}-\frac{22815547}{60984000}$	$\frac{164}{819}$	$-\frac{35}{858}$
$k=-4$	$-\frac{418827051\gamma^2}{29673459200}+\frac{62070544461763\gamma}{73293444224000}+\frac{9236174319103493}{10480962524032000}$	$-\frac{9237\gamma}{366520}-\frac{157349}{146608}$	$\frac{567}{1573}$
$k=-5$	$-\frac{104327\gamma}{462000}-\frac{9833963}{30492000}$	$\frac{661\gamma}{1365}+\frac{4471}{4095}$	$-\frac{30\gamma}{143}-\frac{70}{143}$
$k=-6$		$-\frac{445149\gamma^2}{95190095}-\frac{587573895081\gamma}{1424043821200}+\frac{104968047599}{125190665600}$	$\frac{151551\gamma}{513760}-\frac{14415741}{22605440}$
$k=-7$	$-\frac{8118171\gamma}{50396500}-\frac{114716983}{100793000}$	$-\frac{1179657\gamma}{3335332}-\frac{55771}{46648}$	$\frac{756\gamma}{1573}+\frac{2916}{1573}$
$k=-8$	$\frac{61\gamma^2}{2420}+\frac{27461211\gamma}{70470400}+\frac{3347992607}{4651046400}$	$\frac{34\gamma^2}{1183}+\frac{2901\gamma}{47320}-\frac{4841}{141960}$	$\frac{427\gamma^2}{41184}-\frac{167888961127633\gamma}{166256455622400}-\frac{5482054296482347}{7315284047385600}$
$k=-9$	$\frac{16839\gamma}{304304}-\frac{1085826767}{3682078400}$	$\frac{33678\gamma}{112385}-\frac{4676251}{5439434}$	$\frac{1060857\gamma}{2825680}-\frac{1647800217}{1367629120}$
$k=-10$	$\frac{1149664413\gamma}{5241236000}+\frac{207885400509}{136272136000}$	$\frac{466297149\gamma}{1334132800}+\frac{4846089009}{2668265600}$	$\frac{12555\gamma}{53482}+\frac{131139}{106964}$
$k=-11$	$-\frac{361\gamma^2}{10780}-\frac{3681043\gamma}{8408400}-\frac{19239877}{18498480}$	$-\frac{19\gamma^2}{560}-\frac{18280747\gamma}{54654600}-\frac{24731363}{36436400}$	$-\frac{\gamma^2}{66}-\frac{16589\gamma}{111540}-\frac{125003}{408980}$
$k=-12$	$\frac{30359621107}{147283136000}-\frac{488331\gamma}{12172160}$	$\frac{36101964719}{118362083840}-\frac{18202959\gamma}{244549760}$	$\frac{44618931927}{415759252480}-\frac{20762487\gamma}{859006720}$
$k=-13$	$-\frac{147\gamma}{968}-\frac{646183}{692120}$	$-\frac{9861\gamma}{70070}-\frac{635053}{770770}$	$-\frac{2997\gamma}{78650}-\frac{192537}{865150}$
$k=-14$	$\frac{1803\gamma^2}{74800}+\frac{1337811833\gamma}{4596160800}+\frac{3982665971}{5515392960}$	$\frac{111\gamma^2}{6916}+\frac{1078190297\gamma}{5815249440}+\frac{3055176545}{6978299328}$	$\frac{57\gamma^2}{16016}+\frac{1873881\gamma}{45805760}+\frac{8865797}{91611520}$
$k=-16$	$\frac{31497\gamma}{919600}+\frac{6545551}{52601120}$	$\frac{26589\gamma}{1401400}+\frac{207773}{2802800}$	$\frac{3699\gamma}{1157728}+\frac{317763}{25470016}$
$k=-17$	$-\frac{17\gamma^2}{2200}-\frac{31358782537\gamma}{398333936000}-\frac{195575670263}{1195001808000}$	$-\frac{305\gamma^2}{88088}-\frac{21471961559\gamma}{613434261440}-\frac{136186343437}{1840302784320}$	$-\frac{7\gamma^2}{13728}-\frac{295271\gamma}{57383040}-\frac{623911}{57383040}$
$k=-20$	$\frac{553\gamma^2}{647680}+\frac{2385587089\gamma}{345181056000}+\frac{130642399631}{11390974848000}$	$\frac{281\gamma^2}{1019200}+\frac{501680027\gamma}{217273056000}+\frac{8141798513}{2048574528000}$	$\frac{5\gamma^2}{164736}+\frac{35015\gamma}{137719296}+\frac{1325845}{3029824512}$
$b_{m,k,n,1}$			
$k=-8$			$-\frac{6\gamma^2}{143}-\frac{21\gamma}{143}-\frac{15}{143}$

TABLE 3. Coefficients table for the third-order JRE ($\mathcal{M}^6_\infty$) in 3D (sphere) (only $n\in\{3,5,7\}$ terms). The last row indicates the single logarithmic term.

	$m=0$	$m=1$	$m=2$
q_0	$\frac{3}{2}$	$\frac{3}{2}-\frac{83\mathcal{M}_\infty^2}{220}$	$\frac{3}{2}-\frac{83\mathcal{M}_\infty^2}{220}-\frac{(378591917\gamma-678984653)\mathcal{M}_\infty^4}{5690484800}$
q_2	0	$\frac{-1408394988000\mathcal{M}_\infty^2\left[\mathcal{M}_\infty^2(102855\gamma-333287)-3527160\right]}{\left[(378591917\gamma-678984653)\mathcal{M}_\infty^4-2146864720\mathcal{M}_\infty^2+8535727200\right]^2}$	$\frac{-1408394988000\mathcal{M}_\infty^2\left[\mathcal{M}_\infty^2(102855\gamma-333287)-3527160\right]}{\left[(378591917\gamma-678984653)\mathcal{M}_\infty^4-2146864720\mathcal{M}_\infty^2+8535727200\right]^2}$
q_3	0	$-\frac{7623000\mathcal{M}_\infty^2}{\left(83\mathcal{M}_\infty^2-330\right)^3}$	$\frac{-80144502716101824000000\mathcal{M}_\infty^2\left[\mathcal{M}_\infty^2(6569315\gamma+2958729)-16460080\right]}{\left[(378591917\gamma-678984653)\mathcal{M}_\infty^4-2146864720\mathcal{M}_\infty^2+8535727200\right]^3}$

	$m=3$
q_0	$\frac{3}{2}-\frac{83\mathcal{M}_\infty^2}{220}+\frac{(378591917\gamma-678984653)\mathcal{M}_\infty^4}{5690484800}-\frac{\left(1058753493922135\gamma^2-1417255351884097\gamma+1844488552375516\right)\mathcal{M}_\infty^6}{37338428991864000}$
q_2	$\frac{2940401283109290000\mathcal{M}_\infty^2\left[\left(3584503350615\gamma^2+3753496299366\gamma-1602354228496\right)\mathcal{M}_\infty^4-20622030(102855\gamma-333287)\mathcal{M}_\infty^2+72737199334800\right]}{\left[\left(1058753493922135\gamma^2-1417255351884097\gamma+1844488552375516\right)\mathcal{M}_\infty^6-6561555(378591917\gamma-678984653)\mathcal{M}_\infty^4+14086770937839600\mathcal{M}_\infty^2-56007643487796000\right]^2}$
q_3	$\frac{-54894982258481009533016408280000000\mathcal{M}_\infty^2\left[\left(140520538815030\gamma^2+13019275653153\gamma-97154441684485\right)\mathcal{M}_\infty^4-41244060(6569315\gamma+2958729)\mathcal{M}_\infty^2+678880527124800\right]}{\left[\left(1058753493922135\gamma^2-1417255351884097\gamma+1844488552375516\right)\mathcal{M}_\infty^6-6561555(378591917\gamma-678984653)\mathcal{M}_\infty^4+14086770937839600\mathcal{M}_\infty^2-56007643487796000\right]^3}$

	$m=4$
q_0	$\frac{3}{2}-\frac{83\mathcal{M}_\infty^2}{220}+\frac{(378591917\gamma-678984653)\mathcal{M}_\infty^4}{5690484800}-\frac{\left(1058753493922135\gamma^2-1417255351884097\gamma+1844488552375516\right)\mathcal{M}_\infty^6}{37338428991864000}+\frac{\left(14976238371862724157681480000\gamma^3-54767356061541106493910208468\gamma^2+2147848928400930013001151343 5\gamma-44582394863079130273355228987\right)\mathcal{M}_\infty^8}{(167744945072150781939571968000 0)}$
q_2	$-52839657697727496310965169920000 0\mathcal{M}_\infty^2\left[\left(113812743442673525009760\gamma^3-315711837982991377500861 20\gamma^2-55929937306277740962895557\gamma+2027120618272390659730466 2\right)\mathcal{M}_\infty^6-11231387447280\left(3584503350615\gamma^2+3753496299366\gamma-1602354228496\right)\mathcal{M}_\infty^4+2316140088794315784000(102855\gamma-333287)\mathcal{M}_\infty^2-81693966755917588606934400 0\right]/\left[\left(14976238371862724157681480000\gamma^3-54767356061541106493910208468\gamma^2+2147848928400930013001151343 5\gamma-44582394863079130273355228987\right)\mathcal{M}_\infty^8-44925549789120\left(1058753493922135\gamma^2-1417255351884097\gamma+1844488552375516\right)\mathcal{M}_\infty^6+2947814658465492816 00(378591917\gamma-678984653)\mathcal{M}_\infty^4-63285592913584158640838515200 0\mathcal{M}_\infty^2+2516174176082261729093579520000\right]^2$
q_3	$-29545284927121827046603057602189943700439974003962675200000000 0\mathcal{M}_\infty^2\left[\left(112636348112419486392037728 0\gamma^3-66194676705840945723982158 0\gamma^2-10863105167126087150794286 31\gamma+8042768314886040488899272 46\right)\mathcal{M}_\infty^6-16847081170920\left(140520538815030\gamma^2+13019275653153\gamma-97154441684485\right)\mathcal{M}_\infty^4+6948420266382947352 00(6569315\gamma+2958729)\mathcal{M}_\infty^2-11437155345828462404970816000\right]/\left[\left(14976238371862724157681480000\gamma^3-54767356061541106493910208468\gamma^2+2147848928400930013001151343 5\gamma-44582394863079130273355228987\right)\mathcal{M}_\infty^8-44925549789120\left(1058753493922135\gamma^2-1417255351884097\gamma+1844488552375516\right)\mathcal{M}_\infty^6+2947814658465492816 00(378591917\gamma-678984653)\mathcal{M}_\infty^4-63285592913584158640838515200 0\mathcal{M}_\infty^2+2516174176082261729093579520000\right]^3$

TABLE 4. Coefficients of the first four terms in the Taylor expansion of $q(u)$, for different JRE orders m, at general Mach number and γ.