Gravity-driven film flow down a uniformly heated, smoothly corrugated, rigid substrate - Supplementary Material #2

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Appendix E. RAM–θ_{para} - Basis Function Coefficients

The contributions from higher-order terms to the coefficients in equations (2.35) - written in terms of $\{s, \tilde{a}_0, \tilde{a}_1, \tilde{b}_0, \tilde{b}_1\}$ where $\{a_j, b_j\} = \{\tilde{a}_j, \tilde{b}_j\} + \mathcal{O}(\epsilon)$ - read:

$$
\begin{aligned}
A_0 = {} & \epsilon^2 \Bigg[2\tilde{a}_1 \left(\frac{ds}{dx}\right)^2 - 2\frac{\partial \tilde{a}_0}{\partial x}\frac{ds}{dx} - \tilde{a}_0 \frac{d^2 s}{dx^2} \Bigg] - \epsilon^2 \frac{\partial^2}{\partial x^2}\left[h\tilde{a}_0 + h^2 \tilde{a}_1\right] + \epsilon^4 \frac{h^4}{12}\frac{\partial^4 \tilde{a}_1}{\partial x^4} \\
& + \epsilon^3 Re \Bigg[\frac{1}{3}\frac{\partial \tilde{a}_0}{\partial t}\left\{\left(\frac{\partial^2 h}{\partial x^2}\right)^2 - \left(\frac{ds}{dx}\right)^2\right\} + \frac{h^6}{18}\Bigg(\frac{\partial^2 \tilde{a}_1}{\partial x^2} - \tilde{a}_1 \frac{\partial^3 \tilde{a}_1}{\partial x^3}\Bigg) + h^5 \Bigg(\frac{\tilde{a}_1^2}{5}\frac{d^3 s}{dx^3} \\
& + \frac{\partial}{\partial x}\left[\frac{\tilde{a}_1}{15}\frac{\partial \tilde{a}_1}{\partial x}\right]\frac{ds}{dx} + \frac{1}{3}\left(\frac{\partial \tilde{a}_1}{\partial x}\right)^2 \frac{\partial h}{\partial x} - \frac{\tilde{a}_1}{3}\frac{\partial^2 \tilde{a}_1}{\partial x^2}\frac{\partial h}{\partial x} - \frac{\tilde{a}_1}{10}\frac{\partial^3 \tilde{\tilde{a}}_0}{\partial x^3} - \frac{\tilde{a}_0}{15}\frac{\partial^3 \tilde{a}_1}{\partial x^3} \\
& + \frac{1}{15}\frac{\partial \tilde{a}_1}{\partial x}\frac{\partial^2 \tilde{a}_0}{\partial x^2} + \frac{1}{10}\frac{\partial \tilde{a}_0}{\partial x}\frac{\partial^2 \tilde{a}_1}{\partial x^2} + \frac{7\tilde{a}_1}{15}\frac{\partial \tilde{a}_1}{\partial x}\frac{d^2 s}{dx^2}\Bigg) + h^4 \Bigg(\frac{\tilde{a}_0 \tilde{a}_1}{2}\frac{d^3 s}{dx^3} - \frac{\tilde{a}_0}{8}\frac{\partial^3 \tilde{a}_0}{\partial x^3} \\
& + \frac{5}{6}\frac{\partial \tilde{a}_0}{\partial x}\frac{\partial \tilde{a}_1}{\partial x}\frac{\partial h}{\partial x} + \frac{1}{8}\frac{\partial \tilde{a}_0}{\partial x}\frac{\partial^2 \tilde{a}_0}{\partial x^2} + \frac{2\tilde{a}_0}{3}\frac{\partial \tilde{a}_1}{\partial x}\frac{d^2 s}{dx^2} - \frac{\tilde{a}_0}{3}\frac{\partial^2 \tilde{a}_1}{\partial x^2}\frac{\partial h}{\partial x} + \tilde{a}_1^2 \frac{\partial h}{\partial x}\frac{d^2 s}{dx^2} \\
& + \frac{\partial}{\partial x}\left[\frac{\tilde{a}_0}{6}\frac{\partial \tilde{a}_1}{\partial x}\right]\frac{ds}{dx} + 2\tilde{a}_1 \frac{\partial \tilde{a}_0}{\partial x}\frac{d^2 s}{dx^2} + \frac{\tilde{a}_1}{3}\frac{\partial \tilde{a}_1}{\partial x}\frac{\partial h}{\partial x}\frac{ds}{dx}\Bigg) + h^3 \Bigg(2\tilde{a}_0 \tilde{a}_1 \frac{\partial h}{\partial x}\frac{d^2 s}{dx^2} \\
& + \frac{1}{12}\frac{\partial^3 \tilde{a}_1}{\partial t \partial x^2} + \frac{1}{2}\left(\frac{\partial \tilde{a}_0}{\partial x}\right)^2 \frac{\partial h}{\partial x} - \frac{\tilde{a}_0}{2}\frac{\partial^2 \tilde{a}_0}{\partial x^2}\frac{\partial h}{\partial x} + \frac{2\tilde{a}_0}{3}\frac{\partial \tilde{a}_1}{\partial x}\frac{\partial h}{\partial x}\frac{ds}{dx} + \frac{\tilde{a}_0^2}{3}\frac{d^3 s}{dx^3} \\
& + \frac{5\tilde{a}_0}{6}\frac{\partial \tilde{a}_0}{\partial x}\frac{d^2 s}{dx^2} + \frac{1}{6}\frac{\partial}{\partial x}\left[\tilde{a}_0 \frac{\partial \tilde{a}_0}{\partial x}\right]\frac{ds}{dx}\Bigg) + h^2 \Bigg(\frac{\tilde{a}_0}{2}\frac{\partial \tilde{a}_0}{\partial x}\frac{\partial h}{\partial x}\frac{ds}{dx} + \tilde{a}_0^2 \frac{\partial h}{\partial x}\frac{d^2 s}{dx^2} \\
& + \frac{1}{6}\frac{\partial^3 \tilde{a}_0}{\partial t \partial x^2} + \frac{1}{2}\frac{\partial^2 \tilde{a}_1}{\partial t \partial x}\frac{\partial h}{\partial x} + \frac{1}{4}\frac{\partial \tilde{a}_1}{\partial t}\frac{\partial^2 h}{\partial x^2}\Bigg) + h \Bigg(\frac{2}{3}\frac{\partial^2 \tilde{a}_0}{\partial t \partial x}\frac{\partial h}{\partial x} + \frac{1}{3}\frac{\partial \tilde{a}_0}{\partial t}\frac{\partial^2 h}{\partial x^2} \\
& + \frac{1}{2}\frac{\partial \tilde{a}_1}{\partial t}\left(\frac{\partial h}{\partial x}\right)^2 \Bigg)\Bigg] + \epsilon^4 \Bigg[\frac{h^3}{6}\left(\frac{\partial^4 \tilde{a}_0}{\partial x^4} - 4\frac{\partial^3 \tilde{a}_1}{\partial x^3}\frac{ds}{dx} - 4\frac{\partial \tilde{a}_1}{\partial x}\frac{d^3 s}{dx^3} - 2\tilde{a}_1 \frac{d^4 s}{dx^4}\right) \\
& - h^2 \Bigg(\frac{\partial^3 \tilde{a}_0}{\partial x^3}\frac{ds}{dx} - \frac{\partial^2 \tilde{a}_1}{\partial x^2}\left(\frac{ds}{dx}\right)^2 + 2\frac{\partial \tilde{a}_1}{\partial x}\frac{ds}{dx}\frac{d^2 s}{dx^2} + 2\tilde{a}_1 \frac{ds}{dx}\frac{d^3 s}{dx^3} + \tilde{a}_1 \left(\frac{d^2 s}{dx^2}\right)^2
\end{aligned}
$$

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$$-\frac{\partial \tilde{a}_0}{\partial x}\frac{d^3 s}{dx^3} - \frac{\tilde{a}_0}{2}\frac{d^4 s}{dx^4}\Bigg) + h\Bigg(4\left(\tilde{a}_0 + \tilde{a}_1\right)\frac{ds}{dx}\frac{d^3 s}{dx^3} + \left(3\tilde{a}_0 + 4\tilde{a}_1\right)\left(\frac{d^2 s}{dx^2}\right)^2$$

$$+\left(6\frac{\partial \tilde{a}_0}{\partial x} + 8\frac{\partial \tilde{a}_1}{\partial x}\right)\frac{ds}{dx}\frac{d^2 s}{dx^2} + 2\frac{\partial^2 \tilde{a}_1}{\partial x^2}\left(\frac{ds}{dx}\right)^2 - 2\tilde{a}_1\left(\frac{ds}{dx}\right)^2\frac{d^2 s}{dx^2}\Bigg)\Bigg]$$

$$-2\left(1 - \epsilon\frac{\partial f}{\partial x}\cot\beta\right) - \epsilon^3\frac{\partial^2}{\partial x^2}\left[\frac{(1 - Ma\vartheta)}{Ca}\frac{\frac{\partial f}{\partial x}}{\sqrt{1+\epsilon^2 g}}\right], \tag{E 1}$$

$$A_1 = \epsilon Re\frac{\partial \tilde{a}_0}{\partial t} + \epsilon^3 Re\frac{\partial}{\partial t}\left[\frac{2}{3}\frac{\partial \tilde{a}_0}{\partial x}\frac{ds}{dx} - \frac{\tilde{a}_1}{2}\left(\frac{ds}{dx}\right)^2 + \frac{\tilde{a}_0}{3}\frac{d^2 s}{dx^2}\right] + \epsilon^2\Bigg(\frac{\partial^2 \tilde{a}_0}{\partial x^2}$$

$$-4\frac{\partial \tilde{a}_1}{\partial x}\frac{ds}{dx} - 2\tilde{a}_1\frac{d^2 s}{dx^2}\Bigg) + \epsilon^4\Bigg(4\left(\tilde{a}_0 + \tilde{a}_1\right)\frac{ds}{dx}\frac{d^3 s}{dx^3} + \left(3\tilde{a}_0 + 4\tilde{a}_1\right)\left(\frac{d^2 s}{dx^2}\right)^2$$

$$+2\left(\frac{\partial^2 \tilde{a}_1}{\partial x^2} - \tilde{a}_1\frac{d^2 s}{dx^2}\right)\left(\frac{ds}{dx}\right)^2 + \left(6\frac{\partial \tilde{a}_0}{\partial x} + 6\frac{\partial \tilde{a}_1}{\partial x}\right)\frac{ds}{dx}\frac{d^2 s}{dx^2}\Bigg), \tag{E 2}$$

$$A_2 = \epsilon Re\left(\frac{\partial \tilde{a}_1}{\partial t} + \frac{\tilde{a}_0}{2}\frac{\partial \tilde{a}_0}{\partial x}\right) + \epsilon^2\frac{\partial^2 \tilde{a}_1}{\partial x^2} + \epsilon^3 Re\Bigg(\frac{\tilde{a}_0}{2}\frac{\partial \tilde{a}_0}{\partial x}\left(\frac{ds}{dx}\right)^2 + \tilde{a}_0^2\frac{ds}{dx}\frac{d^2 s}{dx^2}$$

$$-\frac{1}{2}\frac{\partial}{\partial t}\left[\frac{1}{3}\frac{\partial^2 \tilde{a}_0}{\partial x^2} - \frac{\partial \tilde{a}_1}{\partial x}\frac{ds}{dx} - \frac{\tilde{a}_1}{2}\frac{d^2 s}{dx^2}\right]\Bigg) + \epsilon^4\Bigg(2\tilde{a}_1\frac{ds}{dx}\frac{d^3 s}{dx^3} + \tilde{a}_1\left(\frac{d^2 s}{dx^2}\right)^2$$

$$+\frac{\partial^3 \tilde{a}_0}{\partial x^3}\frac{ds}{dx} - \frac{\partial \tilde{a}_0}{\partial x}\frac{d^3 s}{dx^3} - \frac{\tilde{a}_0}{2}\frac{d^4 s}{dx^4} - \frac{\partial^2 \tilde{a}_1}{\partial x^2}\left(\frac{ds}{dx}\right)^2 + 2\frac{\partial \tilde{a}_1}{\partial x}\frac{ds}{dx}\frac{d^2 s}{dx^2}\Bigg), \tag{E 3}$$

$$A_3 = \epsilon Re\frac{2\tilde{a}_0}{3}\frac{\partial \tilde{a}_1}{\partial x} - \epsilon^4\left(\frac{1}{6}\frac{\partial^4 \tilde{a}_0}{\partial x^4} - \frac{2}{3}\frac{\partial^3 \tilde{a}_1}{\partial x^3}\frac{ds}{dx} + \frac{2}{3}\frac{\partial \tilde{a}_1}{\partial x}\frac{d^3 s}{dx^3} + \frac{\tilde{a}_1}{3}\frac{d^4 s}{dx^4}\right)$$

$$-\epsilon^3 Re\Bigg(\frac{1}{12}\frac{\partial^3 \tilde{a}_1}{\partial t\partial x^2} + \frac{2\tilde{a}_0}{3}\frac{\partial^2 \tilde{a}_0}{\partial x^2}\frac{ds}{dx} + \frac{1}{3}\left(\frac{\partial \tilde{a}_0}{\partial x}\right)^2\frac{ds}{dx} - \frac{2\tilde{a}_0}{3}\frac{\partial \tilde{a}_1}{\partial x}\left(\frac{ds}{dx}\right)^2$$

$$+\frac{5\tilde{a}_0}{6}\frac{\partial \tilde{a}_0}{\partial x}\frac{d^2 s}{dx^2} - 2\tilde{a}_0\tilde{a}_1\frac{ds}{dx}\frac{d^2 s}{dx^2} + \frac{\tilde{a}_0^2}{3}\frac{d^3 s}{dx^3}\Bigg), \tag{E 4}$$

$$A_4 = \epsilon Re\frac{\tilde{a}_1}{3}\frac{\partial \tilde{a}_1}{\partial x} + \epsilon^3 Re\Bigg(\frac{\tilde{a}_0}{8}\frac{\partial^3 \tilde{a}_0}{\partial x^3} - \frac{1}{8}\frac{\partial \tilde{a}_0}{\partial x}\frac{\partial^2 \tilde{a}_0}{\partial x^2} + \frac{\tilde{a}_1}{3}\frac{\partial \tilde{a}_1}{\partial x}\left(\frac{ds}{dx}\right)^2$$

$$-\frac{\tilde{a}_0}{2}\frac{\partial^2 \tilde{a}_1}{\partial x^2}\frac{ds}{dx} - \frac{\tilde{a}_1}{2}\frac{\partial^2 \tilde{a}_0}{\partial x^2}\frac{ds}{dx} + \frac{2}{3}\frac{\partial \tilde{a}_0}{\partial x}\frac{\partial \tilde{a}_1}{\partial x}\frac{ds}{dx} - \frac{2\tilde{a}_0}{3}\frac{\partial \tilde{a}_1}{\partial x}\frac{d^2 s}{dx^2}$$

$$-\frac{\tilde{a}_1}{2}\frac{\partial \tilde{a}_0}{\partial x}\frac{d^2 s}{dx^2} + \tilde{a}_1^2\frac{ds}{dx}\frac{d^2 s}{dx^2} - \frac{\tilde{a}_0\tilde{a}_1}{2}\frac{d^3 s}{dx^3}\Bigg) - \frac{\epsilon^4}{12}\frac{\partial^4 \tilde{a}_1}{\partial x^4}, \tag{E 5}$$

$$A_5 = \frac{\epsilon^3 Re}{5}\Bigg(\frac{\tilde{a}_0}{3}\frac{\partial^3 \tilde{a}_1}{\partial x^3} + \frac{\tilde{a}_1}{2}\frac{\partial^3 \tilde{a}_0}{\partial x^3} - \frac{1}{2}\frac{\partial \tilde{a}_0}{\partial x}\frac{\partial^2 \tilde{a}_1}{\partial x^2} - \frac{1}{3}\frac{\partial \tilde{a}_1}{\partial x}\frac{\partial^2 \tilde{a}_0}{\partial x^2}$$

$$-2\tilde{a}_1\frac{\partial^2 \tilde{a}_1}{\partial x^2}\frac{ds}{dx} - \frac{7\tilde{a}_1}{3}\frac{\partial \tilde{a}_1}{\partial x}\frac{d^2 s}{dx^2} + \frac{4}{3}\left(\frac{\partial \tilde{a}_1}{\partial x}\right)^2\frac{ds}{dx} - \tilde{a}_1^2\frac{d^3 s}{dx^3}\Bigg), \tag{E 6}$$

$$A_6 = \frac{\epsilon^3 Re}{18}\left(\tilde{a}_1\frac{\partial^3 \tilde{a}_1}{\partial x^3} - \frac{\partial \tilde{a}_1}{\partial x}\frac{\partial^2 \tilde{a}_1}{\partial x^2}\right); \tag{E 7}$$

$$B_0 = \epsilon^2 \left(2\tilde{b}_1 \left(\frac{ds}{dx} \right)^2 - 2\frac{\partial \tilde{b}_0}{\partial x}\frac{ds}{dx} - \tilde{b}_0 \frac{d^2 s}{dx^2} \right), \tag{E 8}$$

$$B_1 = \epsilon RePr \frac{\partial \tilde{b}_0}{\partial t} + \epsilon^2 \left(\frac{\partial^2 \tilde{b}_0}{\partial x^2} - 4\frac{\partial \tilde{b}_1}{\partial x}\frac{ds}{dx} - 2\tilde{b}_1 \frac{d^2 s}{dx^2} \right), \tag{E 9}$$

$$B_2 = \epsilon RePr \left(\frac{\partial \tilde{b}_1}{\partial t} + \tilde{a}_0 \frac{\partial \tilde{b}_0}{\partial x} - \frac{\tilde{b}_0}{2}\frac{\partial \tilde{a}_0}{\partial x} \right) + \epsilon^2 \frac{\partial^2 \tilde{b}_1}{\partial x^2}, \tag{E 10}$$

$$B_3 = \epsilon RePr \left(\tilde{a}_1 \frac{\partial \tilde{b}_0}{\partial x} + \tilde{a}_0 \frac{\partial \tilde{b}_1}{\partial x} - \tilde{b}_1 \frac{\partial \tilde{a}_0}{\partial x} - \frac{\tilde{b}_0}{3}\frac{\partial \tilde{a}_1}{\partial x} \right), \tag{E 11}$$

$$B_4 = \epsilon RePr \left(\tilde{a}_1 \frac{\partial \tilde{b}_1}{\partial x} - \frac{2}{3}\tilde{b}_1 \frac{\partial \tilde{a}_1}{\partial x} \right). \tag{E 12}$$