Gravity-driven film flow down a uniformly heated, smoothly corrugated, rigid substrate - Supplementary Material #1

G. R. DALY[1]†, P. H. GASKELL[1] and S. VEREMIEIEV[1]

[1]Department of Engineering, Durham University, Durham, DH1 3LE, UK

Appendix D. Derivation of the Fluid Pressure

Taking the z-momentum equation, namely:

$$\frac{\partial p}{\partial z} = -2\epsilon \cot\beta + \epsilon^2 \frac{\partial}{\partial x}\left[\epsilon^2 \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right] + 2\epsilon^2 \frac{\partial^2 w}{\partial z^2} - \epsilon^3 Re \left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right], \quad \text{(D 1)}$$

and integrating *with respect to* z between z and $z = f(x,t)$, yields:

$$\begin{aligned} p = p_0 &+ 2\epsilon\cot\beta \int_z^f dz - 2\epsilon^2 \frac{\partial u}{\partial x} + 2\epsilon^2 \left[1 - \frac{1-\epsilon^2 g}{1+\epsilon^2 g}\right] \frac{\partial u}{\partial x}\bigg|_{z=f} \\ &-\epsilon^2 \int_z^f \frac{\partial}{\partial x}\left[\epsilon^2 \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right] dz + \epsilon^2 \frac{2}{1+\epsilon^2 g}\frac{\partial f}{\partial x}\left[\epsilon^2 \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right]\bigg|_{z=f} \\ &+\epsilon^3 Re \int_z^f \left[\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}(uw) + \frac{\partial}{\partial z}\left(w^2\right)\right] dz - \epsilon^3 \frac{(1 - Ma\vartheta)}{Ca}\frac{\partial}{\partial x}\left[\frac{\frac{\partial f}{\partial x}}{\sqrt{1+\epsilon^2 g}}\right], \quad \text{(D 2)} \end{aligned}$$

where the continuity equation has been utilised to replace $\partial w/\partial z$ by $-\partial u/\partial x$ and the upper limit of integration has been given by the normal stress at the free-surface, namely:

$$p|_{z=f} = p_0 - 2\epsilon^2 \frac{1-\epsilon^2 g}{1+\epsilon^2 g}\frac{\partial u}{\partial x}\bigg|_f + \frac{2\epsilon^2 \frac{\partial f}{\partial x}\left[\epsilon^2 \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right]\Big|_f}{1+\epsilon^2 g} - \epsilon^3 \frac{(1-Ma\vartheta)}{Ca}\frac{\frac{\partial^2 f}{\partial x^2}}{\left[1+\epsilon^2 g\right]^{3/2}}, \quad \text{(D 3)}$$

furthermore, the capillary term has been factorised via the chain rule, like so:

$$\frac{\partial}{\partial x}\left[\frac{\frac{\partial f}{\partial x}}{\sqrt{1+\epsilon^2 g}}\right] = \frac{\frac{\partial^2 f}{\partial x^2}}{\sqrt{1+\epsilon^2 g}} - \frac{\epsilon^2 g \frac{\partial^2 f}{\partial x^2}}{\left[1+\epsilon^2 g\right]^{3/2}} = \frac{\frac{\partial^2 f}{\partial x^2}}{\left[1+\epsilon^2 g\right]^{3/2}}. \quad \text{(D 4)}$$

Expression (D 2) can be simplified via the Leibniz integral rule, namely:

$$\int_z^f \frac{\partial}{\partial x}\left[\epsilon^2 \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right] dz = \frac{\partial}{\partial x}\int_z^f \left[\epsilon^2 \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right] dz - \frac{\partial f}{\partial x}\left[\epsilon^2 \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right]\bigg|_{z=f}, \quad \text{(D 5)}$$

† Email address for correspondence: george.r.daly@durham.ac.uk

which, after collecting terms, gives:

$$p = p_0 + 2\epsilon \cot\beta \int_z^f dz - 2\epsilon^2 \frac{\partial u}{\partial x} + 2\epsilon^2 \frac{2\epsilon^2 g}{1+\epsilon^2 g} \frac{\partial u}{\partial x}\bigg|_{z=f}$$
$$-\epsilon^2 \frac{\partial}{\partial x} \int_z^f \left[\epsilon^2 \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right] dz - \epsilon^2 \frac{1-\epsilon^2 g}{1+\epsilon^2 g} \frac{\partial f}{\partial x} \left[\epsilon^2 \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right]\bigg|_{z=f}$$
$$+\epsilon^3 Re \int_z^f \left[\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}(uw) + \frac{\partial}{\partial z}\left(w^2\right)\right] dz - \epsilon^3 \frac{(1-Ma\vartheta)}{Ca} \frac{\partial}{\partial x}\left[\frac{\frac{\partial f}{\partial x}}{\sqrt{1+\epsilon^2 g}}\right]. \quad \text{(D 6)}$$

Finally, the shear stress boundary condition at the free-surface, given by†:

$$\left[\epsilon^2 \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right]\bigg|_{z=f} = 4 \frac{\epsilon^2 \frac{\partial f}{\partial x} \frac{\partial u}{\partial x}\Big|_{z=f}}{1-\epsilon^2 g} - \epsilon \frac{Ma}{Ca} \frac{\sqrt{1+\epsilon^2 g}}{1-\epsilon^2 g} \frac{\partial \vartheta}{\partial x}, \quad \text{(D 7)}$$

and yet another factorisation of the capillary terms via the chain rule, namely:

$$\frac{\partial}{\partial x}\left[\frac{(1-Ma\vartheta)}{Ca} \frac{\frac{\partial f}{\partial x}}{\sqrt{1+\epsilon^2 g}}\right] = \frac{(1-Ma\vartheta)}{Ca} \frac{\partial}{\partial x}\left[\frac{\frac{\partial f}{\partial x}}{\sqrt{1+\epsilon^2 g}}\right] - \frac{Ma}{Ca} \frac{\frac{\partial f}{\partial x}\frac{\partial \vartheta}{\partial x}}{\sqrt{1+\epsilon^2 g}}, \quad \text{(D 8)}$$

allow equation (D 6) to be reduced to the following expression:

$$p = p_0 + 2\epsilon (f-z) \cot\beta - \epsilon^2 \frac{\partial u}{\partial x} - \epsilon^2 \frac{\partial}{\partial x}\left(u|_{z=f}\right) - \epsilon^3 \frac{\partial}{\partial x}\left[\frac{(1-Ma\vartheta)}{Ca} \frac{\frac{\partial f}{\partial x}}{\sqrt{1+\epsilon^2 g}}\right]$$
$$+\epsilon^3 Re \int_z^f \left[\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}(uw) + \frac{\partial}{\partial z}\left(w^2\right)\right] dz - \epsilon^4 \frac{\partial}{\partial x} \int_z^f \frac{\partial w}{\partial x}\, dz. \quad \text{(D 9)}$$

† The thermo-capillary term in equation (D 7) has been simplified via the total spatial derivative of the free-surface temperature along the free-surface, $\partial\vartheta/\partial x = \partial\theta/\partial x|_{z=f} + (\partial f/\partial x)\, \partial\theta/\partial z|_{z=f}$.