Supplementary Material for "Perturbation analysis of baroclinic torque in low-Mach-number flows"

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A. Detailed proofs

A.1. *Proof for the ansatz* (2.15) *when* $\epsilon < 1$

If a vector field $\boldsymbol{v}$ and a scalar field ϕ satisfy

$$\begin{cases} \mathcal{V}: & \nabla^2\phi = \nabla\cdot\boldsymbol{v}, \\ \partial\mathcal{V}: & \boldsymbol{n}\cdot\nabla\phi = \boldsymbol{n}\cdot\boldsymbol{v}. \end{cases} \tag{A.1}$$

It can be proved that $\|\nabla\phi\| \leqslant \|\boldsymbol{v}\|$:

$$\begin{aligned} &\|\boldsymbol{v}\|^2 - \|\nabla\phi\|^2 \\ &= \int_{\mathcal{V}} (\boldsymbol{v}\cdot\boldsymbol{v} - \nabla\phi\cdot\nabla\phi)\,dV \\ &= \int_{\mathcal{V}} [2(\boldsymbol{v}\cdot\nabla\phi - \nabla\phi\cdot\nabla\phi) + (\boldsymbol{v}-\nabla\phi)\cdot(\boldsymbol{v}-\nabla\phi)]\,dV \\ &\geqslant 2\int_{\mathcal{V}} (\boldsymbol{v}\cdot\nabla\phi - \nabla\phi\cdot\nabla\phi)\,dV \\ &= 2\int_{\mathcal{V}} \left[\nabla\cdot(\phi\boldsymbol{v}) - \phi\nabla\cdot\boldsymbol{v} - \frac{1}{2}\nabla^2\left(\phi^2\right) + \phi\nabla^2\phi\right] dV \\ &= 2\int_{\mathcal{V}} \left[\nabla\cdot(\phi\boldsymbol{v}) - \frac{1}{2}\nabla^2\left(\phi^2\right) + \phi\left(\nabla^2\phi - \nabla\cdot\boldsymbol{v}\right)\right] dV \\ &= 2\int_{\mathcal{V}} \left[\nabla\cdot(\phi\boldsymbol{v}) - \frac{1}{2}\nabla^2\left(\phi^2\right)\right] dV \\ &= 2\oint_{\partial\mathcal{V}} \left[\boldsymbol{n}\cdot(\phi\boldsymbol{v}) - \frac{1}{2}\boldsymbol{n}\cdot\nabla\left(\phi^2\right)\right] dS \\ &= 2\oint_{\partial\mathcal{V}} (\phi\boldsymbol{n}\cdot\boldsymbol{v} - \phi\boldsymbol{n}\cdot\nabla\phi)\,dS \\ &= 2\oint_{\partial\mathcal{V}} [\phi(\boldsymbol{n}\cdot\boldsymbol{v} - \boldsymbol{n}\cdot\nabla\phi)]\,dS \\ &= 0. \end{aligned} \tag{A.2}$$

And since equation (2.16) could be written as

$$\begin{aligned} k=0: &\quad \begin{cases} \mathcal{V}: & \nabla^2\mathit{\Pi}_{\boldsymbol{Y}}^{(0)} = \nabla\cdot(\rho_0\boldsymbol{Y}), \\ \partial\mathcal{V}: & \boldsymbol{n}\cdot\nabla\mathit{\Pi}_{\boldsymbol{Y}}^{(0)} = \boldsymbol{n}\cdot(\rho_0\boldsymbol{Y}), \end{cases} \\ k\geqslant 1: &\quad \begin{cases} \mathcal{V}: & \nabla^2\mathit{\Pi}_{\boldsymbol{Y}}^{(k)} = \nabla\cdot\left(\eta\nabla\mathit{\Pi}_{\boldsymbol{Y}}^{(k-1)}\right), \\ \partial\mathcal{V}: & \boldsymbol{n}\cdot\nabla\mathit{\Pi}_{\boldsymbol{Y}}^{(k)} = \boldsymbol{n}\cdot\left(\eta\nabla\mathit{\Pi}_{\boldsymbol{Y}}^{(k-1)}\right), \end{cases} \end{aligned} \tag{A.3}$$

it is easy to prove that

$$k=0: \; \|\nabla\mathit{\Pi}_{\boldsymbol{Y}}^{(0)}\| \leqslant \|\rho_0\boldsymbol{Y}\|, \tag{A.4}$$

by letting $\boldsymbol{v}=\rho_0\boldsymbol{Y}, \phi=\mathit{\Pi}_{\boldsymbol{Y}}^{(0)}$, and that

$$k\geqslant 1: \ \|\nabla \mathit{\Pi}_{\boldsymbol{Y}}^{(k)}\| \leqslant \|\eta\nabla \mathit{\Pi}_{\boldsymbol{Y}}^{(k-1)}\|, \tag{A.5}$$

by letting $\boldsymbol{v}=\eta\nabla \mathit{\Pi}_{\boldsymbol{Y}}^{(k-1)}, \phi=\mathit{\Pi}_{\boldsymbol{Y}}^{(k)}$.

Recalling that $|\eta(\boldsymbol{x},t)|\leqslant 1$, it is easy to prove from equation (A.5) that

$$k\geqslant 1: \ \|\nabla \mathit{\Pi}_{\boldsymbol{Y}}^{(k)}\| \leqslant \|\nabla \mathit{\Pi}_{\boldsymbol{Y}}^{(k-1)}\|. \tag{A.6}$$

Straightforwardly, equation (A.4), (A.6) and mathematical induction gives that

$$\forall k\geqslant 0: \ \|\nabla \mathit{\Pi}_{\boldsymbol{Y}}^{(k)}\| \leqslant \|\rho_0\boldsymbol{Y}\|. \tag{A.7}$$

Therefore, according to the Minkowski inequality, the Cauchy criterion is satisfied if $\epsilon<1$:

$$\begin{aligned}
&\forall\delta>0, \exists k_1>\max\left[0,\log_\epsilon\frac{(1-\epsilon)\,\delta}{\|\rho_0\boldsymbol{Y}\|}\right], \text{ s.t. } \forall k_2>k_1: \\
&\left\|\sum_{k=k_1}^{k_2}\left(\epsilon^k\nabla \mathit{\Pi}_{\boldsymbol{Y}}^{(k)}\right)\right\| \leqslant \sum_{k=k_1}^{k_2}\left(\epsilon^k\|\nabla \mathit{\Pi}_{\boldsymbol{Y}}^{(k)}\|\right) \\
&\qquad\leqslant \|\rho_0\boldsymbol{Y}\|\sum_{k=k_1}^{k_2}\epsilon^k \\
&\qquad= \|\rho_0\boldsymbol{Y}\|\epsilon^{k_1}\frac{1-\epsilon^{k_2-k_1+1}}{1-\epsilon} \\
&\qquad< \|\rho_0\boldsymbol{Y}\|\frac{\epsilon^{k_1}}{1-\epsilon}<\delta,
\end{aligned} \tag{A.8}$$

proving that the infinite series in expansion (2.15) is L^2 convergent when $\epsilon<1$.

In order to prove that the infinite series converges to $\nabla \mathit{\Pi}_{\boldsymbol{Y}}$ when $\epsilon<1$, define

$$\nabla \mathit{\Pi}'_{\boldsymbol{Y}}=\sum_{k=0}^{\infty}\left(\epsilon^k\nabla \mathit{\Pi}_{\boldsymbol{Y}}^{(k)}\right)-\nabla \mathit{\Pi}_{\boldsymbol{Y}}, \tag{A.9}$$

which satisfies

$$\begin{cases} \mathcal{V}: & \nabla\cdot\left[(1-\epsilon\eta)\,\nabla \mathit{\Pi}'_{\boldsymbol{Y}}\right]=0, \\ \partial\mathcal{V}: & \boldsymbol{n}\cdot\left[(1-\epsilon\eta)\,\nabla \mathit{\Pi}'_{\boldsymbol{Y}}\right]=0. \end{cases} \tag{A.10}$$

Consequently there is

$$\begin{aligned}
&\int_{\mathcal{V}}(1-\epsilon\eta)\,\nabla \mathit{\Pi}'_{\boldsymbol{Y}}\cdot\nabla \mathit{\Pi}'_{\boldsymbol{Y}}dV \\
=&\int_{\mathcal{V}}\left\{\nabla\cdot\left[\mathit{\Pi}'_{\boldsymbol{Y}}\,(1-\epsilon\eta)\,\nabla \mathit{\Pi}'_{\boldsymbol{Y}}\right]-\mathit{\Pi}'_{\boldsymbol{Y}}\nabla\cdot\left[(1-\epsilon\eta)\,\nabla \mathit{\Pi}'_{\boldsymbol{Y}}\right]\right\}dV \\
=&\int_{\mathcal{V}}\nabla\cdot\left[\mathit{\Pi}'_{\boldsymbol{Y}}\,(1-\epsilon\eta)\,\nabla \mathit{\Pi}'_{\boldsymbol{Y}}\right]dV \\
=&\oint_{\partial\mathcal{V}}\mathit{\Pi}'_{\boldsymbol{Y}}\boldsymbol{n}\cdot\left[(1-\epsilon\eta)\,\nabla \mathit{\Pi}'_{\boldsymbol{Y}}\right]dS \\
=&0.
\end{aligned} \tag{A.11}$$

Therefore, when $\epsilon<1$, there is $\nabla \mathit{\Pi}'_{\boldsymbol{Y}}=0$ since $1-\epsilon\eta>0$.

A.2. *Proof for the accuracy of buoyancy terms*

A.2.1. $\rho'\boldsymbol{g}/\rho_0$

For general flows, it is reasonable to assume that

$$\exists\alpha\in\mathbb{R}:\begin{cases}0<\lim\limits_{\lambda\to0^+}\left\{\left\|\nabla\eta\times\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(0)}\right\|\Big/\epsilon^{\alpha}\right\}<\infty,\\ 0<\lim\limits_{\lambda\to0^+}\left\{\left\|\nabla\eta\times\left(2\eta\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(0)}-\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(1)}\right)\right\|\Big/\epsilon^{\alpha}\right\}<\infty.\end{cases}\tag{A.12}$$

Therefore the relative error of the partial buoyancy term $\rho'\boldsymbol{g}/\rho_0$ is $O(\epsilon)$:

$$\begin{aligned}
&\lim_{\lambda\to0^+}\left\{\frac{\left\|\nabla\times\left(\rho'\rho_0^{-1}\boldsymbol{g}\right)-\rho^{-2}\nabla\rho\times\nabla\mathit{\Pi}_{\boldsymbol{f}}\right\|}{\left\|\rho^{-2}\nabla\rho\times\nabla\mathit{\Pi}_{\boldsymbol{f}}\right\|}\Bigg/\epsilon\right\}\\
&=\lim_{\lambda\to0^+}\left\{\frac{\left\|\nabla\times\left(\frac{\rho'}{\rho}\frac{\rho}{\rho-\rho'}\rho_0^{-1}\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(0)}\right)-\rho^{-2}\nabla\rho\times\nabla\mathit{\Pi}_{\boldsymbol{f}}\right\|}{\left\|\rho^{-2}\nabla\rho\times\nabla\mathit{\Pi}_{\boldsymbol{f}}\right\|}\Bigg/\epsilon\right\}\\
&=\lim_{\lambda\to0^+}\left\{\frac{\left\|\epsilon\rho_0^{-1}\nabla\frac{\eta}{1-\epsilon\eta}\times\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(0)}-\epsilon\rho_0^{-1}\nabla\eta\times\sum\limits_{k=0}^{\infty}\left(\epsilon^k\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(k)}\right)\right\|}{\left\|\epsilon\rho_0^{-1}\nabla\eta\times\sum\limits_{k=0}^{\infty}\left(\epsilon^k\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(k)}\right)\right\|}\Bigg/\epsilon\right\}\\
&=\lim_{\lambda\to0^+}\left\{\frac{\left\|\epsilon\rho_0^{-1}\frac{1}{(1-\epsilon\eta)^2}\nabla\eta\times\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(0)}-\epsilon\rho_0^{-1}\nabla\eta\times\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(0)}-\epsilon\rho_0^{-1}\nabla\eta\times\sum\limits_{k=1}^{\infty}\left(\epsilon^k\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(k)}\right)\right\|}{\left\|\epsilon\rho_0^{-1}\nabla\eta\times\sum\limits_{k=0}^{\infty}\left(\epsilon^k\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(k)}\right)\right\|}\Bigg/\epsilon\right\}\\
&=\lim_{\lambda\to0^+}\left\{\frac{\left\|\epsilon^2\rho_0^{-1}\frac{2\eta-\epsilon\eta^2}{(1-\epsilon\eta)^2}\nabla\eta\times\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(0)}-\epsilon\rho_0^{-1}\nabla\eta\times\sum\limits_{k=1}^{\infty}\left(\epsilon^k\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(k)}\right)\right\|}{\left\|\epsilon\rho_0^{-1}\nabla\eta\times\sum\limits_{k=0}^{\infty}\left(\epsilon^k\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(k)}\right)\right\|}\Bigg/\epsilon\right\}\\
&=\lim_{\lambda\to0^+}\left\{\frac{\left\|\epsilon^2\rho_0^{-1}\nabla\eta\times\left(2\eta\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(0)}-\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(1)}\right)\right\|}{\left\|\epsilon\rho_0^{-1}\nabla\eta\times\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(0)}\right\|}\Bigg/\epsilon\right\}\\
&=\lim_{\lambda\to0^+}\frac{\left\|\nabla\eta\times\left(2\eta\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(0)}-\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(1)}\right)\right\|}{\left\|\nabla\eta\times\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(0)}\right\|}.
\end{aligned}\tag{A.13}$$

A.2.2. $-\rho'\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})/\rho_0$

For general flows, it is reasonable to assume that

$$\exists\alpha\in\mathbb{R}:\begin{cases}0<\lim\limits_{\lambda\to0^+}\left\{\left\|\nabla\eta\times\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(0)}\right\|\Big/\epsilon^{\alpha}\right\}<\infty,\\ 0<\lim\limits_{\lambda\to0^+}\left\{\left\|\nabla\eta\times\left(2\eta\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(0)}-\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(1)}\right)\right\|\Big/\epsilon^{\alpha}\right\}<\infty,\end{cases}\tag{A.14}$$

Therefore the relative error of the partial buoyancy term $-\rho'\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})/\rho_0$ is $O(\epsilon)$ (following equation (A.13)):

$$\lim_{\lambda\to0^+}\left\{\frac{\left\|\nabla\times\boldsymbol{B_f}-\rho^{-2}\nabla\rho\times\nabla\Pi_{\boldsymbol{f}}\right\|}{\left\|\rho^{-2}\nabla\rho\times\nabla\Pi_{\boldsymbol{f}}\right\|}\Big/\epsilon\right\}=\lim_{\lambda\to0^+}\frac{\left\|\nabla\eta\times\left(2\eta\nabla\Pi_{\boldsymbol{f}}^{(0)}-\nabla\Pi_{\boldsymbol{f}}^{(1)}\right)\right\|}{\left\|\nabla\eta\times\nabla\Pi_{\boldsymbol{f}}^{(0)}\right\|}. \tag{A.15}$$

A.2.3. $-\rho'\boldsymbol{u}\cdot\nabla\boldsymbol{u}/\rho_0$

For general flows, it is reasonable to assume that

$$\exists\alpha\in\mathbb{R}:\begin{cases}0<\lim\limits_{\lambda\to0^+}\left\{\left\|\nabla\eta\times\nabla\Pi_{\boldsymbol{C}}^{(0)}\right\|\Big/\epsilon^{\alpha}\right\}<\infty,\\ 0<\lim\limits_{\lambda\to0^+}\left\{\left\|\nabla\times\left[\eta\left(\rho_0\boldsymbol{C}-\nabla\Pi_{\boldsymbol{C}}^{(0)}\right)\right]\right\|\Big/\epsilon^{\alpha}\right\}<\infty.\end{cases} \tag{A.16}$$

Therefore the relative error of the partial buoyancy term $-\rho'\boldsymbol{u}\cdot\nabla\boldsymbol{u}/\rho_0$ is $O(1)$:

$$\begin{aligned}&\lim_{\lambda\to0^+}\left\{\frac{\left\|\nabla\times\left(\rho'\rho_0^{-1}\boldsymbol{C}\right)-\rho^{-2}\nabla\rho\times\nabla\Pi_{\boldsymbol{C}}\right\|}{\left\|\rho^{-2}\nabla\rho\times\nabla\Pi_{\boldsymbol{C}}\right\|}\right\}\\ &=\lim_{\lambda\to0^+}\left\{\frac{\left\|\epsilon\nabla\times\left(\frac{\eta}{1-\epsilon\eta}\boldsymbol{C}\right)-\epsilon\rho_0^{-1}\nabla\times\sum\limits_{k=0}^{\infty}\left(\epsilon^k\eta\nabla\Pi_{\boldsymbol{C}}^{(k)}\right)\right\|}{\left\|\epsilon\rho_0^{-1}\nabla\times\sum\limits_{k=0}^{\infty}\left(\epsilon^k\eta\nabla\Pi_{\boldsymbol{C}}^{(k)}\right)\right\|}\right\}\\ &=\lim_{\lambda\to0^+}\left\{\frac{\left\|\epsilon\nabla\times(\eta\boldsymbol{C})-\epsilon\rho_0^{-1}\nabla\times\left(\eta\nabla\Pi_{\boldsymbol{C}}^{(0)}\right)\right\|}{\left\|\epsilon\rho_0^{-1}\nabla\times\left(\eta\nabla\Pi_{\boldsymbol{C}}^{(0)}\right)\right\|}\right\}\\ &=\lim_{\lambda\to0^+}\frac{\left\|\nabla\times\left[\eta\left(\rho_0\boldsymbol{C}-\nabla\Pi_{\boldsymbol{C}}^{(0)}\right)\right]\right\|}{\left\|\nabla\eta\times\nabla\Pi_{\boldsymbol{C}}^{(0)}\right\|}.\end{aligned} \tag{A.17}$$

A.2.4. $2\rho'\boldsymbol{u}\times\boldsymbol{\Omega}/\rho_0$

For general flows, it is reasonable to assume that

$$\exists\alpha\in\mathbb{R}:\begin{cases}0<\lim\limits_{\lambda\to0^+}\left\{\left\|\nabla\eta\times\nabla\Pi_{\boldsymbol{f}}^{(0)}\right\|\Big/\epsilon^{\alpha}\right\}<\infty,\\ 0<\lim\limits_{\lambda\to0^+}\left\{\left\|\nabla\times\left[\eta\left(\rho_0\boldsymbol{f}-\nabla\Pi_{\boldsymbol{f}}^{(0)}\right)\right]\right\|\Big/\epsilon^{\alpha}\right\}<\infty.\end{cases} \tag{A.18}$$

Therefore the relative error of the partial buoyancy term $2\rho'\boldsymbol{u}\times\boldsymbol{\Omega}/\rho_0$ is $O(1)$ (following equation (A.17)):

$$\lim_{\lambda\to0^+}\left\{\frac{\left\|\nabla\times\left(\rho'\rho_0^{-1}\boldsymbol{f}\right)-\rho^{-2}\nabla\rho\times\nabla\Pi_{\boldsymbol{f}}\right\|}{\left\|\rho^{-2}\nabla\rho\times\nabla\Pi_{\boldsymbol{f}}\right\|}\right\}=\lim_{\lambda\to0^+}\frac{\left\|\nabla\times\left[\eta\left(\rho_0\boldsymbol{f}-\nabla\Pi_{\boldsymbol{f}}^{(0)}\right)\right]\right\|}{\left\|\nabla\eta\times\nabla\Pi_{\boldsymbol{f}}^{(0)}\right\|}. \tag{A.19}$$

A.2.5. $\boldsymbol{B}^{(n)},\ n \geqslant 1$

For general flows, it is reasonable to assume that

$$\exists \alpha \in \mathbb{R}: \begin{cases} 0 < \lim\limits_{\lambda \to 0^+} \left\{ \left\| \nabla \eta \times \nabla \mathit{\Pi}^{(0)} \right\| \Big/ \epsilon^{\alpha} \right\} < \infty, \\ 0 < \lim\limits_{\lambda \to 0^+} \left\{ \left\| \nabla \eta \times \nabla \mathit{\Pi}^{(n)} \right\| \Big/ \epsilon^{\alpha} \right\} < \infty. \end{cases} \tag{A.20}$$

Therefore the relative error of the buoyancy term $\boldsymbol{B}^{(n)}$ is $O(\epsilon^n)$:

$$\begin{aligned} &\lim_{\lambda \to 0^+} \left\{ \frac{\left\| \nabla \times \boldsymbol{B}^{(n)} - \rho^{-2} \nabla \rho \times \nabla \mathit{\Pi} \right\|}{\left\| \rho^{-2} \nabla \rho \times \nabla \mathit{\Pi} \right\|} \Bigg/ \epsilon^n \right\} \\ &= \lim_{\lambda \to 0^+} \left\{ \frac{\left\| \epsilon \rho_0^{-1} \nabla \eta \times \sum\limits_{k=0}^{n-1} \left(\epsilon^k \nabla \mathit{\Pi}^{(k)} \right) - \epsilon \rho_0^{-1} \nabla \eta \times \sum\limits_{k=0}^{\infty} \left(\epsilon^k \nabla \mathit{\Pi}^{(k)} \right) \right\|}{\left\| \epsilon \rho_0^{-1} \nabla \eta \times \sum\limits_{k=0}^{\infty} \left(\epsilon^k \nabla \mathit{\Pi}^{(k)} \right) \right\|} \Bigg/ \epsilon^n \right\} \\ &= \lim_{\lambda \to 0^+} \left\{ \frac{\left\| \epsilon \rho_0^{-1} \nabla \eta \times \sum\limits_{k=n}^{\infty} \left(\epsilon^k \nabla \mathit{\Pi}^{(k)} \right) \right\|}{\left\| \epsilon \rho_0^{-1} \nabla \eta \times \sum\limits_{k=0}^{\infty} \left(\epsilon^k \nabla \mathit{\Pi}^{(k)} \right) \right\|} \Bigg/ \epsilon^n \right\} \\ &= \lim_{\lambda \to 0^+} \left\{ \frac{\left\| \epsilon^{n+1} \rho_0^{-1} \nabla \eta \times \nabla \mathit{\Pi}^{(n)} \right\|}{\left\| \epsilon \rho_0^{-1} \nabla \eta \times \nabla \mathit{\Pi}^{(0)} \right\|} \Bigg/ \epsilon^n \right\} \\ &= \lim_{\lambda \to 0^+} \frac{\left\| \nabla \eta \times \nabla \mathit{\Pi}^{(n)} \right\|}{\left\| \nabla \eta \times \nabla \mathit{\Pi}^{(0)} \right\|}. \end{aligned} \tag{A.21}$$

A.2.6. $\hat{\boldsymbol{B}}^{(n)},\ n \geqslant 1$

For general flows with $\boldsymbol{f} = \nabla \mathit{\Phi}$ and $(\|\boldsymbol{f}\| / \|\boldsymbol{A} + \boldsymbol{C} + \boldsymbol{D}\|) \sim O(\epsilon^{-1})$, it is reasonable to assume that

$$\exists \alpha \in \mathbb{R}: \begin{cases} 0 < \lim\limits_{\lambda \to 0^+} \left\{ \left\| \nabla \eta \times \nabla \mathit{\Pi}_{\boldsymbol{f}}^{(0)} \right\| \Big/ \epsilon^{\alpha} \right\} < \infty, \\ 0 < \lim\limits_{\lambda \to 0^+} \left\{ \left\| \nabla \eta \times \nabla \left(\epsilon \mathit{\Pi}_{\boldsymbol{f}}^{(n)} + \mathit{\Pi}_{\boldsymbol{A}}^{(n-1)} + \mathit{\Pi}_{\boldsymbol{C}}^{(n-1)} + \mathit{\Pi}_{\boldsymbol{D}}^{(n-1)} \right) \right\| \Big/ \epsilon^{\alpha+1} \right\} < \infty. \end{cases} \tag{A.22}$$

Therefore the relative error of the buoyancy term $\hat{\boldsymbol{B}}^{(n)}$ is $O(\epsilon^n)$:

$$\begin{aligned} &\lim_{\lambda \to 0^+} \left\{ \frac{\left\| \nabla \times \hat{\boldsymbol{B}}^{(n)} - \rho^{-2} \nabla \rho \times \nabla \mathit{\Pi} \right\|}{\left\| \rho^{-2} \nabla \rho \times \nabla \mathit{\Pi} \right\|} \Bigg/ \epsilon^n \right\} \\ &= \lim_{\lambda \to 0^+} \left\{ \frac{\left\| \epsilon \rho_0^{-1} \nabla \eta \times \left[\mathit{\Pi}_{\boldsymbol{f}}^{(0)} + \sum\limits_{k=1}^{n-1} \left(\epsilon^k \nabla \hat{\mathit{\Pi}}^{(k)} \right) \right] - \epsilon \rho_0^{-1} \nabla \eta \times \sum\limits_{k=0}^{\infty} \left(\epsilon^k \nabla \mathit{\Pi}^{(k)} \right) \right\|}{\left\| \epsilon \rho_0^{-1} \nabla \eta \times \sum\limits_{k=0}^{\infty} \left(\epsilon^k \nabla \mathit{\Pi}^{(k)} \right) \right\|} \Bigg/ \epsilon^n \right\} \\ &= \lim_{\lambda \to 0^+} \left\{ \frac{\left\| \epsilon \rho_0^{-1} \nabla \eta \times \left\{ \sum\limits_{k=n}^{\infty} \left(\epsilon^k \nabla \mathit{\Pi}_{\boldsymbol{f}}^{(k)} \right) + \sum\limits_{k=n-1}^{\infty} \left[\epsilon^k \nabla \left(\mathit{\Pi}_{\boldsymbol{A}}^{(k)} + \mathit{\Pi}_{\boldsymbol{C}}^{(k)} + \mathit{\Pi}_{\boldsymbol{D}}^{(k)} \right) \right] \right\} \right\|}{\left\| \epsilon \rho_0^{-1} \nabla \eta \times \sum\limits_{k=0}^{\infty} \left[\epsilon^k \nabla \left(\mathit{\Pi}_{\boldsymbol{f}}^{(k)} + \mathit{\Pi}_{\boldsymbol{A}}^{(k)} + \mathit{\Pi}_{\boldsymbol{C}}^{(k)} + \mathit{\Pi}_{\boldsymbol{D}}^{(k)} \right) \right] \right\|} \Bigg/ \epsilon^n \right\} \end{aligned}$$

$$
= \lim_{\lambda\to0^+}\left\{ \frac{\left\|\epsilon^{n+1}\rho_0^{-1}\nabla\eta\times\left[\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(n)}+\epsilon^{-1}\nabla\left(\mathit{\Pi}_{\boldsymbol{A}}^{(n-1)}+\mathit{\Pi}_{\boldsymbol{C}}^{(n-1)}+\mathit{\Pi}_{\boldsymbol{D}}^{(n-1)}\right)\right]\right\|}{\left\|\epsilon\rho_0^{-1}\nabla\eta\times\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(0)}\right\|}\Big/\epsilon^n \right\}
$$

$$
= \lim_{\lambda\to0^+}\frac{\left\|\nabla\eta\times\left[\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(n)}+\epsilon^{-1}\nabla\left(\mathit{\Pi}_{\boldsymbol{A}}^{(n-1)}+\mathit{\Pi}_{\boldsymbol{C}}^{(n-1)}+\mathit{\Pi}_{\boldsymbol{D}}^{(n-1)}\right)\right]\right\|}{\left\|\nabla\eta\times\nabla\mathit{\Pi}_{\boldsymbol{f}}^{(0)}\right\|}. \tag{A.23}
$$

B. Computation procedures of type-I and type-II buoyancy terms

B.1. *Type-I buoyancy terms*

Following the definition (2.20), the n^{th}-order type-I buoyancy term with $n \geqslant 1$ can be written as

$$
\boldsymbol{B}^{(n)} = \frac{\rho'}{\rho_0\rho}\sum_{k=0}^{n-1}\left(\epsilon^k\nabla\mathit{\Pi}^{(k)}\right),\ n\geqslant 1. \tag{B.1}
$$

with $\mathit{\Pi}^{(k)} = \mathit{\Pi}_{\boldsymbol{A}}^{(k)} + \mathit{\Pi}_{\boldsymbol{C}}^{(k)} + \mathit{\Pi}_{\boldsymbol{D}}^{(k)} + \mathit{\Pi}_{\boldsymbol{f}}^{(k)}$. And $\mathit{\Pi}^{(k)}$ could be computed by solving the following Poisson equations, which are the linear combinations of equations (2.16):

$$
\begin{aligned}
k=0: &\begin{cases} \mathcal{V}: & \nabla^2\mathit{\Pi}^{(0)} = \rho_0\left[\nabla\cdot(\boldsymbol{C}+\boldsymbol{D}+\boldsymbol{f})+\dfrac{\partial}{\partial t}\left(\dfrac{Q}{\rho}\right)\right], \\ \partial\mathcal{V}: & \boldsymbol{n}\cdot\nabla\mathit{\Pi}^{(0)} = \rho_0\left[\boldsymbol{n}\cdot(\boldsymbol{C}+\boldsymbol{D}+\boldsymbol{f})-\dfrac{\partial(\boldsymbol{n}\cdot\boldsymbol{u})}{\partial t}+\dfrac{\partial\boldsymbol{n}}{\partial t}\cdot\boldsymbol{u}\right], \end{cases} \\
k\geqslant 1: &\begin{cases} \mathcal{V}: & \nabla^2\mathit{\Pi}^{(k)} = \nabla\cdot\left(\eta\nabla\mathit{\Pi}^{(k-1)}\right), \\ \partial\mathcal{V}: & \boldsymbol{n}\cdot\nabla\mathit{\Pi}^{(k)} = \boldsymbol{n}\cdot\left(\eta\nabla\mathit{\Pi}^{(k-1)}\right). \end{cases}
\end{aligned} \tag{B.2}
$$

Therefore, for $n \geqslant 1$, computing $\boldsymbol{B}^{(n)}$ requires solving n Poisson equations for $\mathit{\Pi}^{(k)}$ with $0 \leqslant k \leqslant n-1$.

B.2. *Type-II buoyancy terms*

Following the definition (2.21), the n^{th}-order type-II buoyancy term with $n \geqslant 2$ can be written as

$$
\hat{\boldsymbol{B}}^{(n)} = \frac{\rho'}{\rho}\boldsymbol{f} + \frac{\rho'}{\rho_0\rho}\sum_{k=1}^{n-1}\left[\epsilon^k\nabla\hat{\mathit{\Pi}}^{(k)}\right], \tag{B.3}
$$

where $\hat{\mathit{\Pi}}^{(k)} = \mathit{\Pi}_{\boldsymbol{f}}^{(k)} + \epsilon^{-1}\left(\mathit{\Pi}_{\boldsymbol{A}}^{(k-1)} + \mathit{\Pi}_{\boldsymbol{C}}^{(k-1)} + \mathit{\Pi}_{\boldsymbol{D}}^{(k-1)}\right)$ could be computed by solving the following Poisson equations, which are also the linear combinations of equations (2.16):

$$
\begin{aligned}
k=1: &\begin{cases} \mathcal{V}: & \nabla^2\hat{\mathit{\Pi}}^{(1)} = \epsilon^{-1}\rho_0\left[\nabla\cdot(\boldsymbol{C}+\boldsymbol{D}+\epsilon\eta\boldsymbol{f})+\dfrac{\partial}{\partial t}\left(\dfrac{Q}{\rho}\right)\right], \\ \partial\mathcal{V}: & \boldsymbol{n}\cdot\nabla\hat{\mathit{\Pi}}^{(1)} = \epsilon^{-1}\rho_0\left[\boldsymbol{n}\cdot(\boldsymbol{C}+\boldsymbol{D}+\epsilon\eta\boldsymbol{f})+\dfrac{\partial(\boldsymbol{n}\cdot\boldsymbol{u})}{\partial t}-\dfrac{\partial\boldsymbol{n}}{\partial t}\cdot\boldsymbol{u}\right], \end{cases} \\
k\geqslant 2: &\begin{cases} \mathcal{V}: & \nabla^2\hat{\mathit{\Pi}}^{(k)} = \nabla\cdot\left(\eta\nabla\hat{\mathit{\Pi}}^{(k-1)}\right), \\ \partial\mathcal{V}: & \boldsymbol{n}\cdot\nabla\hat{\mathit{\Pi}}^{(k)} = \boldsymbol{n}\cdot\left(\eta\nabla\hat{\mathit{\Pi}}^{(k-1)}\right). \end{cases}
\end{aligned} \tag{B.4}
$$

Therefore, for $n \geqslant 2$, computing $\hat{\boldsymbol{B}}^{(n)}$ only requires solving $n-1$ Poisson equations for $\hat{\Pi}^{(k)}$ with $1 \leqslant k \leqslant n-1$, and computing $\hat{\boldsymbol{B}}^{(1)}$ does not require solving any extra Poisson equation.