# Supplementary Material: Scale-similar structures of homogeneous isotropic non-mirror-symmetric turbulence based on the Lagrangian closure theory

**Kazuhiro Inagaki†**

Institute of Industrial Science, The University of Tokyo, Komaba, Meguro-ku, Tokyo 153-8505, Japan

## S. Contributions from outside the similarity range to the response function equation and energy and helicity fluxes

In the scale-similar analysis demonstrated in §3, we neglect the contributions from outside the similarity range. Here, we verify the conditions for this negligibility. We decompose the response function and spectral densities of energy and helicity into $G(p,t,s) = G_S(p,t,s) + G_O(p,t,s)$, $Q(p,t,t) = Q_S(p,t,t) + Q_O(p,t,t)$, and $Q^H(p,t,t) = Q_S^H(p,t,t) + Q_O^H(p,t,t)$ where $G_S(p,t,s)$, $Q_S(p,t,t)$, and $Q_S(p,t,t)$ are those given by (3.1), (3.2), and (3.4); namely, $G_S(p,t,s) = G_S((p/k)^\ell \omega_p(t-s))$ obeying (3.11), $\omega_p^{-1} = \varepsilon^{1/3-\ell}(\varepsilon^H)^{-2/3+\ell}p^{-\ell}$, $Q_S(p,t,t) = [C_K^{(n)}/(2\pi)]\varepsilon^{7/3-n}(\varepsilon^H)^{-5/3+n}p^{-n-2}$, and $Q_S^H(p,t,t) = [C_H^{(m)}/(4\pi)]\varepsilon^{4/3-m}(\varepsilon^H)^{-2/3+n}p^{-m-2}$. On the other hand, $G_O(p,t,s)$, $Q_O(p,t,t)$, and $Q_O^H(p,t,t)$ denote those defined outside the similarity range. When the lower and upper bounds of the wavenumber of the similarity range are denoted by $k_{\mathrm{lower}}$ and $k_{\mathrm{upper}}$, we have $G_S(p,t,s) = Q_S(p,t,t) = Q_S^H(p,t,t) = 0$ in $p < k_{\mathrm{lower}}$ and $p > k_{\mathrm{upper}}$. Similarly, $G_O(p,t,s) = Q_O(p,t,t) = Q_O^H(p,t,t) = 0$ in $k_{\mathrm{lower}} \leqslant p \leqslant k_{\mathrm{upper}}$. Hereafter, we consider that $k$ lies in the similarity range.

### S.1. *Response function equation*

Considering the contributions from outside the similarity range, $\eta(k,t,s)$ given by (2.41) can be decomposed into

$$\eta(k,t,s) = \eta_L(k,t,s) + \eta_S(k,t,s) + \eta_U(k,t,s), \tag{S.1}$$

where

$$\eta_L(k,t,s) = k\int_0^{k_{\mathrm{lower}}} \mathrm{d}q\, q^3 J\left(\frac{q}{k}\right)\int_s^t \mathrm{d}s' G_O(q,t,s')Q_O(q,s',s'), \tag{S.2$a$}$$

$$\eta_S(k,t,s) = k\int_{k_{\mathrm{lower}}}^{k_{\mathrm{upper}}} \mathrm{d}q\, q^3 J\left(\frac{q}{k}\right)\int_s^t \mathrm{d}s' G_S(q,t-s')Q_S(q), \tag{S.2$b$}$$

$$\eta_U(k,t,s) = k\int_{k_{\mathrm{upper}}}^{\infty} \mathrm{d}q\, q^3 J\left(\frac{q}{k}\right)\int_s^t \mathrm{d}s' G_O(q,t-s')Q_O(q). \tag{S.2$c$}$$

† Present address: Research and Education Center for Natural Sciences, Keio University, 4-1-1 Hiyoshi, Kohoku-ku, Yokohama 223-8521, Japan. Email address for correspondence: inagakik@keio.jp

Because $J(x) \geqslant 0$, $G(p,t,s) \geqslant 0$ for $t \geqslant s$ if $Q(p,s,s) \geqslant 0$ (Kaneda 1986). When the similarity range is sufficiently large such that one can simultaneously take both limits $k_{\mathrm{lower}}/k \to 0$ and $k_{\mathrm{upper}}/k \to \infty$, we have

$$\lim_{\substack{k_{\mathrm{lower}}/k\to 0 \\ k_{\mathrm{upper}}/k\to\infty}} \eta_S(k,t,s) = \gamma\omega_k (k^H/k)^{-3+n+2\ell} \int_0^\infty \mathrm{d}w\; w^{-n+1} J(w) \int_0^\tau \mathrm{d}\sigma\; \overline{G}(w^\ell \sigma), \quad \text{(S.3)}$$

where $w = q/k$, $\sigma = \gamma^{-1}\omega_k(t-s')$, and $\overline{G}(\tau) = G(\gamma\tau)$ with $\gamma = (2\pi/C_K^{(n)})^{1/2}$. As discussed in §3.2, this integral converges when $-3+n+2\ell = 0$ and $0 < \ell < 2$. Consequently, we have $\eta_S(k,t,s) \sim \omega_k$. If $\eta_L(k,t,s)$ and $\eta_U(k,t,s)$ become negligible to $\eta_S(k,t,s)$ in the limit $k_{\mathrm{lower}}/k \to 0$ and $k_{\mathrm{upper}}/k \to \infty$, we can justify the negligibility of the contributions from outside the similarity range to the response function.

According to Linkmann (2018), $\beta(= \varepsilon L/u^3)$ remains finite in homogeneous helical turbulence forced at large scales, although the value of $\beta$ decreases as the relative helicity of forcing increases. Here, $L$ and $u$ denote the integral length scale and root-mean-square velocity fluctuation, respectively. This fact indicates that the energy spectral density is bounded by $Q_O(q) \lesssim u^2 k_L^{-3} \simeq \varepsilon^{2/3} k_L^{-11/3}$ at $q \leqslant k_{\mathrm{lower}}$ where $k_L(= L^{-1})$ denotes the integral wavenumber scale. Several numerical simulations of homogeneous turbulence are consistent with this evaluation (André & Lesieur 1977; Borue & Orszag 1997; Chen *et al.* 2003; Mininni *et al.* 2006; Baerenzung *et al.* 2008; Sahoo *et al.* 2017; Alexakis 2017). Besides, $\int_s^t \mathrm{d}s' G_O(q,t,s')$ at $q \leqslant k_{\mathrm{lower}}$ will be bounded by the largest time scale, which is the turn-over time of the largest eddy. Namely, we have $\int_s^t \mathrm{d}s' G_O(q,t,s') \lesssim (uk_L)^{-1} \simeq \varepsilon^{-1/3} k_L^{-2/3}$ at $q \leqslant k_{\mathrm{lower}}$. Furthermore, we have $J(x) = 16\pi x/15$ for small $x$ (Kaneda 1986). Consequently, we can evaluate $\eta_L(k,t,s)$ as follows:

$$\eta_L(k,t,s) \lesssim k \int_0^{k_{\mathrm{lower}}} \mathrm{d}q\; q^3 \times \frac{q}{k} \times \varepsilon^{-1/3} k_L^{-2/3} \times \varepsilon^{2/3} k_L^{-11/3} \simeq \varepsilon^{1/3} k_{\mathrm{lower}}^{2/3} (k_{\mathrm{lower}}/k_L)^{13/3}, \quad \text{(S.4)}$$

where we omit constant factors such as $16\pi/15$ for simplicity. Hence, we can evaluate that

$$\eta_L(k,t,s)/\eta_S(k,t,s) \lesssim (k^H/k)^{-2/3+\ell} (k_{\mathrm{lower}}/k)^{2/3} (k_{\mathrm{lower}}/k_L)^{13/3}. \quad \text{(S.5)}$$

For this ratio to be negligible at $k^H/k \ll 1$ and $k_{\mathrm{lower}}/k \ll 1$, it is required that $\ell \geqslant 2/3$ and $k_{\mathrm{lower}}/k_L$ remains finite. The power-law exponent $\ell = 2/3$ obtained in §3.3 is consistent with the former condition $\ell \geqslant 2/3$. The ratio $k_{\mathrm{lower}}/k_L$ is expected to increase if the helicity is injected at large scales because the helicity hinders the energy transfer to small scales (André & Lesieur 1977; Morinishi *et al.* 2001; Kessar *et al.* 2015; Stepanov *et al.* 2015). Nevertheless, several numerical simulations of homogeneous turbulence suggest that the relative helicity $E^H(k)/[2kE(k)]$ rapidly decreases almost proportional to $k^{-1}$ as it goes away from the integral scale $k_L$ (André & Lesieur 1977; Borue & Orszag 1997; Mininni *et al.* 2006; Baerenzung *et al.* 2008) unless one injects helicity to a wide range of scales (Kessar *et al.* 2015; Plunian *et al.* 2020). Therefore, the strongly helical range is confined only at large scales where the helicity is injected. When the relative helicity rapidly decreases, we expect that the effect of helicity also rapidly decreases. In such a case, the ratio $k_{\mathrm{lower}}/k_L$ will remain finite, even though it slightly increases compared with the mirror-symmetric case. Further verification is needed to evaluate the ratio $k_{\mathrm{lower}}/k_L$ in more general helical turbulent flows. Consequently, for $\ell \geqslant 2/3$, we can evaluate that $\eta_L(k,t,s)/\eta_S(k,t,s) \to 0$ in the limit $k^H/k \to 0$ and $k_{\mathrm{lower}}/k \to 0$ with the assumption that $k_{\mathrm{lower}}/k_L$ remains finite.

Similarly, at $q \geqslant k_{\text{upper}}$, the energy spectrum density and time integral of the response function will be bounded by $Q_O(q) \lesssim \varepsilon^{7/3-n}(\varepsilon^H)^{-5/3+n}q^{-n-2}$ and $\int_s^t \mathrm{d}s' G_O(q,t,s') \lesssim \varepsilon^{1/3-\ell}(\varepsilon^H)^{-2/3+\ell}q^{-\ell}$. Even if one considers the bottleneck region (Ishihara *et al.* 2016), the error of the bound on $Q_O(q)$ is at most a few times. Hence, when we employ $-3+n+2\ell=0$ (3.10), we have

$$\eta_U(k,t,s) \lesssim k\int_{k_{\text{upper}}}^{\infty} \mathrm{d}q\ q^3\frac{k}{q} \times \varepsilon^{1/3-\ell}(\varepsilon^H)^{-2/3+\ell}q^{-\ell} \times \varepsilon^{-2/3+2\ell}(\varepsilon^H)^{4/3-2\ell}q^{-5+2\ell}$$
$$\simeq \varepsilon^{-1/3+\ell}(\varepsilon^H)^{2/3-\ell}k^2 k_{\text{upper}}^{-2+\ell}, \tag{S.6}$$

where we use $J(x)=J(1/x)=16\pi/15/x$ for $x \gg 1$ and assume $\ell<2$. Then, we have

$$\eta_U(k,t,s)/\eta_S(k,t,s) \lesssim (k_{\text{upper}}/k)^{-2+\ell}. \tag{S.7}$$

If $\ell<2$, the right-hand side converges to zero in the limit $k_{\text{upper}}/k \to \infty$. Consequently, if $2/3 \leqslant \ell < 2$ with $-3+n+2\ell=0$ (3.10), both $\eta_L(k,t,s)$ and $\eta_U(k,t,s)$ become negligible to $\eta_S(k,t,s)$ in the limit $k_{\text{lower}}/k \to 0$ and $k_{\text{upper}}/k \to \infty$.

### S.2. *Energy flux*

For the LRA, the energy flux reads

$$\Pi(k) = 4\pi^2\int_k^{\infty}\mathrm{d}k'\int_0^k \mathrm{d}p'\int_{\max(p',k'-p')}^{k'+p'}\mathrm{d}q'\int_{-\infty}^t \mathrm{d}s\ G(k',t,s)G(p',t,s)G(q',t,s)$$
$$\times\left[f_b(k',p',q',s)+f_c(k',p',q',s)\right], \tag{S.8}$$

where

$$f_b(k',p',q',s) = k'^3p'q'\left\{b_{k'p'q'}\left[Q(p',s,s)-Q(k',s,s)\right]Q(q',s,s)\right.$$
$$\left.+\ b_{k'q'p'}\left[Q(q',s,s)-Q(k',s,s)\right]Q(p',s,s)\right\}, \tag{S.9}$$
$$f_c(k',p',q',s) = k'p'q'\left\{c_{k'p'q'}\left[Q^H(p',s,s)-Q^H(k',s,s)\right]Q^H(q',s,s)\right.$$
$$\left.+\ c_{k'q'p'}\left[Q^H(q',s,s)-Q^H(k',s,s)\right]Q^H(p',s,s)\right\}, \tag{S.10}$$

and we consider $t_0 \to -\infty$. For the contributions from outside the similarity range to be negligible, the integrals that at least one of $(k',p',q')$ is outside the similarity range must be negligible to $\varepsilon$.

In the integration range of the energy flux, $\max(p',k'-p') \leqslant k/2$ with equality if and only if $k'=k$ and $p'=k/2$. Because we consider the limit $k_{\text{lower}}/k \to 0$, we need not consider the case that $k_{\text{lower}} > k/2$. Hence, the integration range of $q'(\geqslant k/2)$ does not involve $k_{\text{lower}}$ and $k_L$. Besides, $k'+p' \leqslant k_{\text{upper}}$ when $k' \leqslant k_{\text{upper}}-k$. Considering these conditions, we decompose the energy flux into the following form:

$$\Pi(k)/(4\pi^2) = I_{Sb}+I_{O1b}+I_{O2b}+I_{Sc}+I_{O1c}+I_{O2c}, \tag{S.11}$$

where

$$I_{Sa} = \int_k^{k_{\text{upper}}-k}\mathrm{d}k'\int_{k_{\text{lower}}}^k \mathrm{d}p'\int_{\max(p',k'-p')}^{k'+p'}\mathrm{d}q'\int_{-\infty}^t \mathrm{d}s\ G_S(k',t,s)G_S(p',t,s)G_S(q',t,s)$$
$$\times f_a(k',p',q',s), \tag{S.12$a$}$$

$$I_{O1a} = \int_k^{\infty}\mathrm{d}k'\int_0^{k_{\text{lower}}}\mathrm{d}p'\int_{\max(p',k'-p')}^{k'+p'}\mathrm{d}q'\int_{-\infty}^t \mathrm{d}s\ G(k',t,s)G_O(p',t,s)G(q',t,s)$$
$$\times f_a(k',p',q',s), \tag{S.12$b$}$$

$$I_{O2a} = \int_{k_{\rm upper}-k}^{\infty} {\rm d}k' \int_{k_{\rm lower}}^{k} {\rm d}p' \int_{\max(p',k'-p')}^{k'+p'} {\rm d}q' \int_{-\infty}^{t} {\rm d}s\ G(k',t,s)G_S(p',t,s)G(q',t,s) \times f_a(k',p',q',s), \tag{S.12$c$}$$

and $a = b, c$. Here, the integration range of $I_{Sa}$ is composed of the wavenumbers only inside the similarity range, whereas $I_{O1a}$ and $I_{O2a}$ involve the contributions from $p' \leqslant k_{\rm lower}$ and/or $k', q' \geqslant k_{\rm upper}$.

Let us consider $I_{O1b}$. Considering $k' \geqslant k \gg k_{\rm lower} \geqslant p'$, it is enough to consider a small $p'/k'$. Furthermore, we have $\max(p', k'-p') = k'-p'$ in this integration range. Putting $q' = k' + \xi p'$ with $\xi \in [-1, 1]$, we have

$$b_{k'p'q'} = \xi(1-\xi^2)p'/k' + \frac{1}{2}(1+2\xi^2-3\xi^4)(p'/k')^2 + O((p'/k')^3), \tag{S.13$a$}$$

$$b_{k'q'p'} = (1-\xi^2) + \xi(1-\xi^2)p'/k' + \frac{1}{4}(-5+3\xi^2-\xi^4)(p'/k')^2 + O((p'/k')^3), \tag{S.13$b$}$$

for a small $p'/k'$. When we assume that $k'^d\partial^d Q(k', s, s)/\partial k'^d$ and $k'^d\partial^d G(k', t, s)/\partial k'^d$ with $d = 1, 2, ...$ are finite in the limit $k_{\rm lower}/k \to 0$ and $k_{\rm upper}/k \to \infty$, $I_{O1b}$ can be evaluated as

$$\begin{aligned}
I_{O1b} &= \int_{k}^{\infty} {\rm d}k' \int_{0}^{k_{\rm lower}} {\rm d}p' \int_{-1}^{1} {\rm d}\xi\ p' \int_{-\infty}^{t} {\rm d}s\ G(k',t,s)G_O(p',t,s)G(k'+\xi p',t,s) \\
&\quad \times k'^4 p'(1+\xi p'/k') \Big\{ \Big[\xi(1-\xi^2)(p'/k') + \frac{1}{2}(1+2\xi^2-3\xi^4)(p'/k')^2 + O((p'/k')^3)\Big] \\
&\qquad \times [Q_O(p',s,s) - Q(k',s,s)]\, Q(k'+\xi p',s,s) \\
&\qquad + \left[1-\xi^2+\xi(1-\xi^2)(p'/k') + O((p'/k')^2)\right] \\
&\qquad \times [Q(k'+\xi p',s,s) - Q(k',s,s)]\, Q_O(p',s,s) \Big\} \\
&= \frac{2}{15}\int_{k}^{\infty} {\rm d}k' \int_{0}^{k_{\rm lower}} {\rm d}p' \int_{-\infty}^{t} {\rm d}s\ G(k',t,s)^2 G_O(p',t,s) \\
&\quad \times k'^2 p'^4 \Big\{ 7\,[Q_O(p',s,s) - Q(k',s,s)]\, Q(k',s,s) \\
&\qquad + Q_O(p',s,s) k' \frac{\partial Q(k',s,s)}{\partial k'} + O(p'/k') \Big\} \\
&\quad - \frac{2}{15}\int_{0}^{k_{\rm lower}} {\rm d}p' \int_{-\infty}^{t} {\rm d}s\ G(k,t,s)^2 G_O(p',t,s) \\
&\quad \times k^3 p'^4 \left\{ [Q_O(p',s,s) - Q(k,s,s)]\, Q(k,s,s) + Q_O(p',s,s) k \frac{\partial Q(k,s,s)}{\partial k} \right\},
\end{aligned} \tag{S.14}$$

where we use $\int_{-1}^{1} {\rm d}\xi\ \xi(1-\xi^2) = 0$ and

$$\begin{aligned}
&\int_{k}^{\infty} {\rm d}k'\ k'^3 \frac{\partial G(k',t,s)}{\partial k'} G(k',t,s) G_O(p',t,s) \\
&\times \left\{ [Q_O(p',s,s) - Q(k',s,s)]\, Q(k',s,s) + k'\frac{\partial Q(k',s,s)}{\partial k'} Q_O(p',s,s) \right\} \\
&= -\frac{1}{2} k^3 G(k,t,s)^2 G(p,t,s) \Big\{ [Q_O(p',s,s) - Q(k,s,s)]\, Q(k,s,s)
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + Q_O(p',s,s) k \frac{\partial Q(k,s,s)}{\partial k} \Bigg\} \\
&- \frac{1}{2}\int_k^\infty \mathrm{d}k'\ k'^2 G(k',t,s)^2 G_O(p',t,s) \\
&\quad \times \Bigg\{ 3\left[Q_O(p',s,s) - Q(k',s,s)\right] Q(k',s,s) \\
&\quad + \left[5Q_O(p',s,s) - 2Q(k',s,s)\right] k' \frac{\partial Q(k',s,s)}{\partial k'} + Q_O(p',s,s) k'^2 \frac{\partial^2 Q(k',s,s)}{\partial k'^2} \Bigg\}.
\end{aligned} \tag{S.15}
$$

To derive (S.15), we assume that $\lim_{k'\to\infty} k'^3 G(k',t,s)^2 Q(k',s,s) = 0$ and $\lim_{k'\to\infty} k'^4 \times G(k',t,s)^2 \partial Q(k',s,s)/\partial k' = 0$, which are justified considering that the energy spectral density exponentially decreases at large $k'$. For $k' \geqslant k > p'$, we can evaluate $Q(k',s,s) \leqslant Q(k,s,s) < Q_O(p',s,s) \lesssim \varepsilon^{2/3} k_L^{-11/3}$. Besides, we can evaluate $Q(k',s,s) \lesssim \varepsilon^{7/3-n}(\varepsilon^H)^{-5/3+n} k'^{-n-2}$ for $k' \geqslant k$. Because $Q(k',s,s)$ exponentially decreases at large $k'$, we can also evaluate that $|k'\partial Q(k',s,s)/\partial k'| \lesssim (n+2)\varepsilon^{7/3-n}(\varepsilon^H)^{-5/3+n} k'^{-n-2}$ for $k' \geqslant k$. The time integral of the response function will be bounded by $\int_{-\infty}^t \mathrm{d}s\ G(k',t,s)^2 G_O(p',t,s) \lesssim \varepsilon^{-1/3} k_L^{-2/3}$ for $k' \geqslant k \gg k_{\mathrm{lower}} \geqslant p'$. Consequently, we can evaluate the absolute value of (S.14) as follows:

$$
\begin{aligned}
|I_{O1b}| &\leqslant \frac{2}{15}\int_k^\infty \mathrm{d}k' \int_0^{k_{\mathrm{lower}}} \mathrm{d}p' \int_{-\infty}^t \mathrm{d}s\ G(k',t,s)^2 G_O(p',t,s) \\
&\quad \times k'^2 p'^4 \Bigg\{ 7|Q_O(p',s,s) - Q(k',s,s)| Q(k',s,s) \\
&\qquad\qquad + Q_O(p',s,s) \left| k' \frac{\partial Q(k',s,s)}{\partial k'} \right| + O(p'/k') \Bigg\} \\
&\quad + \frac{2}{15}\int_0^{k_{\mathrm{lower}}} \mathrm{d}p' \int_{-\infty}^t \mathrm{d}s\ G(k,t,s)^2 G_O(p',t,s) \\
&\qquad \times k^3 p'^4 \left\{ |Q_O(p',s,s) - Q(k,s,s)| Q(k,s,s) + Q_O(p',s,s) \left| k \frac{\partial Q(k,s,s)}{\partial k} \right| \right\}, \\
&\lesssim \int_k^\infty \mathrm{d}k' \int_0^{k_{\mathrm{lower}}} \mathrm{d}p'\ \varepsilon^{-1/3} k_L^{-2/3} \times k'^2 p'^4 \times \varepsilon^{2/3} k_L^{-11/3} \times \varepsilon^{7/3-n} (\varepsilon^H)^{-5/3+n} k'^{-n-2} \\
&\quad + \int_0^{k_{\mathrm{lower}}} \mathrm{d}p'\ \varepsilon^{-1/3} k_L^{-2/3} \times k^3 p'^4 \times \varepsilon^{2/3} k_L^{-11/3} \times \varepsilon^{7/3-n} (\varepsilon^H)^{-5/3+n} k^{-n-2} \\
&\quad \simeq \varepsilon (k^H/k)^{-5/3+n} (k_{\mathrm{lower}}/k)^{2/3} (k_{\mathrm{lower}}/k_L)^{13/3}
\end{aligned} \tag{S.16}
$$

where we assume $n > 1$ and omit constant factors. Hence, we can evaluate that $|I_{O1b}|/\varepsilon \to 0$ in the limit $k^H/k \to 0$ and $k_{\mathrm{lower}}/k \to 0$ if $n \geqslant 5/3$ and $k_{\mathrm{lower}}/k_L$ remains finite.

For $I_{O2b}$, we also have $k' \geqslant k_{\mathrm{upper}} - k \gg k \geqslant p'$ and $\max(p', k'-p') = k'-p'$ because we consider the case that $k_{\mathrm{upper}} \gg k$. Therefore, we can apply the same asymptotic analysis as $I_{O1b}$ to $I_{O2b}$. Considering the inequalities $Q(k',s,s) < Q_S(p',s,s)$, $Q(k',s,s) \lesssim \varepsilon^{7/3-n}(\varepsilon^H)^{-5/3+n} k'^{-n-2}$, $|k'\partial Q(k',s,s)/\partial k'| \lesssim (n+2)\varepsilon^{7/3-n}(\varepsilon^H)^{-5/3+n} k'^{-n-2}$, and $\int_{-\infty}^t \mathrm{d}s\ G(k',t,s)^2 G_S(p',t,s) \lesssim \varepsilon^{1/3-\ell}(\varepsilon^H)^{-2/3+\ell} p'^{-\ell}$ with $-4 + 2n + \ell = 0$ (3.18) for $k' > k \geqslant p'$, we have

$$
|I_{O2b}| \lesssim \int_{k_{\mathrm{upper}}-k}^\infty \mathrm{d}k' \int_{k_{\mathrm{lower}}}^k \mathrm{d}p'\ \varepsilon^{-11/3+2n} (\varepsilon^H)^{10/3-2n} p'^{-4+2n} \times k'^2 p'^4
$$

$$
\begin{aligned}
&\times \varepsilon^{7/3-n}(\varepsilon^H)^{-5/3+n}p'^{-n-2} \times \varepsilon^{7/3-n}(\varepsilon^H)^{-5/3+n}k'^{-n-2} \\
&+ \int_{k_{\mathrm{lower}}}^{k} \mathrm{d}p'\ \varepsilon^{-11/3+2n}(\varepsilon^H)^{10/3-2n}p'^{-4+2n} \times (k_{\mathrm{upper}}-k)^3 p'^4 \\
&\quad \times \varepsilon^{7/3-n}(\varepsilon^H)^{-5/3+n}p'^{-n-2} \times \varepsilon^{7/3-n}(\varepsilon^H)^{-5/3+n}(k_{\mathrm{upper}}-k)^{-n-2} \\
&\simeq \varepsilon(k/k_{\mathrm{upper}})^{-1+n}[1-(k_{\mathrm{lower}}/k)^{1-n}](1-k/k_{\mathrm{upper}})^{1-n},
\end{aligned}
\tag{S.17}
$$

where we assume $n > 1$. If $n > 1$, the right-hand side of (S.17) vanishes in the limit $k_{\mathrm{lower}}/k \to 0$ and $k_{\mathrm{upper}}/k \to \infty$. The conventional inertial range scaling $n = 5/3$ agrees with this condition. Consequently, we can evaluate that $|I_{O2b}|/\varepsilon \to 0$ in the limit $k_{\mathrm{lower}}/k \to 0$ and $k_{\mathrm{upper}}/k \to \infty$ if $n > 1$.

The asymptotic analyses of $I_{O1c}$ and $I_{O2c}$ are the same as those of $I_{O1b}$ and $I_{O2b}$. Putting $q' = k' + \xi p'$ with $\xi \in [-1, 1]$, the asymptotes of $c_{k'p'q'}$ and $c_{k'q'p'}$ for a small $p'/k'$ yield

$$
c_{k'p'q'} = -\xi(1-\xi^2)p'/k' + \frac{1}{2}(1-\xi^4)(p'/k')^2 + O((p'/k')^3), \tag{S.18$a$}
$$

$$
c_{k'q'p'} = (1-\xi^2) - \frac{3}{4}(1-\xi^2)^2(p'/k')^2 + O((p'/k')^3). \tag{S.18$b$}
$$

Then, we have

$$
\begin{aligned}
I_{O1c} =& \int_k^\infty \mathrm{d}k' \int_0^{k_{\mathrm{lower}}} \mathrm{d}p' \int_{-1}^{1} \mathrm{d}\xi\ p' \int_{-\infty}^{t} \mathrm{d}s\ G(k',t,s)G_O(p',t,s)G(k'+\xi p',t,s) \\
&\times k'^2 p'(1+\xi p'/k') \Big\{ \Big[ -\xi(1-\xi^2)(p'/k') + \frac{1}{2}(1-\xi^4)(p'/k')^2 + O((p'/k')^3) \Big] \\
&\qquad \times \left[ Q_O^H(p',s,s) - Q^H(k',s,s) \right] Q^H(k'+\xi p',s,s) \\
&\qquad + \left[ 1-\xi^2 + O((p'/k')^2) \right] \\
&\qquad \times \left[ Q^H(k'+\xi p',s,s) - Q^H(k',s,s) \right] Q_O^H(p',s,s) \Big\} \\
=& \frac{2}{15}\int_k^\infty \mathrm{d}k' \int_0^{k_{\mathrm{lower}}} \mathrm{d}p' \int_{-\infty}^{t} \mathrm{d}s\ G(k',t,s)^2 G_O(p',t,s) \\
&\times p'^4 \Big\{ 5\left[ Q_O^H(p',s,s) - Q^H(k',s,s) \right] Q^H(k',s,s) \\
&\qquad - Q_O^H(p',s,s)k' \frac{\partial Q^H(k',s,s)}{\partial k'} + O(p'/k') \Big\} \\
&+ \frac{2}{15}\int_0^{k_{\mathrm{lower}}} \mathrm{d}p' \int_{-\infty}^{t} \mathrm{d}s\ G(k,t,s)^2 G_O(p',t,s) \\
&\times kp'^4 \left\{ \left[ Q_O^H(p',s,s) - Q^H(k,s,s) \right] Q^H(k,s,s) - Q_O^H(p',s,s) k \frac{\partial Q^H(k,s,s)}{\partial k} \right\},
\end{aligned}
\tag{S.19}
$$

where we also assume that $\lim_{k'\to\infty} k'^3 G(k',t,s)^2 Q^H(k',s,s) = 0$ and $\lim_{k'\to\infty} k'^4 \times G(k',t,s)^2 \partial Q^H(k',s,s)/\partial k' = 0$. The realisability condition (1.1) accompanied by $\varepsilon L/u^3 =$ const. suggests that the helicity spectral density in $p' \leqslant k_{\mathrm{lower}}$ is bounded by $|Q_O^H(p',s,s)| \lesssim \varepsilon^{2/3} k_L^{-8/3}$. Considering the inequalities, $|Q^H(k',s,s)| \leqslant |Q^H(k,s,s)| < |Q_O^H(p',s,s)|$, $|Q(k',s,s)| \lesssim \varepsilon^{4/3-m}(\varepsilon^H)^{-2/3+m}k'^{-m-2}$, $|k'\partial Q^H(k',s,s)/\partial k'| \lesssim (m+2)\varepsilon^{4/3-n}(\varepsilon^H)^{-2/3+m}k'^{-m-2}$, and $\int_{-\infty}^{t} \mathrm{d}s\ G(k',t,s)^2 G_O(p',t,s) \lesssim \varepsilon^{-1/3}k_L^{-2/3}$ for $k' \geqslant$

$k \gg k_{\rm lower} \geqslant p'$, we can evaluate $|I_{O1c}|$ as

$$
\begin{aligned}
|I_{O1c}| \leqslant& \frac{2}{15}\int_k^\infty \mathrm{d}k' \int_0^{k_{\rm lower}} \mathrm{d}p' \int_{-\infty}^t \mathrm{d}s\ G(k',t,s)^2 G_O(p',t,s) \\
&\times p'^4 \Big\{ 5\left|Q_O^H(p',s,s) - Q^H(k',s,s)\right| |Q^H(k',s,s)| \\
&\qquad + |Q_O^H(p',s,s)| \left| k' \frac{\partial Q^H(k',s,s)}{\partial k'} \right| + O(p'/k') \Big\} \\
&+ \frac{2}{15}\int_0^{k_{\rm lower}} \mathrm{d}p' \int_{-\infty}^t \mathrm{d}s\ G(k,t,s)^2 G_O(p',t,s) \\
&\times kp'^4 \Big\{ \left|Q_O^H(p',s,s) - Q^H(k,s,s)\right| Q^H(k,s,s) \\
&\qquad + |Q_O^H(p',s,s)| \left| k \frac{\partial Q^H(k,s,s)}{\partial k} \right| \Big\} \\
\lesssim& \int_k^\infty \mathrm{d}k' \int_0^{k_{\rm lower}} \mathrm{d}p'\ \varepsilon^{-1/3} k_L^{-2/3} \times p'^4 \times \varepsilon^{2/3} k_L^{-8/3} \times \varepsilon^{4/3-m} (\varepsilon^H)^{-2/3+m} k'^{-m-2} \\
&+ \int_0^{k_{\rm lower}} \mathrm{d}p'\ \varepsilon^{-1/3} k_L^{-2/3} \times kp'^4 \times \varepsilon^{2/3} k_L^{-8/3} \times \varepsilon^{4/3-m} (\varepsilon^H)^{-2/3+m} k^{-m-2} \\
\simeq& \varepsilon (k^H/k)^{-2/3+m} (k_{\rm lower}/k)^{5/3} (k_{\rm lower}/k_L)^{10/3}, \qquad \text{(S.20)}
\end{aligned}
$$

where we assume $m > -1$. Consequently, we can verify that $|I_{O1c}|/\varepsilon \to 0$ in the limit $k^H/k \to 0$ and $k_{\rm lower}/k \to 0$ if $m \geqslant 2/3$ and $k_{\rm lower}/k_L$ remains finite. Similarly, the asymptotic analysis of $|I_{O2c}|$ yields

$$
\begin{aligned}
|I_{O2c}| \lesssim& \int_{k_{\rm upper}-k}^\infty \mathrm{d}k' \int_{k_{\rm lower}}^k \mathrm{d}p'\ \varepsilon^{-11/3+2n} (\varepsilon^H)^{10/3-2n} p'^{-4+2n} \times p'^4 \\
&\times \varepsilon^{4/3-m} (\varepsilon^H)^{-2/3+m} p'^{-m-2} \times \varepsilon^{4/3-m} (\varepsilon^H)^{-2/3+m} k'^{-m-2} \\
&+ \int_{k_{\rm lower}}^k \mathrm{d}p'\ \varepsilon^{-11/3+2n} (\varepsilon^H)^{10/3-2n} p'^{-4+2n} \times (k_{\rm upper}-k) p'^4 \\
&\times \varepsilon^{4/3-m} (\varepsilon^H)^{-2/3+m} p'^{-m-2} \times \varepsilon^{4/3-m} (\varepsilon^H)^{-2/3+m} (k_{\rm upper}-k)^{-m-2} \\
\simeq& \varepsilon (k^H/k)^{2(1-n+m)} (k/k_{\rm upper})^{1+m} [1 - (k_{\rm lower}/k)^{-1+2n-m}] (1 - k/k_{\rm upper})^{-1-m}, \qquad \text{(S.21)}
\end{aligned}
$$

where we assume $m > -1$ and employ the inequalities $|Q^H(k',s,s)| < |Q_S^H(p',s,s)|$, $|Q^H(k',s,s)| \lesssim \varepsilon^{4/3-m}(\varepsilon^H)^{-2/3+m} k'^{-m-2}$, $|k'\partial Q^H(k',s,s)/\partial k'| \lesssim (m+2)\varepsilon^{4/3-m} \times (\varepsilon^H)^{-2/3+m} k'^{-m-2}$, and $\int_{-\infty}^t \mathrm{d}s\ G(k',t,s)^2 G_S(p',t,s) \lesssim \varepsilon^{1/3-\ell} (\varepsilon^H)^{-2/3+\ell} p'^{-\ell}$ with $-4+2n+\ell = 0$ (3.18) for $k' > k \geqslant p'$. Consequently, we can verify that $|I_{O2c}|/\varepsilon \to 0$ in the limit $k^H/k \to 0$, $k_{\rm lower}/k \to 0$, and $k_{\rm upper}/k \to \infty$ if $1-n+m \geqslant 0$, $-1+2n-m \geqslant 0$, and $m > -1$.

By putting $k' = k/v$, $p' = k'r$, $q' = k'w$, and $\gamma\tau = \omega_{k'}(t-s)$, $I_{Sb} + I_{Sc}$ yields

$$
\begin{aligned}
I_{Sb} + I_{Sc} =& \gamma \int_{1/(k_{\rm upper}/k-1)}^1 \mathrm{d}v \int_{vk_{\rm lower}/k}^v \mathrm{d}r \int_{\max(r,1-r)}^{1+r} \mathrm{d}w \int_0^\infty \mathrm{d}\tau G_S(\tau) G_S(r^\ell \tau) G_S(w^\ell \tau) \\
&\times \omega_{k/v}^{-1} \frac{1}{v} \left(\frac{k}{v}\right)^3 \left[ f_b(k/v, rk/v, wk/v) + f_c(k/v, rk/v, wk/v) \right]. \qquad \text{(S.22)}
\end{aligned}
$$

Equation (S.22) multiplied by $4\pi^2$ yields (3.17) in the limit $k_{\mathrm{lower}}/k \to 0$ and $k_{\mathrm{upper}}/k \to \infty$ when the similarity laws (3.1), (3.2), and (3.4) are substituted. Note that the $I_{Sc}$ part will vanish in the limit $k^H/k \to 0$ as discussed in §3.3. Finally, we obtain the energy flux composed of the contributions solely from the similarity range wavenumbers in these limits. The required conditions are $k_{\mathrm{lower}}/k_L$ remains finite, $n \geqslant 5/3$, $m \geqslant 2/3$, $1-n+m \geqslant 0$, and $-1+2n-m \geqslant 0$ with $-4+2n+\ell = 0$ (3.18).

## S.3. *Helicity flux*

The helicity flux can also be decomposed into

$$\varPi^H(k)/(4\pi^2) = I_S^H + I_{O1}^H + I_{O2}^H, \tag{S.23}$$

where

$$I_S^H = \int_k^{k_{\mathrm{upper}}-k} \mathrm{d}k' \int_{k_{\mathrm{lower}}}^{k} \mathrm{d}p' \int_{\max(p',k'-p')}^{k'+p'} \mathrm{d}q' \int_{-\infty}^{t} \mathrm{d}s \; G_S(k',t,s) G_S(p',t,s) G_S(q',t,s) \times f^H(k',p',q',s), \tag{S.24$a$}$$

$$I_{O1}^H = \int_k^{\infty} \mathrm{d}k' \int_0^{k_{\mathrm{lower}}} \mathrm{d}p' \int_{\max(p',k'-p')}^{k'+p'} \mathrm{d}q' \int_{-\infty}^{t} \mathrm{d}s \; G(k',t,s) G_O(p',t,s) G(q',t,s) \times f^H(k',p',q',s), \tag{S.24$b$}$$

$$I_{O2}^H = \int_{k_{\mathrm{upper}}-k}^{\infty} \mathrm{d}k' \int_{k_{\mathrm{lower}}}^{k} \mathrm{d}p' \int_{\max(p',k'-p')}^{k'+p'} \mathrm{d}q' \int_{-\infty}^{t} \mathrm{d}s \; G(k',t,s) G_S(p',t,s) G(q',t,s) \times f^H(k',p',q',s), \tag{S.24$c$}$$

and

$$\begin{aligned} f^H(k',p',q',s) = k'^3 p' q' \Bigg[ & \left( b_{k'p'q'} - \frac{q'^2}{k'^2} c_{k'q'p'} \right) Q^H(p',s,s) Q(q',s,s) \\ & + \left( b_{k'q'p'} - \frac{p'^2}{k'^2} c_{k'p'q'} \right) Q(p',s,s) Q^H(q',s,s) \\ & - b_{k'p'q'} Q^H(k',s,s) Q(q',s,s) + c_{k'p'q'} Q(k',s,s) Q^H(q',s,s) \\ & - b_{k'q'p'} Q^H(k',s,s) Q(p',s,s) + c_{k'q'p'} Q(k',s,s) Q^H(p',s,s) \Bigg]. \end{aligned} \tag{S.25}$$

Using the same asymptotic analysis demonstrated in §S.2, we have

$$\begin{aligned} I_{O1}^H = & \frac{2}{15} \int_k^{\infty} \mathrm{d}k' \int_0^{k_{\mathrm{lower}}} \mathrm{d}p' \int_{-\infty}^{t} \mathrm{d}s \; G(k',t,s)^2 G_O(p',t,s) \\ & \times k'^2 p'^4 \Bigg\{ 7 Q_O^H(p',s,s) Q(k',s,s) + Q_O^H(p',s,s) k' \frac{\partial Q(k',s,s)}{\partial k'} \\ & \qquad + Q_O^H(p',s,s) k'^2 \frac{\partial^2 Q(k',s,s)}{\partial k'^2} - Q_O(p',s,s) k'^2 \frac{\partial^2 Q^H(k',s,s)}{\partial k'^2} \Bigg\} \\ & + \frac{2}{15} \int_0^{k_{\mathrm{lower}}} \mathrm{d}p' \int_{-\infty}^{t} \mathrm{d}s \; G(k,t,s)^2 G_O(p',t,s) \\ & \times k^3 p'^4 \Big\{ - \left[ Q_O^H(p',s,s) + 2 Q^H(k,s,s) \right] Q(k,s,s) \end{aligned}$$

$$- Q_O^H(p', s, s) k \frac{\partial Q(k, s, s)}{\partial k} + Q_O(p', s, s) k \frac{\partial Q^H(k, s, s)}{\partial k} \bigg\}. \qquad \text{(S.26)}$$

Owing to the inequalities the same as used in §S.2, we can evaluate that

$$\begin{aligned}
|I_{O1}^H| \lesssim & \int_k^\infty \mathrm{d}k' \int_0^{k_{\mathrm{lower}}} \mathrm{d}p'\ \varepsilon^{-1/3} k_L^{-2/3} \times k'^2 p'^4 \\
& \times \Big[ \varepsilon^{2/3} k_L^{-8/3} \times \varepsilon^{7/3-n} (\varepsilon^H)^{-5/3+n} k'^{-n-2} \\
& \quad + \varepsilon^{2/3} k_L^{-11/3} \times \varepsilon^{4/3-m} (\varepsilon^H)^{-2/3+m} k'^{-m-2} \Big] \\
& \quad + \int_0^{k_{\mathrm{lower}}} \mathrm{d}p'\ \varepsilon^{-1/3} k_L^{-2/3} \times k^3 p'^4 \\
& \quad \times \Big[ \varepsilon^{2/3} k_L^{-8/3} \times \varepsilon^{7/3-n} (\varepsilon^H)^{-5/3+n} k^{-n-2} \\
& \qquad + \varepsilon^{4/3-m} (\varepsilon^H)^{-2/3+m} k^{-m-2} \times \varepsilon^{7/3-n} (\varepsilon^H)^{-5/3+n} k^{-n-2} \\
& \qquad + \varepsilon^{2/3} k_L^{-11/3} \times \varepsilon^{4/3-m} (\varepsilon^H)^{-2/3+m} k^{-m-2} \Big] \\
& \simeq \varepsilon^H (k_{\mathrm{lower}}/k_L)^{13/3} (k_{\mathrm{lower}}/k)^{2/3} (k^H/k)^{-5/3+m} \\
& \quad \times \Big[ (k^H/k)^{n-m} k_L/k^H + (k^H/k)^{-5/3+n} (k_L/k)^{11/3} + 1 \Big], \qquad \text{(S.27)}
\end{aligned}$$

where we assume $n > 1$ and $m > 1$. Even if $n = m = 5/3$, we have to additionally require that $k_L/k^H$ remains finite for $|I_{O1}^H|/\varepsilon^H \to 0$ in the limit $k_{\mathrm{lower}}/k \to 0$ and $k^H/k \to 0$. This result comes from the crude bound on the helicity spectral density based on the realizability condition; namely, $Q_O^H(p', s, s) \lesssim k_L^{-1} Q_O(p', s, s) \simeq \varepsilon^{2/3} k_L^{-8/3}$ for $p' \leqslant k_{\mathrm{lower}}$. In the decaying homogeneous helical turbulence, Briard & Gomez (2017) suggested that $\varepsilon \langle \boldsymbol{u}^2 \rangle \sim \varepsilon^H \langle \boldsymbol{u} \cdot \boldsymbol{\omega} \rangle$ in terms of the EDQNM. In such a case, the helicity spectrum density is bounded by $Q_O^H(p', s, s) \lesssim (k^H)^{-1} Q_O(p', s, s) \simeq \varepsilon^{2/3} k_L^{-8/3} (k_L/k^H)$. If we employ this inequality, we can evaluate that $|I_{O1}^H|/\varepsilon^H \to 0$ in the limit $k_{\mathrm{lower}}/k \to 0$, $k_L/k \to 0$, and $k^H/k \to 0$ if $n \geqslant m \geqslant 5/3$ without assuming that $k_L/k^H$ remains finite. Note that several numerical simulations suggest that $k_L/k^H \simeq O(1)$ (Borue & Orszag 1997; Baerenzung *et al.* 2008; Mininni & Pouquet 2009; Deusebio & Lindborg 2014). In such a case, the evaluation that $|I_{O1}^H|/\varepsilon^H \to 0$ in the limit $k_{\mathrm{lower}}/k \to 0$, $k_L/k \to 0$, and $k^H/k \to 0$ may be reasonable. Further verification is needed to evaluate the ratio $k_L/k^H$ in more general helical turbulent flows.

Similarly, we can evaluate $|I_{O2}^H|$ as

$$\begin{aligned}
|I_{O2}^H| \lesssim & \int_k^\infty \mathrm{d}k' \int_{k_{\mathrm{lower}}}^k \mathrm{d}p'\ \varepsilon^{-11/3+n+m} (\varepsilon^H)^{10/3-n-m} p'^{-4+n+m} \times k'^2 p'^4 \\
& \times \Big[ \varepsilon^{4/3-m} (\varepsilon^H)^{-2/3+m} p'^{-m-2} \times \varepsilon^{7/3-n} (\varepsilon^H)^{-5/3+n} k'^{-n-2} \\
& \quad + \varepsilon^{7/3-n} (\varepsilon^H)^{-5/3+n} p'^{-n-2} \times \varepsilon^{4/3-m} (\varepsilon^H)^{-2/3+m} k'^{-m-2} \Big] \\
& + \int_{k_{\mathrm{lower}}}^k \mathrm{d}p'\ \varepsilon^{-11/3+n+m} (\varepsilon^H)^{10/3-n-m} p'^{-4+n+m} \times (k_{\mathrm{upper}} - k)^3 p'^4 \\
& \times \Big[ \varepsilon^{4/3-m} (\varepsilon^H)^{-2/3+m} p'^{-m-2} \times \varepsilon^{7/3-n} (\varepsilon^H)^{-5/3+n} (k_{\mathrm{upper}} - k)^{-n-2} \\
& \quad + \varepsilon^{4/3-m} (\varepsilon^H)^{-2/3+m} (k_{\mathrm{upper}} - k)^{-m-2} \times \varepsilon^{7/3-n} (\varepsilon^H)^{-5/3+n} (k_{\mathrm{upper}} - k)^{-n-2} \\
& \quad + \varepsilon^{7/3-n} (\varepsilon^H)^{-5/3+n} p'^{-n-2} \times \varepsilon^{4/3-m} (\varepsilon^H)^{-2/3+m} (k_{\mathrm{upper}} - k)^{-m-2} \Big]
\end{aligned}$$

$$\simeq \varepsilon^H \left\{ \left[ (1 - (k_{\mathrm{lower}}/k)^{n-1} \right] (1 - k/k_{\mathrm{upper}})^{1-n} + \left[ (1 - (k_{\mathrm{lower}}/k)^{1+n+m} \right] (1 - k/k_{\mathrm{upper}})^{-1-n-m} + \left[ (1 - (k_{\mathrm{lower}}/k)^{m-1} \right] (1 - k/k_{\mathrm{upper}})^{1-m} \right\}, \tag{S.28}$$

where we assume $n > 1$ and $m > 1$ and use $-4 + n + m + \ell = 0$ (3.32). Consequently, we can verify that $|I^H_{O2}|/\varepsilon^H \to 0$ in the limit $k_{\mathrm{lower}}/k \to 0$ and $k_{\mathrm{upper}}/k \to \infty$ if $n > 1$ and $m > 1$.

For $I^H_S$, we readily confirm that $8\pi^2 I^H_S$ yields (3.31) in the limit $k_{\mathrm{lower}}/k \to 0$, and $k_{\mathrm{upper}}/k \to \infty$ using the same procedure as used in deriving (S.22). Finally, we obtain the helicity flux composed of the contributions solely from the similarity range wavenumbers in these limits. The required conditions are $k_{\mathrm{lower}}/k_L$ and $k_L/k^H$ remain finite, $n \geqslant 5/3$, $m \geqslant 5/3$, and $n - m \geqslant 0$ with $-4 + n + m + \ell = 0$ (3.32).

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