Other supplementary material

In this supplement (Supplements A–E), we present the technical details required to reproduce the details of our analysis.

Supplement A. Hakkinen–Rott wake: higher-order scheme

To elucidate the structure of (3.2) and (3.5) in more detail, we have to match the latter in the limit $Z \to 0$ to the appropriate expansion

$$F := \frac{\Psi}{\Lambda_0^{1/3} X^{2/3}} \sim f_{\mathrm{HR}}(\eta) + \frac{c_1 X^{1/3}}{\Lambda_0^{4/3}}[\ln X F_1(\eta) + F_2(\eta)] + O\big(X^{2/3}(\ln X)^2\big) \quad (A\,1)$$

holding for $\eta = O(1)$ as $X \to 0+$ and with $F_{1,2}$ to be found. The logarithmic contributions to (3.2)–(3.4) and (A 1), by anticipating $c_{1,2} \neq 0$, comply with the representation

$$\Psi \sim \Lambda_0 Z^2/2 + P'(0-)Z^3/6 + (P - P_-)[\Lambda_0^{-1} + P'(0-)\Lambda_0^{-2} Z \ln Z + O(Z)] \quad (A\,2)$$

of (3.5). This is found with the help of (2.16), in the overlap $1 \gg Z \gg X^{1/3}$ or $X^{-1/3} \gg \eta \gg 1$, see (3.1), with the expansion (A 1). There they cause a reordering of its terms arising for large values of η, as seen from expressing Z in (A 2) in terms of η:

$$F \sim \frac{\eta^2}{2} + \frac{P - P_-}{\Lambda_0^{4/3} X^{2/3}} + \frac{P'(0-) X^{1/3} \eta}{3\Lambda_0^{4/3}} \left\{ \frac{\eta^2}{2} + \frac{P - P_-}{\Lambda_0^{4/3} X^{2/3}} \big[\ln X + 3\ln\eta + O(1)\big] \right\}. \quad (A\,3)$$

As it is the HRW where the inertia terms in (2.14a) are fully restored, the Z-independent contribution to Ψ in (A 2) is again in agreement with (2.14c). Thus, matching the X-independent terms in (A 1) and (A 3) confirms that $f_{\mathrm{HR}} - \eta^2/2 \sim p_{\mathrm{HR}} + \mathrm{TST}$ ($\eta \to \infty$); matching the higher-order terms requires the successive emergence of the logarithmic terms.

To this end, we substitute (A 1) into (2.14a)–(2.14c) and exploit (A 3) with $\eta \gg 1$ kept fixed. This reveals two inhomogeneous linear problems governing F_i ($i = 1, 2$):

$$f'_{\mathrm{HR}} F'_i - f''_{\mathrm{HR}} F_i - 2 f_{\mathrm{HR}} F''_i/3 = F'''_i - G_i(\eta), \quad F_i(0) = F''_i(0) = 0, \quad (A\,4a)$$

$$G_1 := 1, \quad G_2 := 1 + c_2/c_1 + f'_{\mathrm{HR}} F'_1 - f''_{\mathrm{HR}} F_1. \quad (A\,4b)$$

The well-known behaviour of f_{HR} entails the essentially algebraic growth

$$F_1 \sim \gamma_1(\eta^3 + 6\, p_{\mathrm{HR}} \eta \ln\eta) + \delta_1 \eta - 6\gamma_1 + G_1 + O(\eta^{-1}) \quad (\eta \to \infty) \quad (A\,5)$$

with some constants γ_1, δ_1. This is confirmed by expanding (A 3) up to $O(X^{1/3} \ln X)$ provided that $\gamma_1 = 0$ and $c_1 = p_{\mathrm{HR}} P'(0-)/(3\delta_1)$. Then $F''_1(\infty) = 0$, and (A 4) has a unique non-trivial solution in the case $i = 1$. The numerical method and discretisation we used to compute f_{HR} gives $\delta_1 \simeq -2.6110$, implying $c_1 > 0$. (We also note that $F'_1(0) \simeq -1.8422$.) In turn, $G_2 = 1 + c_2/c_1 + \mathrm{TST}$ ($\eta \to \infty$) so that (A 5) holds also for the corresponding quantities having the subscript 2. Accordingly, expanding (A 3) up to $O(X^{1/3})$ then gives

$\gamma_2 = P'(0-)/(6c_1) = \delta_1/(2\,p_{\rm HR})$. This fixes the missing BC as $F_2'' \sim 3\delta_1\eta/p_{\rm HR}$ ($\eta \to \infty$). Adjusting $F_2'(0)$, however, allows this condition to be satisfied for any c_2, which can be determined by proceeding in this manner and considering the abbreviated remainder terms in (A 1) and (A 3).

Supplement *B*. Outer Rayleigh problem: wavenumber space

We investigate the Rayleigh problem (3.22) in the distinguished limit $T \to 1-$ and $k \to 0$, for $|\mathrm{Re}\,k| \to \infty$ and the eigenspace (poles) of $\mathcal{V}$ in detail.

B.1. *Singular limits* $k \to 0$, $T \to 1-$

The analysis of the long-wave limit $k \to 0$ is closely related to the discussion of (3.15) with (3.16). We substitute the expansion $\mathcal{V} \sim \sum_{i=0}^{\infty} \kappa_i(k)\mathcal{V}_i(y)$ ($\kappa_{i+1}/\kappa_i \to 0$) with, at first, unknown gauge functions κ_i and $O(1)$-functions $\mathcal{V}_i$ into (3.22). At leading order, the problem (3.22) then only permits the homogeneous solution parametrised by κ_0, thus $\mathcal{V}_0 = \psi_0'$. To allow for deviations from this, the analysis of $\mathcal{V}_1$ requires $\kappa_1 = O(k^2\kappa_0)$. Specifying $\kappa_1 := k^2\kappa_0$ yields the inhomogeneous problem

$$\mathcal{V}_1'' - (\psi_0'''/\psi_0')\mathcal{V}_1 = \psi_0', \quad \mathcal{V}_1(0) = -\kappa_1^{-1}, \quad u_0^+\mathcal{V}_1'(h_0) = TJ, \tag{B1}$$

where we have anticipated that $\kappa_1 = O(1)$, including the alternative $\kappa_1 \to \infty$ as $k \to 0$ as a limiting case. By using the first BC here and after some rearrangements,

$$\frac{\mathcal{V}_1(y)}{\psi_0'(y)} = \alpha_1 - \lambda\,\kappa_1^{-1}\int_y^{h_0}\frac{\mathrm{d}t}{\psi_0'^2(t)} + \int_0^y \frac{\mathrm{d}t}{\psi_0'^2(t)}\int_0^t \psi_0'^2(s)\,\mathrm{d}s. \tag{B2}$$

The initially arbitrary constant α_1 indicates again a homogeneous solution and the first BC in (B1) is met in the limit $y \to 0$. The second BC in (B1) represents a solvability condition for (B1) as it gives

$$\lambda\,\kappa_1^{-1} = (T-1)J \tag{B3}$$

to fix $\kappa_0 = \kappa_1/k^2$ with $\kappa_1 = \lambda/[(T-1)J]$.

This analysis reveals a double pole of $\mathcal{V}$ at $k = 0$, the strength of which becomes unbounded as $T \to 1$. It refers to an apparent solution $|\bar{X}|\psi_0'(y)$ ($\bar{X} \neq 0$) to the homogeneous problem formed by (3.14a,c). Since k^2 enters (3.22) linearly, the above expansion is now specified as

$$\mathcal{V} \sim \kappa_0(k)\sum_{i=0}^{\infty} k^{2i}\mathcal{V}_i(y), \quad \mathcal{V}_0 = \psi_0'. \tag{B4}$$

and breaks down passively where $y = O(k^2)$. We solve the resulting hierarchy of the inhomogeneous problems

$$\mathcal{V}_i'' - (\psi_0'''/\psi_0')\mathcal{V}_i = \mathcal{V}_{i-1}, \quad \mathcal{V}_i(0) = 0, \quad u_0^{+2}\,\mathcal{V}_i'(h_0) = TJ\mathcal{V}_{i-1}(h_0) \quad (i > 1) \tag{B5}$$

using the approach that leads to (B2):

$$\frac{\mathcal{V}_i(y)}{\psi_0'(y)} = \alpha_i + \int_0^y \frac{\mathrm{d}t}{\psi_0'^2(t)}\int_0^t \psi_0'(s)\mathcal{V}_{i-1}(s)\,\mathrm{d}s \tag{B6}$$

where the last BC in (B5) fixes the constant α_{i-1} as a function of T in terms of a solvability condition for the problem governing $\mathcal{V}_{i-1}$. We finally write this

constraint after some manipulations as the recursive relationship

$$\frac{\mathcal{V}_i(h_0)}{u_0^+} = \frac{\mathcal{I}_i}{1-T}, \quad \mathcal{I}_i := \frac{1}{J}\int_0^{h_0} \psi_0'^2(y)\,\mathrm{d}y \int_y^{h_0} \frac{\mathrm{d}t}{\psi_0'^2(t)} \int_0^t \psi_0'(s)\mathcal{V}_{i-1}(s)\,\mathrm{d}s. \tag{B 7}$$

Also, $\mathcal{V}_i$ $(i > 0)$ is inversely proportional to $1 - T$.

Inspection of (3.14) reveals immediately the expansion $\bar{V} \sim \bar{a}_+\psi_0'(y) - \sum_{i=1}^{\infty} \bar{a}_+^{(2i)}\mathcal{V}_i(y)$ $(\bar{X} \to \infty)$, which completes (3.15) subject to (3.16) as the reciprocal form to (B 4). From this, one infers that $\bar{b}_+ = -\alpha_1$.

The above analysis ceases to be valid when $1 - T$ is so small that $\kappa_1^{-1} = O(k^2)$ and thus no longer enters (B 1) but, instead, the problem governing $\mathcal{V}_2$. However, the asymptotic series (B 4) captures this shift of the lower BC formally when we introduce the (positive) parameter

$$\mathcal{T} := (1-T)/k^2 = O(1) \tag{B 8}$$

to quantify the consequent least-degenerate distinguished limit. Then κ_1^{-1} is replaced by 0 and T by 1 in (B 1), (B 2) such that (B 3) is satisfied identically. Most importantly, the BCs in (B 5) are modified to

$$\mathcal{V}_2(0) = -\kappa_2^{-1}, \quad \mathcal{V}_i(0) = 0 \ \ (i > 2), \quad u_0^{+2}\,\mathcal{V}_i'(h_0) = J[\mathcal{V}_{i-1} - \mathcal{T}\mathcal{V}_{i-2}](h_0) \ \ (i > 1) \tag{B 9}$$

where we take $\kappa_2 = k^4\kappa_0$ as of $O(1)$. Hence,

$$\frac{\mathcal{V}_2(y)}{\psi_0'(y)} = \alpha_2 - \lambda\,\kappa_2^{-1}\int_y^{h_0} \frac{\mathrm{d}t}{\psi_0'^2(t)} + \int_0^y \frac{\mathrm{d}t}{\psi_0'^2(t)} \int_0^t \psi_0'(s)\mathcal{V}_1(s)\,\mathrm{d}s. \tag{B 10}$$

The special form of the dynamic BC in (B 9) determines the value of the constant α_{i-2} for $i > 2$. It is sufficient for our purposes to concentrate on this BC for $i = 2$. As $\mathcal{V}_1$ is given by (B 6) for $i = 1$, this solvability condition for (B 10) yields, with $\mathcal{V}_0$ specified in (B 4) and the definitions of $\mathcal{I}_1$ in (B 7) and $\mathcal{T}$ in (B 8), and after some rearrangements, an expression for κ_2 and thus κ_0, independent of the value of α_1:

$$J\kappa_0/\lambda = k^{-2}[k^2\mathcal{I}_1 - (1-T)]^{-1}. \tag{B 11}$$

We find that the distinguished limit (B 8) is rich enough to disclose the behaviour of $\mathcal{V}$ near the critical point $k = 0$ and $T = 1$ (although the last situation is excluded in this study). First, one readily finds that $\mathcal{V}$ admits a regular expansion in $T - 1$ as $T \to 1$ for $k \neq 0$ or $\mathcal{T} \to 0$. It is seen that $\mathcal{V}$ attains a fourth-order pole in $k = 0$, $T = 1$ which morphs into a double pole for $T < 1$, here recovered in the limit $\mathcal{T} \to \infty$ where the two forms of (B 4) considered match. This behaviour is associated with the bifurcation of a simple pole for k becoming positive, whose location we trace in the (T, k)-plane as

$$k = k_u(T), \tag{B 12}$$

indicating the existence of undamped capillary oscillations, where $\mathcal{T} \sim \mathcal{I}_1$ or

$$T \sim 1 - \mathcal{I}_1 k_u^2 + O(k_u^4) \quad (k_u \to 0). \tag{B 13}$$

As suggested by these asymptotic findings, our numerical study predicts exactly one value of $k^2 = k_u^2$ for each value of T in the relevant interval $0 < T < 1$. We also infer from $\mathcal{V} \sim \mathrm{Res}_{k=k_u}(\mathcal{V})/(k - k_u)$ $(k \to k_u)$, (B 4), (B 11) and (B 13) that

$$\mathrm{Res}_{k=k_u}(\mathcal{V})/\psi_0'(y) \sim \lambda/(2J\mathcal{I}_1 k_u^3) \sim \lambda\,\mathcal{I}_1^{1/2}/\left[2J(1-T)^{3/2}\right] \quad (k_u \to 0). \tag{B 14}$$

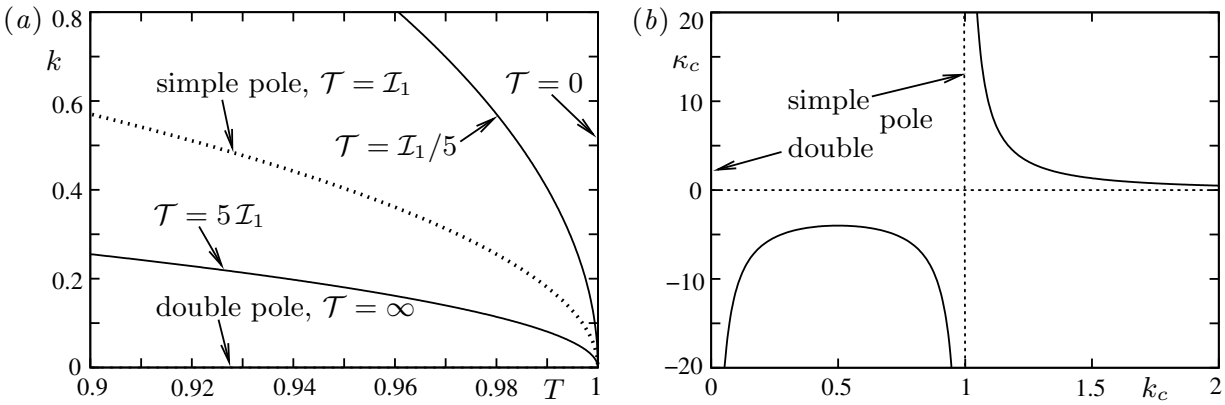

Figure S 1: Plots of (a) poles and (b) variation of $\mathcal{V}$ near $k = 0$, $T = 1$.

The amplitude $\bar{a}_u$, see (3.25), varies predominantly with $k_u^{-14/3}$ or $(1-T)^{-7/3}$ in this limit. To exploit the above results numerically, we specify ψ_0' by Watson's flow, using (2.7) with $x_v = -1$. Then $\mathcal{I}_1$ can be transformed into a single integral, and we add

$$J = \lambda, \quad \mathcal{I}_1 = \frac{2}{u_0^+} - \frac{1}{\lambda h_0} - \frac{1}{\lambda} + \lambda \int_0^{h_0} \left[\frac{1}{\psi_0'^2(y)} - \frac{1}{(\lambda y)^2}\right] \mathrm{d}y \simeq 0.307059. \quad (B\,15)$$

The relationships (B 13)–(B 15) set the basis for (3.26a).

The above results can be nicely captured on the basis of (B 8) and (B 11) cast in the normal form $\kappa_c = 1/[k_c^2(k_c^2-1)]$ employing the canonical variables $k_c := [\mathcal{I}_1/(1-T)]^{1/2}k$, $\kappa_c := J(1-T)^2\kappa_0/(\lambda\mathcal{I}_1)$: see figure S 1.

B.2. *Singular short-wave limit* $|Re\, k| \to \infty$

For $|\mathrm{Re}\, k| \to \infty$, $\mathcal{V}$ obviously varies exponentially weakly with k. We first identify a viscous sublayer $\zeta := ky = O(1)$ where we take $\mathcal{V}$ as a function of ζ and k. There (3.22a) and (2.5) give $\partial_{\zeta\zeta}\mathcal{V} - \mathcal{V} \sim \omega\zeta\mathcal{V}/k^3 + O(k^{-6})$. In turn,

$$\mathcal{V} \sim e\,\mathrm{e}^{-\zeta}\left[1 - \frac{\omega}{4k^3}(\zeta+\zeta^2) + O(k^{-6})\right] - (1+e)\,\mathrm{e}^{\zeta}\left[1 - \frac{\omega}{4k^3}(\zeta-\zeta^2) + O(k^{-6})\right] \quad (B\,16)$$

where e is some function of k satisfying $e(-k) \equiv -e(k) - 1$ as (3.22) enforces symmetry of $\mathcal{V}$ with respect to k. For $y = O(1)$, the exponential variation in (B 16) is morphed into a rapid one, typically captured by a Wentzel–Kramers–Brillouin–Jeffreys (WKBJ) ansatz. Inserting this into (3.22a,c) yields after some manipulations and exploiting the above symmetry property intrinsic to (3.22)

$$\frac{\mathcal{V}}{E(k)} \sim \mathrm{e}^{-k(y-h_0)}\left[1 - \frac{K_0 - K(y)}{2k} + O(k^{-2})\right] - \mathrm{e}^{k(y-h_0)}\left[1 + \frac{K_0 - K(y)}{2k} + O(k^{-2})\right], \quad (B\,17)$$

skew-symmetric in k and depending solely on the homogeneous BC (3.22c). The asymptotic relationship $E(k) \equiv -E(-k)$ follows from matching (B 16) and (B 17) as $e(k)$, and

$$K(y) := \int_y^{h_0} \frac{\psi_0'''(t)}{\psi_0'(t)}\,\mathrm{d}t, \quad K_0 := \frac{2u_0^{+2}}{TJ}. \quad (B\,18)$$

The last constant of integration ensures that (3.22*c*) is satisfied with the accuracy specified in (*B* 17) and (*B* 16), where the terms varying algebraically with k originate in the vorticity of the base flow.

Expanding K for $y \to 0$ with the help of (2.5) confirms the match of (*B* 16) and (*B* 17), up to contributions of respectively $O(k^{-3})$ and $O(k^{-1})$ in the brackets. This first yields two relationships involving e and E: $e \sim E\,\mathrm{e}^{kh_0}[1 - K_0/(2k) + O(k^{-2})]$, $1 + e \sim E\,\mathrm{e}^{-kh_0}[1 + K_0/(2k) + O(k^{-2})]$. From these we infer

$$e^{-1} \sim -1 + \mathrm{e}^{-2kh_0}\mathcal{A}(-k)/\mathcal{A}(k), \quad E^{-1} \sim \mathrm{e}^{-kh_0}\mathcal{A}(-k) - \mathrm{e}^{kh_0}\mathcal{A}(k) \tag{B 19}$$

where we abbreviate the algebraic variations with k in terms of

$$\mathcal{A}(k) \sim 1 - K_0/(2k) + O(k^{-2}) \quad (k \to \pm\infty). \tag{B 20}$$

The first of the relations (*B* 19) verifies that $\mathcal{V} = O(1)$ in (*B* 16) and the second the exponential smallness of $\mathcal{V}$ for $y \gg 1/|k|$. From (*B* 17), there

$$\mathcal{V} \sim -\mathrm{e}^{\mp ky}\left[1 - \frac{K(y)}{K_0 \mp 2k} + O(k^{-2})\right] + \mathrm{e}^{\mp k(2h_0 - y)}\left[1 - \frac{2K_0 - K(y)}{K_0 \mp 2k} + O(k^{-2})\right] \tag{B 21}$$

where the exponentially weak remainder term is of $O\big(\mathrm{e}^{\mp k(y+2h_0)}\big)$. Therefore,

$$\mathcal{V}(k, h_0) \sim -2\,\mathrm{e}^{\mp kh_0}\big[K_0/(K_0 \mp 2k) + O(k^{-2})\big] + O\big(\mathrm{e}^{\mp 3kh_0}\big) \tag{B 22}$$

with the aid of (*B* 18).

As seen from (*B* 19) and (*B* 20), the $O(k^{-2})$-terms in (*B* 21) and (*B* 22) are correct as long as $E \sim -\mathrm{e}^{-kh_0}/\mathcal{A}(k)$ or $|\mathcal{A}(k)| \gg |\mathrm{e}^{-2kh_0}|$ for $\mathrm{Re}\,k > 0$ and $E \sim \mathrm{e}^{kh_0}/\mathcal{A}(-k)$ or $|\mathcal{A}(-k)| \gg |\mathrm{e}^{2kh_0}|$ for $\mathrm{Re}\,k < 0$. However, they increase up to $O(k^{-1})$ when these constraints are violated, that is, when both contributions to E^{-1} in (*B* 19) become of the same order of magnitude or

$$k \sim \pm K_0/2 + O(T). \tag{B 23}$$

The expression for K_0 in (*B* 18) heralds this possibility when T is so small that $Tk \sim \pm u_0^{+2}/J + O(T^2)$. Moreover, E^{-1} might change sign then, which reveals the emergence of a real pole of $\mathcal{V}$ that represents the small-T asymptote of (*B* 12). Hence, the trace of the pole is known in a first approximation as weak deviations must account for its weak straining due to the higher-order corrections in (*B* 21):

$$Tk_u[1 + O(k_u^{-1})] = u_0^{+2}/J = \sqrt{3/(4\pi)}\,\Gamma(\tfrac{1}{3})/\Gamma(\tfrac{5}{6}) \simeq 1.15960 \quad (k_u \to \infty) \tag{B 24}$$

when u_0^+ and J are evaluated for Watson's flow by (2.8). Concomitantly, (*B* 21) yields with $K_0 \sim 2k_u + O(1)$ from (*B* 23)

$$\mathrm{Res}_{k=k_u}(\mathcal{V}) \sim -\,\mathrm{e}^{-k_u y}\big[K(y)/2 + O(k_u^{-1})\big] + \mathrm{e}^{k_u(y-2h_0)}[2k_u + O(1)]. \tag{B 25}$$

In turn, and as also obtained directly from (*B* 22),

$$\mathrm{Res}_{k=k_u}[\mathcal{V}(k, h_0)] \sim \mathrm{e}^{-k_u h_0}\big[2k_u + O(k_u^{-1})\big] + O\big(\mathrm{e}^{-3k_u h_0}\big). \tag{B 26}$$

These delicate consequences of matching exponentially varying terms verify a-posteriori the inclusion of the algebraically varying ones.

The asymptotic behaviours (*B* 24) and (*B* 26) are condensed into (3.26*b*) as substituting them into (3.25), specified for Watson's flow, results in

$$\bar{a}_u k_u^{2/3}\exp(k_u h_0) \to 8\pi\,\Gamma(\tfrac{2}{3})\big[\Gamma(\tfrac{5}{6})/\Gamma(\tfrac{1}{3})\big]^2 \simeq 6.0422 \quad (k_u \to \infty). \tag{B 27}$$

Dividing this limiting value by u_0^+ gives $16\pi^2\Gamma(2/3)[\Gamma(5/6)/\Gamma(1/3)]^2 \simeq 6.7402$, which is finally used in (3.27) when evaluated for $T \ll 1$.

B.3. *Eigenspace and poles*

As an important aspect shown next, for any $T \geqslant 0$, the homogeneous version of (3.22) is solvable only for a countable, infinite set of real eigenvalues of k^2 bounded from above.

We let $\bar{\mathcal{V}}_k(y)$ symbolise the space of eigenfunctions,

$$\bar{\mathcal{V}}_k'' = (k^2 + \psi_0'''/\psi_0')\bar{\mathcal{V}}_k, \quad y = 0\colon \quad \bar{\mathcal{V}}_k = 0, \quad y = h_0\colon \quad \psi_0'^2\bar{\mathcal{V}}_k' = TJk^2\bar{\mathcal{V}}_k. \qquad (B\,28)$$

In § B.1, we considered the eigenvalue $k = 0$ and the associated double pole. All other eigenvalues are expected to define the simple poles of $\mathcal{V}$ in the k-plane: an infinite number of conjugate imaginary and an, at most finite, number of real ones, these associated with isolated neutral capillary modes. To demonstrate these fundamental properties, we first consider two twice differentiable functions $\mathcal{U}(y)$, $\mathcal{W}(y)$ and the typical inner product $\int_0^{h_0} \mathcal{U}\,\overline{\mathcal{W}}\,\mathrm{d}y$; in the following, overbars unambiguously indicate complex-conjugates. One readily confirms that the operator $(\mathrm{d}^2/\mathrm{d}y^2 - \psi_0'''/\psi_0')\mathcal{U}$ subject to $\mathcal{U}(0) = \mathcal{U}'(h_0) = 0$ is self-adjoint, but the appearance of the eigenvalue k in the BC at $y = h_0$ impedes proving its typically expected properties in standard fashion if $T > 0$: k^2 is real; members of $\mathcal{V}_k$ for different eigenvalues are orthogonal with respect to the above inner product. Rather, if $\mathcal{U}$ and $\mathcal{W}$ now denote eigenfunctions for different eigenvalues k_1 and k_2, say, we obtain from (B 28) via integration by parts

$$(k_1^2 - \bar{k}_2^2)\left[\frac{TJ(\mathcal{U}\,\overline{\mathcal{W}})(h_0)}{\psi_0'^2(h_0)} - \int_0^{h_0} \mathcal{U}\,\overline{\mathcal{W}}\,\mathrm{d}y\right] = 0. \qquad (B\,29)$$

This prompts us to seek a transformation of (B 28) such that k no longer enters the BC for $y = h_0$. To this end, we introduce the transformed eigenfunctions $\mathcal{F}_k := \psi_0'\mathcal{V}_k' - \psi_0''\mathcal{V}_k$. We then obtain from (B 28)

$$[\psi_0'^2(\bar{\mathcal{V}}_k/\psi_0')']' \equiv \mathcal{F}_k' = k^2\psi_0'\bar{\mathcal{V}}_k \qquad (B\,30)$$

and, since $\psi_0''(h_0) = 0$, $\psi_0'^2\mathcal{F}_k = TJ\mathcal{F}_k'$ for $y = h_0$. Differentiation of (B 30) after division by $\psi_0'^2$ casts (B 28) into the form

$$(-\mathcal{F}_k'/\psi_0'^2)' = -k^2\mathcal{F}_k/\psi_0'^2, \quad y = 0\colon \quad \mathcal{F}_k = 0, \quad y = h_0\colon \quad \psi_0'^2\mathcal{F}_k = TJ\mathcal{F}_k'. \qquad (B\,31)$$

Adopting the signs in the usual notation, (B 31) represents a traditional self-adjoint Sturm–Liouville eigenvalue problem with the (for $y \to 0$ singular) weight function $\psi_0'^{-2}$ for the eigenvalues of $-k^2$. According to classical results, these indeed form a discrete set $k^2 = k_i^2(T)$ $(i = 0, 1, \ldots)$ bounded from below and satisfying the Weyl asymptotics $-k_i^2 \sim (\pi i/h_0)^2 + O(i)$ $(i \to \infty)$, controlled by the right-side of the BC for $y = h_0$ (cf. Teschl 2012). Here $k_0^2 = k_u^2 > 0$, referring to the single neutral mode considered in § B.2, so that k_i^2 is set to $-\mu_i^2(T) < 0$ for $i > 0$. It is also noteworthy that this sequence μ_i does not collapse in the limiting case $T = 1$ as $\mu_1(1) \simeq 0.015569$.

Supplement C. Outer Rayleigh problem: diffusive overlayer

Let us take ψ, p, h_+ as functions of $\bar{X}$, ξ, ϵ. We rectify the Maclaurin expansion of ψ and p for $\xi = O(1)$ justified by (2.3*c*), (3.10), (3.11) by adding an $O(\epsilon^{5/3})$-term

(and resultant higher-order corrections) that involves the $O(1)$-functions Ψ^*, P^* so as to account for (2.3d):

$$[\psi, p] \sim [1, P_0(\bar{X})] + \sum_{i=1}^{\infty} [\Psi_i, P_i](\bar{X};\epsilon)\,\frac{\epsilon^{i/2}\xi^i}{i!} + \epsilon^{5/3}[\Psi^*, P^*](\bar{X},\xi) + O(\epsilon^{12/7}). \quad (C1)$$

The structure of ($C1$) is explained in the following.

The bounded coefficient functions Ψ_i, P_i ($i \geqslant 0$) ensue from expanding (3.10), (3.11) together with (2.6) and the resultant property $\psi_0''''(h_0) = 0$ (see SBP18). One obtains

$$\Psi_1 \sim u_0^+ + \epsilon^{4/7} m[\psi_* - m\psi_0'''(h_0)A^2(0)/2] + \epsilon^{2/3}\,\bar{\Psi}_y(\bar{X}, h_0) + O(\epsilon^{5/7}), \quad (C2a)$$

$$\Psi_2 \sim \epsilon^{4/7} m[\psi_*'' - \psi_*\psi_0'''/u_0^+](h_0) + \epsilon^{2/3}[\bar{\psi}_{yy}(\bar{X}, h_0) + \bar{H}\psi_0'''(h_0)] + O(\epsilon^{5/7}), \quad (C2b)$$

$$[P_0', P_1'] \sim \epsilon^{2/3}[\bar{P}_{\bar{X}}, \bar{P}_{y\bar{X}}](\bar{X}, h_0) + O(\epsilon^{5/7}), \quad P_{1,2} = O(\epsilon^{4/7}), \quad h_{+,\bar{X}} = O(\epsilon^{3/7}). \quad (C2c)$$

Substituting ($C1$) into the shear-layer approximation of (2.3a),

$$\psi_\xi\psi_{\xi\bar{X}} - \psi_{\bar{X}}\psi_{\xi\xi} \sim -\epsilon\, p_{\bar{X}} - \epsilon^{1/2} h_{+,\bar{X}}\, p_\xi + \epsilon^{1/2}\psi_{\xi\xi\xi} \quad (C3)$$

(the Prandtl shift preserves the convective operator), and collecting powers of ξ yields a hierarchy of relations involving Ψ_i and P_i. Insertion of ($C2$) into the first two,

$$\Psi_1\Psi_{1,\bar{X}} \sim -P_{0,\bar{X}} - h_{+,\bar{X}}P_1 - \epsilon\,\Psi_3, \quad \Psi_1\Psi_{2,\bar{X}} \sim -P_1' - h_{+,\bar{X}}P_2 - \epsilon\,\Psi_4, \quad (C4)$$

just confirms the two-terms expansion of the streamwise momentum equation in (3.12) for $\xi = O(1)$. On the other hand, the left-hand side of (2.3d) reduces to $\Psi_2 - u_0^+ h_{+,\bar{X}\bar{X}} + O(\epsilon^{23/21})$ within the accuracy provided by (3.10), (3.11) and ($C2$). Evaluating (2.3d) by using ($C2b$) and (3.10) shows that this BC is satisfied up to $O(\epsilon^{4/7})$ once $m[\psi_*''-\psi_*\psi_0'''/u_0^+](h_0)+ml^2u_0^+(G-P_-) = 0$. This constraint for ψ_* must already be provided by the surrounding shear layer addressed in SBP18. However, the follow-up contributions of $O(\epsilon^{2/3})$ to (2.3d) yield in connection with (3.14c) the residual $(\bar{\Psi}_{yy} - \bar{\Psi}_{\bar{X}\bar{X}})(\bar{X}, h_0)$. Compensating for this requires the perturbation stream function Ψ^* to enter ($C1$) at the same order of approximation as the $O(\epsilon^{2/3})$-contribution to P_2. In turn, ($C3$) subject to (2.3c) and (2.3d) yields with the aid of ($C2$) the diffusion problem

$$u_0^+\Psi^*_{\xi\bar{X}} = \Psi^*_{\xi\xi\xi}, \quad \xi = 0\colon\ \Psi^* = 0,\ \Psi^*_{\xi\xi} = \bar{\Psi}_{\bar{X}\bar{X}} - \bar{\Psi}_{yy}, \quad \xi \to -\infty\colon\ \Psi^*_{\xi\xi} \to 0. \quad (C5)$$

This also implies, for $\xi \to -\infty$, a vanishing velocity perturbation Ψ^*_ξ but finite viscous displacement exerted in the bulk flow, measured by $\Psi^*(\bar{X}, \infty)$. The far-upstream and far-downstream asymptotes of Ψ^* are found to be forced by the inhomogeneous BC. Therefore, Ψ^* dies out exponentially for $\bar{X} \to -\infty$ and grows algebraically for $\bar{X} \to \infty$. We then use (3.13), (3.15) and (3.16) to describe the merge with the original overlayer (see SBP18) in this limit by $\Psi^* \sim 27\lambda\psi_0'''(h_0)/(80Mu_0^+)\bar{X}^{11/3}F_*(\eta_*) + O(\bar{X}^{5/3})$. Herein, the typical Rayleigh variable $\eta_* := \xi\sqrt{u_0^+/\bar{X}}$ is of $O(1)$ as F_* satisfies

$$19F_*'/6 - \eta_* F_*''/2 = F_*''', \quad F_*(0) = 0, \quad F_*''(0) = 1, \quad F_*''(-\infty) = 0. \quad (C6)$$

The solution to this problem can be expressed in terms of Kummer's confluent

hypergeometric function, M:

$$F_*(\eta_*) = (3/11)[M(-\tfrac{11}{3}, \tfrac{1}{2}, -\eta_*^2/4) - 1] - \eta_* M(-\tfrac{19}{6}, \tfrac{3}{2}, -\eta_*^2/4)/\Gamma(\tfrac{25}{6}). \qquad (C7)$$

Far downstream, the viscous displacement is quantified by $F_*(-\infty) = -3/11$. Adopting the results of § 3.3.2, one may expresses the full solution to ($C5$) as the Fourier integral

$$\Psi^* = \int_{\mathcal{C}} \frac{[k^2\mathcal{V} + \mathcal{V}_{yy}](k, h_0)}{u_0^+ k^2} \Big(1 - \mathrm{e}^{\sqrt{\mathrm{i}u_0^+ k}\,\xi}\Big)\, \mathrm{e}^{\mathrm{i}k\bar{X}}\, \mathrm{d}k. \qquad (C8)$$

A final remark on the higher-order corrections in ($C1$) reinforces the self-consistency of the above flow description. The kinematic BC (2.3c) induces an $O(\epsilon^{5/3})$-disturbance in the expansions of h^+ in (3.10) and thus the capillary pressure jump in (2.3d) such a term in (3.11). This produces the non-zero P^* in ($C1$). In addition, (2.3b) gives $P_\xi^* \equiv 0$.

Supplement *D*. Extended Jefferey–Hamel limit

We address briefly two formal aspects of the JH limit.

At first, inspection of (4.4) suggests the local expansion

$$\bar{\psi} \sim \bar{g}(\vartheta) + \bar{r}^\sigma G(\vartheta) + o(\bar{r}^\sigma) + c.c., \quad \mathrm{Re}\,\sigma > 0. \qquad (D1)$$

Here σ denotes the eigenvalue and G the corresponding eigenfunction satisfying the resulting eigenvalue problem

$$\big[\sigma^2(\sigma-2)\bar{g}' - \sigma\bar{g}'''\big]G - 2\bar{g}''G' + (\sigma-2)\bar{g}'G'' = \big[(\sigma-2)^2 + \mathrm{d}^2/\mathrm{d}\vartheta^2\big](\sigma^2 G + G''), \qquad (D2a)$$

$$G(0) = G'(0) = G(\pi) = G'(\pi) = 0. \qquad (D2b)$$

In ($D1$), σ then specifies the member of the discrete series of eigenvalues with minimum positive real part. The validity of the JH solution and thus the local representation ($D1$) of the full NS solution depends on the existence of this value of σ.

Secondly, we envisage $\bar{p}$ and $\bar{h}$, related via the dynamic BC in (4.3c), near $\bar{r} = 0$. The pressure gradient ensues from the momentum equations (4.3a,b) in the form $\bar{p}_{\bar{r}} \sim (\bar{g}''' + \bar{g}'^2)/\bar{r}^3 = [\bar{g}'''(\pi) - 4\bar{g}']/\bar{r}^3$, where the last equality follows from (4.6) upon integration, and $\bar{p}_\vartheta \sim 2\bar{g}''/\bar{r}^2$. Finally,

$$\bar{p} \sim \frac{4\bar{g}' - \bar{g}'''(\pi)}{2\bar{r}^2} + o(\bar{r}^{-2}) \quad (\bar{r} \to 0), \qquad \bar{h} \sim \frac{\bar{g}'''(\pi)}{2\tau} \ln \bar{x} \quad (\bar{x} \to 0+) \qquad (D3)$$

with $g'''(\pi) \simeq 87.9545$ ($\bar{g}'''(\pi) \simeq 19.6983$) for the attached (detached) eddy and for any finite value of τ. These singularities are much stronger than those found for the alternative, preferred Stokes limit elucidated in § 4.2.2 and below in Supplement *E*. Accordingly, their resolution would take place in a further NS region defined by the smallest scales, describing the microscopic resolution of the trailing edge.

Supplement *E*. Extended Stokes limit

Separation of variables in (4.7) yields

$$\bar{\psi} \sim \sum\nolimits_{i=0}^{\infty} \bar{\psi}_i + O(\bar{r}^{\sigma_n+\sigma_q}) + c.c., \quad \bar{\psi}_i := \bar{r}^{\sigma_i} f_i(\vartheta), \quad \operatorname{Re}\sigma_{i+1} > \operatorname{Re}\sigma_i > 0. \quad (E1)$$

Herein, σ_i denotes the i-th eigenvalue, f_i the corresponding eigenfunction of the azimuthal variation, and the remainder term arises from the dominant contribution to the quadratic inertial terms in (4.4), not captured by the Stokes balance and of $O(\bar{r}^{\sigma_n+\sigma_q-3})$. Therefore, n and q stand for the lowest indices i such that f_n, f_q are both non-trivial and their coupling produces a non-trivial inhomogeneity.

E.1. *Discussion of eigensolutions and their inertially induced response*

The expansion (*E*1) casts the biharmonic problem into the series of eigenvalue problems

$$\mathcal{S}_i\{f_i\} = f_i(0) = f_i''(0) = f_i(\pi) = f_i'(\pi) = 0, \quad (E2a)$$

$$\mathcal{S}_i\{Q\} := \left[(\sigma_i - 2)^2 + \mathrm{d}^2/\mathrm{d}\vartheta^2\right]\left(\sigma_i^2 + \mathrm{d}^2/\mathrm{d}\vartheta^2\right)\{Q\} \quad (E2b)$$

for any function Q. The reduced Stokes operator $\mathcal{S}_i$ acting on ϑ and parametrised by the discrete eigenvalues is already seen in (4.6) as this replaces (*E*1) for $\sigma_i = 0$. One readily finds that (*E*2) has no solution in the degenerate cases $\sigma_i = 1$ and $\sigma_i = 2$. In any other case, the first two BCs in (*E*2*a*) yield

$$f_i = a_i \sin(\sigma_i\vartheta) + b_i \sin[(\sigma_i - 2)\vartheta], \quad (E3a)$$

where the constants a_i and b_i are functions of Λ_0, are determined by the global solution to (4.3) and must not all be zero. Notably, the $\sin(\sigma_i\vartheta)$-term refers to a potential-flow contribution. The eigenvalue relation $\sin(2\sigma_i\pi) = 0$ equivalent to the last two BCs implies

$$\sigma_i = (1 + i)/2. \quad (E3b)$$

In turn, (*E*3*a*) holds for some real a_i and b_i satisfying

$$b_i = -a_i \quad (i = 0, 2, 4, \ldots), \quad (2 - \sigma_i)b_i = \sigma_i a_i \quad (i = 1, 3, 5, \ldots), \quad (E3c)$$

which confirms that $f_1 \equiv f_3 \equiv 0$ and $a_2 f_0 \equiv a_0 f_2$ and, a-posteriori, the validity of (*E*1). Here we refer to the subsequent discussion in § *E*.2. As σ_i take on integer values for i being odd, it is readily seen that exactly this category refers to regular eigensolutions $\bar{\psi}_i$. Their series, ordered by ascending integer powers in $\bar{x}$ and $\bar{y}$,

$$\bar{\psi}_5 = -4a_5\bar{y}^3, \ \bar{\psi}_7 = -8a_7\bar{x}\bar{y}^3, \ \psi_9 = 40a_9\bar{y}^3(\bar{y}^2/15 - \bar{x}^2/3), \ \ldots, \quad (E4)$$

ensues systematically from expressing $\bar{\Delta}^2$ accordingly and using the BCs. We also infer from (4.7), (*E*2) and (*E*3*a*) that

$$\bar{p} - \bar{p}_0 \sim \sum\nolimits_{i=0,\, i\neq 3}^{\infty} \bar{r}^{\sigma_i - 2} p_i(\vartheta) + O\big(\bar{r}^{\sigma_n+\sigma_q-2}\big), \quad p_i = 4b_i(\sigma_i - 1)\cos[(\sigma_i - 2)\vartheta]. \quad (E5)$$

The constant $\bar{p}_0$ is again to be extracted from the complete NS solution.

The symmetry of f_i in i around $i = 1$ and its azimuthal symmetry/antisymmetry with respect to $\vartheta = \pi/2$ for odd/even values of σ_i (i.e. odd values of i) deserve a comment. The first eight members of the series f_i with a_i set to unity are plotted in figure S 2. One then finds that $f_0 \geqslant 0$, $f_5 \leqslant 0$, and $f_i(\vartheta)$ changes its sign $(i - 2)/2$ times if $i = 4, 6, \ldots$ and $(i - 5)/2$ times if $i = 7, 9, \ldots$ over the interval

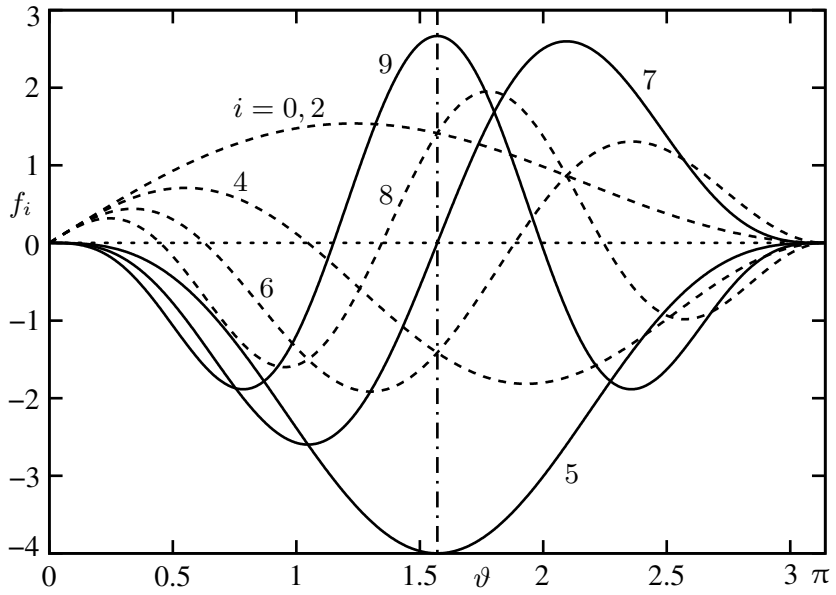

Figure S 2: Eigenfunctions f_i of Stokes operator and their symmetry properties, see ($E3$): $a_i = 1$, labels indicate $i = 0, 2, 4, 6, 8$ (dashed), $i = 5, 7, 9$ (solid).

$(0, \pi)$. Therefore, exactly three alternatives can accommodate the forward-flow condition (4.5):

(A) $a_0 > 0$;
(B) $a_0 = 0$ and $a_2 > 0$;
(C) $a_0 = a_2 = a_4 = 0$ and $a_5 < 0$.

We now proceed to demonstrating that case (C) is the appropriate choice. To this end, we conveniently restate ($E1$) in general more precisely as

$$\bar{\psi} \sim \sum\nolimits_{i=0}^{\infty} \bar{r}^{\sigma_i} g_i(\vartheta), \quad g_0 := f_0, \tag{$E6$}$$

where the functions g_i represent the solutions to the hierarchy of inhomogeneous Stokes problems provoked by the inertia terms in (4.4), which cause the remainder term in ($E1$). As we now demonstrate, the forcing of these eigensolutions of the Stokes operator by the higher-order, convective terms controls the selection of the leading non-zero coefficient a_i of the homogeneous contribution f_i to g_i. Substituting ($E6$) into (4.4) and collecting powers of $\bar{r}$ results in the inhomogeneous extension of ($E2$) for $i > 0$:

$$\mathcal{S}_i\{g_i\} = I_i(\vartheta) := \sum\nolimits_{j=0}^{i-1} I_{i,j}(\vartheta), \tag{$E7a$}$$

$$I_{i,j}(\vartheta) := \left[(\sigma_j - 2)g_k' - \sigma_k g_k \,\mathrm{d}/\mathrm{d}\vartheta\right](\sigma_j^2 g_j + g_j''), \quad k := i - j - 1, \tag{$E7b$}$$

$$g_i(0) = g_i''(0) = g_i(\pi) = g_i'(\pi) = 0, \tag{$E7c$}$$

where ($E7b$) is consistent with the identity $\sigma_j + \sigma_k \equiv \sigma_{j+k+1}$, see ($E3b$). The self-adjointness of the homogeneous Stokes operator defined by ($E2$) gives

$$0 = \int_0^{\pi} \mathcal{S}_i\{g_i\} f_i(\vartheta)\,\mathrm{d}\vartheta = S_i := \int_0^{\pi} I_i(\vartheta) f_i(\vartheta)\,\mathrm{d}\vartheta. \tag{$E8$}$$

This describes the well-known three alternatives: the solution of ($E7$) is unique if $f_i \equiv 0$; it is non-unique if f_i is non-trivial and $S_i = 0$; it does not exist otherwise.

Thus ($E8$) establishes the following analysis of the possible cases concerning the solvability of ($E7$).

Case: $a_0 \neq 0$. In this least-degenerate scenario including case (A) above, ($E6$) is specified as $\bar{\psi} \sim \bar{r}^{1/2} f_0 + \bar{r} g_1 + \bar{r}^{3/2} g_2 + O(\bar{r}^2)$; ($E3$) implies $f_0 = a_0[\sin(\vartheta/2) + \sin(3\vartheta/2)]$ and ($E7b$) $I_1 = 3a_0^2[\sin(2\vartheta)/2 + \sin(3\vartheta)]$. Since $f_1 \equiv 0$, we construct for $i = 1$ the unique solution $g_1 = a_0^2[37\sin(\vartheta) + 32\sin(2\vartheta) + 9\sin(3\vartheta)]/192$ of ($E7$). In turn, we specify ($E7$) and ($E8$) for $i = 2$. A tedious but straightforward calculation involving g_1 and $f_2 \equiv a_2 f_0/a_0$ yields $S_2 = -25\pi a_0^2 a_2/128$ so that g_2 does not exist. We are thus left with the following less singular situation.

Case: $a_i = 0$ $(0 \leqslant i < n)$, $a_n \neq 0$. Here n identifies the index of the dominant non-trivial eigensolution so that $g_n = f_n$. Accordingly, ($E7b$) produces non-trivial $I_{i,j}$ for $j, k \geqslant n$ only. This confirms that $I_i \equiv 0$ $(0 \leqslant i \leqslant 2n)$ as the self-coupling of f_n yields the potential lowest-order inhomogeneity

$$I_{2n+1} = I_{2n+1,n} = -4b_n(\sigma_n - 1)(\sigma_n - 2)\{a_n\sigma_n \sin(2\vartheta) + b_n \sin[(2\sigma_n - 4)\vartheta]\}, \quad (E9)$$

which corresponds to the case $n = q$ in ($E1$). It is emphasised that I_{2n+1} vanishes identically only for $n = 1$ $(\sigma_1 = 1)$, $n = 3$ $(\sigma_3 = 2)$ and $n = 5$ $(\sigma_5 = 3)$. Furthermore, ($E8$) gives after some standard manipulations, involving σ_n and and σ_{2n+1} specified by ($E3b$),

$$S_{2n+1} = \begin{cases} 0 & (n \neq 2), \\ 3\pi a_2^2 a_5/2 & (n = 2). \end{cases} \quad (E10)$$

The last statement requires $a_2 = 0$. This renders the forward-flow case (B) also not possible. Hence, the scenario (C) motivates the following discussion of the special case $n = 5$.

Case: $a_i = 0$ $(0 \leqslant i < 5)$, $a_5 \neq 0$. The result ($E10$) includes that the here dominant eigensolution of the Stokes operator $\bar{\psi}_5$ given by ($E4$), describing a non-degenerate flow profile at separation, trivially generates a vanishing inhomogeneity I_{11}. That said, ($E6$) then degenerates and reads more accurately, with the help of ($E3b$),

$$\bar{\psi} \sim \bar{\psi}_5 + \sum\nolimits_{i=q}^{\infty} \bar{r}^{(1+i)/2} f_i + \bar{r}^{(7+q)/2} g_{6+q} + o\big(\bar{r}^{(7+q)/2}\big) \quad (q > 5, \;\; a_q \neq 0). \quad (E11)$$

At first, any index $q > 5$ is conceivable. If ($E11$) initiates the solution to the full NS problem, such an index indicating the non-trivial follow-up term to $\bar{r}^3 f_5$ must exist. As a central observation, the lowest-order inhomogeneity in ($E7a$) specified by ($E7b$) is $I_{6+q} = I_{6+q,5} + I_{6+q,q}$ and produces g_{6+q}, where the eigenfunction f_{6+q} corresponds to to the eigenvalue $\sigma_{q+6} = (7+q)/2$. Inserting these findings into ($E8$) yields indeed $S_{6+q} \equiv 0$ as for ($E10$) but for any q, where we skip the technical details. This guarantees the existences of g_{q+6} and, in turn, of ($E11$).

As an important step, the above analysis determines the least singular (most-degenerate) local representation of $\bar{\psi}$ to be given by case (C) extended by ($E11$) as the sole reliable option satisfying (4.5). It should be emphasised that the impact of the interactive flow on flow detachment on the NS scales is condensed into the aforementioned (pending) dependence of the coefficient a_5 on Λ_0.

The shear rate at the plate immediately upstream of detachment observed on the global scale reads $u_y|_{y=0} \sim \bar{\psi}_{\vartheta\vartheta}(\bar{r}, \pi)/\bar{r}^2$, according to (4.1) and (4.3d). Since ($E3$) entails $f_i''(\pi) = 0$ for odd i, cf. ($E4$), and $f_i''(\pi) = 2a_i(-1)^{i/2}(1-i)$ for even i, it is dominantly fixed either by the eigenfunction f_j of the smallest even index, $j \geqslant 6$, that enters ($E11$) or g_{6+q}. With dots abbreviating smaller terms, that shear

rate tends to zero in the form

$$u_y|_{y=0} \sim 2a_j(-1)^{j/2}(1-j)(-\bar{x})^{(j-1)/2} + (-\bar{x})^{(3+q)/2} g''_{6+q}(\pi) + \cdots \quad (\bar{x} \to 0-). \tag{E12}$$

Since (4.5) requires $u_y|_{y=0} > 0$ here, we expect either $a_j > 0$ ($a_j < 0$) for $j = 6, 10, 14, \ldots$ ($j = 8, 12, 16, \ldots$) or $f''_{6+q}(\pi)$ behaves such that $g''_{6+q}(\pi) > 0$. Typically, the adverse pressure gradient predicted by (E5) in the form $\bar{p} - \bar{p}_0 \sim -24a_5\bar{x} + O\big(\bar{r}^{(q-3)/2}\big)$ initiates flow detachment. Finally, this together with (E11) and (4.3c) results in (4.8).

E.2. *Eigensolutions of the Stokes operator having weakly non-algebraic radial variation?*

Given the absence of a reference length and velocity of the Stokes limit considered, typical dimensional reasoning predicts, in general, algebraic–logarithmic variations of the gauge functions in (E1) with $\bar{r}$. Nonetheless, the following analysis confirms that factors with sub-algebraic (logarithmic) dependence on $\bar{r}$ do indeed not contribute to (E1).

Seeking eigensolutions of the biharmonic operator in the limit $\bar{r} \to 0$ first leaves one with a generalisation of the expansion (E1) into eigenfunctions:

$$\bar{\psi} \sim \sum\nolimits_{i=j=0}^{\infty} \bar{r}^{\sigma_i} \chi_{i,j}(\bar{r}) f_{i,j}(\vartheta), \quad \chi_{i,j+1} = o(\chi_{i,j}). \tag{E13}$$

Herein, $f_{i,j}$ indicates the double series of eigenfunctions, with $f_{i,0} = f_i$ as we found so far, due to the corresponding sought gauge functions $\chi_{i,j}$ exhibiting sub-algebraic variation, including the previous situation $\chi_{i,0} \equiv 1$ and $f_{i,j} \equiv \chi_{i,j} \equiv 0$ for $j > 0$. Following the analysis of Scheichl (2014) of the Laplace operator, for any such function $\Xi_{i,j} := \bar{r}\chi'_{i,j}$ is of $o(\chi_{i,j})$ and again belongs to this family of functions. With this relation in mind, we obtain after some rearrangements

$$\bar{\Delta}^2(\bar{r}^{\sigma_i}\chi_{i,j} f_{i,j}) \sim \Xi_{i,j} J_{i,j}(\vartheta) + o(\Xi_{i,j}), \quad J_{i,j}(\vartheta) := 4(\sigma_i - 1)[f''_{i,j} + \sigma_i(\sigma_i - 2) f_{i,j}] \tag{E14}$$

(agreeing with the symmetry of $f_{i,j}$ in σ_i with respect to $\sigma_i = 1$). In turn, we specify $\chi_{i,1} = -\Xi_{i,0}$. If $\chi_{i,1}$ does not vanish identically, the homogeneous problem determining f_i yields the inhomogeneous follow-up problem fixing $f_{i,1}$ according to (E2)

$$\mathcal{S}_i\{f_{i,1}\} = J_{i,0}(\vartheta), \quad f_{i,1}(0) = f''_{i,1}(0) = f_{i,1}(\pi) = f'_{i,1}(\pi) = 0. \tag{E15}$$

Since the homogeneous operator in (E15) and defined by (E2) is self-adjoint,

$$0 = \int_0^\pi \mathcal{S}_i\{f_{i,1}\} f_i(\vartheta)\,\mathrm{d}\vartheta = \int_0^\pi J_{i,0}(\vartheta) f_i(\vartheta)\,\mathrm{d}\vartheta = 8\pi(-1)^i(\sigma_i - 1)a_i b_i \tag{E16}$$

with the aid of (E3). This contradiction implies $\chi_{i,0} \equiv 1$, $\chi_{i,1} \equiv 0$ and, by iteration, $\chi_{i,j} \equiv 0$ for all $j > 0$. Consequently, the appearance of sub-algebraic factors in (E1) is indeed ruled out.