Supplementary material

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This supplementary material is a summary of the detailed mathematical process leading to the results in the paper entitled 'Eigenvalue bounds for compressible stratified magneto-shear flows varying in two transverse directions'.

Appendix A. The derivation of the pressure equation

The governing equations (2.5) can be reduced to a single equation for $\widetilde{q}$ as follows. Denoting the divergence of the displacement vector as $\mathcal{D} = i\alpha\xi + \eta_y + \zeta_z$, from (2.8c) and (2.9a) the link between $\eta, \zeta, \widetilde{q}$ and $\mathcal{D}$ can be found as

$$(s^2 + a^2)\Lambda_c \mathcal{D} = -U^2\widetilde{q} - \Lambda_a(G_y\eta + G_z\zeta), \tag{A 1}$$

where

$$\Lambda_c \equiv \overline{\rho}(U^2 - c_T^2), \qquad \Lambda_a \equiv \overline{\rho}(U^2 - a^2). \tag{A 2}$$

Here c_T is the local cusp (tube) wave speed defined in section 3.2. Equation (A 1) can then be employed to eliminate $\mathcal{D}$ and $\widetilde{\rho}$ from (2.8b), (2.9b) and (2.9c):

$$\begin{bmatrix} k^2\Lambda_a - \overline{\rho}\mathcal{N}_1^2 & -\overline{\rho}\mathcal{N}_{12}^2 \\ -\overline{\rho}\mathcal{N}_{12}^2 & k^2\Lambda_a - \overline{\rho}\mathcal{N}_2^2 \end{bmatrix} \begin{bmatrix} \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} \widetilde{q}_y - \frac{G_y\overline{\rho}U^2\widetilde{q}}{(s^2+a^2)\Lambda_c} \\ \widetilde{q}_z - \frac{G_z\overline{\rho}U^2\widetilde{q}}{(s^2+a^2)\Lambda_c} \end{bmatrix}. \tag{A 3}$$

The components of the matrix are defined using

$$\mathcal{N}_1^2 \equiv N_1^2 + \frac{\overline{\rho}U^2}{\Lambda_c}\frac{c_T^2 G_y^2}{s^4}, \qquad \mathcal{N}_2^2 \equiv N_2^2 + \frac{\overline{\rho}U^2}{\Lambda_c}\frac{c_T^2 G_z^2}{s^4}, \qquad \mathcal{N}_{12}^2 \equiv N_{12}^2 + \frac{\overline{\rho}U^2}{\Lambda_c}\frac{c_T^2 G_y G_z}{s^4}.$$

Finally, we use

$$0 = (U^2 - s^2)\widetilde{q} + (s^2 + a^2)\Lambda_c(\eta_y + \zeta_z) + U^2\overline{\rho}(G_y\eta + G_z\zeta) \tag{A 4}$$

that can be found by eliminating ξ from (2.8c) and (2.9a). Equation (A 4) becomes a single equation for $\widetilde{q}$ when η and ζ are expressed by $\widetilde{q}$ using (A 3).

Appendix B. The Euler-Lagrange equations

The Euler-Lagrange equations associated with the optimisation problem (3.14) are

$$R^2\xi = (\overline{u} - r_c)^2\xi - (\mathbf{l}_1^\dagger\mathbf{x})/\overline{\rho}k^2, \tag{B 1a}$$

$$R^2\eta = (\overline{u} - r_c)^2\eta - \{\mathbf{l}_2^\dagger\mathbf{x} - \partial_y(\mathbf{l}_4^\dagger\mathbf{x})\}/\overline{\rho}k^2, \tag{B 1b}$$

$$R^2\zeta = (\overline{u} - r_c)^2\zeta - \{\mathbf{l}_3^\dagger\mathbf{x} - \partial_z(\mathbf{l}_4^\dagger\mathbf{x})\}/\overline{\rho}k^2. \tag{B 1c}$$

In principle, for fixed r_c, the optimised value R^2 can be found by solving those equations using some numerical eigenvalue solver. Here $\mathbf{x}$ is the transpose of $[\xi, \eta, \zeta, \eta_y + \zeta_z]$ and

$$\mathbf{l}_1^\dagger = \overline{\rho}[k^2 s^2, -ikG_y, -ikG_z, -iks^2], \tag{B 2a}$$

$$\mathbf{l}_2^\dagger = \overline{\rho}[ikG_y, k^2a^2 + G_y\frac{\overline{\rho}_y}{\overline{\rho}}, G_y\frac{\overline{\rho}_z}{\overline{\rho}}, G_y], \tag{B 2b}$$

$$\mathbf{l}_3^\dagger = \overline{\rho}[ikG_z, G_z\frac{\overline{\rho}_y}{\overline{\rho}}, k^2a^2 + G_z\frac{\overline{\rho}_z}{\overline{\rho}}, G_z], \tag{B 2c}$$

$$\mathbf{l}_4^\dagger = \overline{\rho}[iks^2, G_y, G_z, s^2 + a^2]. \tag{B 2d}$$

The daggers describe a Hermitian transpose. As commented in the main text, calculating R^2 in this way is inefficient.

The computational cost could be reduced by using the fact that the terms in the energy equation can be written in a quadratic form; there exists the Hermitian matrix $\mathbb{M}$ such that

$$\langle(\overline{u} - r_c)^2 Q\rangle - \langle\mathcal{L}\rangle - R^2\langle Q\rangle = \langle\overline{\rho}\mathbf{x}^\dagger\mathbb{M}\mathbf{x}\rangle. \tag{B 3}$$

A straightforward algebra yields

$$\mathbb{M} = \overline{\rho}k^2\{(\overline{u} - r_c)^2 - R^2\}\mathrm{diag}(1,1,1,0) - \begin{bmatrix} \mathbf{l}_1^\dagger \\ \mathbf{l}_2^\dagger \\ \mathbf{l}_3^\dagger \\ \mathbf{l}_4^\dagger \end{bmatrix}, \tag{B 4}$$

using (B 2). If the values of R^2, r_c are given, the four eigenvalues of this matrix ($\lambda_i(y,z)$, $i = 1,2,3,4$ say) can be easily found numerically at each point (y,z). Then the minimum value of R^2 that realises

$$0 = \max_{\Omega}\{\max(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\} \tag{B 5}$$

gives the radius of our interest. The above condition ensures the negative definiteness of $\mathbf{x}^\dagger\mathbb{M}\mathbf{x}$, and therefore (3.13) follows using (B 3). Note that the eigenvalue bound found in the matrix method may be looser than the Euler-Lagrange bound, because in the former method the link between η, ζ, and $\eta_y + \zeta_z$ is lost. This matrix idea can be further advanced to yield the analytic bound summarised in Theorem 1.

Appendix C. Optimisation of σ in the inner envelope bound

The best inner envelope bound can be obtained by choosing $\sigma \in (0, s^2 + a^2]$ that minimises $\lambda \equiv (\lambda_2 + k^{-1}\lambda_1 + k^{-2}\lambda_0)$ at each point y, z. At $\sigma = s^2$ the largest eigenvalue λ_1 changes its form (see (3.16)) so we must consider two intervals $0 < \sigma < s^2$ and $s^2 \le \sigma \le s^2 + a^2$ separately; we shall shortly see that the minimum of λ can always be obtained by the second interval.

Hereafter we denote

$$\widehat{N}_1^2 \equiv \frac{G_y\overline{\rho}_y}{\overline{\rho}} - \frac{G_y^2}{\sigma}, \quad \widehat{N}_2^2 \equiv \frac{G_z\overline{\rho}_z}{\overline{\rho}} - \frac{G_z^2}{\sigma}. \tag{C 1}$$

	$k < k_h$	$k \geq k_h$
$0 < N_1^2 + N_2^2 < N_{1a}^2 + N_{2a}^2$	0	$c_T^2(\frac{k_h}{k} - 1)$
$N_1^2 + N_2^2 < 0 < N_{1a}^2 + N_{2a}^2$	$-\frac{N_1^2+N_2^2}{k_h^2}(\frac{k_h}{k} - 1)$	$c_T^2(\frac{k_h}{k} - 1)$
$N_1^2 + N_2^2 < N_{1a}^2 + N_{2a}^2 < 0$	$c_T^2(\frac{k_h}{k} - 1) - \frac{N_{a1}^2+N_{a2}^2}{k^2}$	$c_T^2(\frac{k_h}{k} - 1) - \frac{N_{a1}^2+N_{a2}^2}{k^2}$

TABLE 1. The summary of the optimised values of $\lambda - (\overline{u} - r_c)^2$.

For the interval $s^2 \leq \sigma \leq s^2 + a^2$, the largest eigenvalues are found as

$$\lambda_2 = (\overline{u} - r_c)^2 - s^2(1 - \frac{s^2}{\sigma}), \tag{C 2}$$

$$\lambda_1 = (1 - \frac{s^2}{\sigma})\sqrt{G_y^2 + G_z^2}, \tag{C 3}$$

$$\lambda_0 = \max(0, -(\widehat{N}_1^2 + \widehat{N}_2^2)). \tag{C 4}$$

Thus

$$\lambda(\sigma) = \begin{cases} \lambda_+(\sigma) & \text{if} \quad \widehat{N}_1^2 + \widehat{N}_2^2 \geq 0, \\ \lambda_-(\sigma) & \text{if} \quad \widehat{N}_1^2 + \widehat{N}_2^2 < 0, \end{cases} \tag{C 5}$$

where

$$\lambda_+(\sigma) \equiv (\overline{u} - r_c)^2 + s^2\left(1 - \frac{s^2}{\sigma}\right)\left(\frac{k_h}{k} - 1\right), \tag{C 6a}$$

$$\lambda_-(\sigma) \equiv (\overline{u} - r_c)^2 + s^2\left(1 - \frac{s^2}{\sigma}\right)\left(\frac{k_h}{k} - 1\right) + \frac{s^4}{\sigma}\frac{k_h^2}{k^2} - \frac{G_y\overline{\rho}_y + G_z\overline{\rho}_z}{\overline{\rho}k^2}. \tag{C 6b}$$

Here k_h is the wavenumber associated with the density scale height defined in section 3.1. From (C 6) we can show for any $\sigma_1 < \sigma_2$ that

$$\lambda_+(\sigma_1) < \lambda_+(\sigma_2), \quad \lambda_-(\sigma_1) > \lambda_-(\sigma_2), \qquad \text{if} \ \ k < k_h, \tag{C 7a}$$

$$\lambda_+(\sigma_1) > \lambda_+(\sigma_2), \quad \lambda_-(\sigma_1) > \lambda_-(\sigma_2), \qquad \text{if} \ \ k > k_h. \tag{C 7b}$$

For the range of σ under consideration, $N_1^2 \leq \widehat{N}_1^2 \leq N_{1a}^2$ and $N_2^2 \leq \widehat{N}_2^2 \leq N_{2a}^2$, where N_{1a}^2, N_{2a}^2 are buoyancy frequencies defined in section 3.1.

When $k > k_h$, (C 7*b*) implies that the larger the value of σ, the smaller the associated value of λ. This means that the best choice of σ is the largest possible value $s^2 + a^2$, without regarding the sign of $(\widehat{N}_1^2 + \widehat{N}_2^2)$. Thus, noting $\lambda_+(s^2+a^2) = (\overline{u}-r_c)^2 + c_T^2\left(\frac{k_h}{k} - 1\right)$ and $\lambda_-(s^2 + a^2) = (\overline{u} - r_c)^2 + c_T^2\left(\frac{k_h}{k} - 1\right) - k^{-2}(N_{1a}^2 + N_{2a}^2)$, we have the optimums summarised at the rightmost column of Table 1.

When $k < k_h$, the situation is more complicated as expected from (C 7*a*). Let $s^2 \leq \sigma \leq s^2 + a^2$ and $k < k_h$. We shall deduce σ that gives the minimum value of λ. Here it is convenient to introduce $\widehat{N}^2(\sigma) = \widehat{N}_1^2 + \widehat{N}_2^2$. Of course, $\widehat{N}^2(s^2) = N^2$ and $\widehat{N}^2(s^2 + a^2) = N_a^2$, writing $N^2 = N_1^2 + N_2^2$ and $N_a^2 = N_{1a}^2 + N_{2a}^2$. Moreover, $\widehat{N}^2(\sigma) = 0$ when $\sigma = \sigma_0 \equiv \frac{\overline{\rho}s^4k_h^2}{G_y\overline{\rho}_y + G_z\overline{\rho}_z}$. Depending on the sign of N^2 and N_a^2, following three cases are possible.

(*a*) When $0 < N^2 < N_a^2$, $\widehat{N}^2(\sigma) = \widehat{N}_1^2 + \widehat{N}_2^2$ is positive. The optimum must be found by λ_+, and thus from (C 7*a*) we select the smallest possible value $\sigma = s^2$. Noting $\lambda_+(s^2) = (\overline{u} - r_c)^2$, we have the result shown in Table 1.

(*b*) When $N^2 < N_a^2 < 0$, $\widehat{N}^2(\sigma) = \widehat{N}_1^2 + \widehat{N}_2^2$ is negative. The optimum must be found by λ_-, and thus from (C 7a) we select the largest possible value $\sigma = s^2 + a^2$. The result is unchanged from the $k \geq k_h$ case, as shown in Table 1.

(c) When $N^2 < 0 < N_a^2$, we need to split the interval of σ into two parts using σ_0. For $s^2 < \sigma < \sigma_0$, $\widehat{N}(\sigma) < 0$ and thus λ_- must be used to compute the optimum, while for $\sigma_0 < \sigma < s^2 + a^2$, $\widehat{N}(\sigma) > 0$ and we should use λ_+. From (C 7a) the optimum is $\lambda_-(\sigma_0) = \lambda_+(\sigma_0) = -\frac{N^2}{k_h^2}\left(\frac{k_h}{k} - 1\right)$. Note that $c_T^2 - \frac{N_a^2}{k_h^2} = -\frac{N^2}{k_h^2}$ and thus the optimum changes continuously; see Table 1.

Next we show that the consideration of the other interval $0 < \sigma \leq s^2$ does not change the optimum values given in Table 1. From (3.16) the explicit expression of λ becomes (C 5) with

$$\lambda_+(\sigma) \equiv (\overline{u} - r_c)^2 - s^2\left(1 - \frac{s^2}{\sigma}\right)\left(\frac{k_h}{k} + 1\right), \tag{C 8a}$$

$$\lambda_-(\sigma) \equiv (\overline{u} - r_c)^2 - s^2\left(1 - \frac{s^2}{\sigma}\right)\left(\frac{k_h}{k} + 1\right) + \frac{s^4}{\sigma}\frac{k_h^2}{k^2} - \frac{G_y\overline{\rho}_y + G_z\overline{\rho}_z}{\overline{\rho}k^2}. \tag{C 8b}$$

Clearly, the optimum can always be found by the largest possible value $\sigma = s^2$. This means that the optimised value is $\lambda - (\overline{u} - r_c)^2 = -\frac{1}{k^2}\min(N^2, 0)$. It is easy to see this optimum value is larger than those found in Table 1, noting the identities

$$-\frac{N^2}{k_h^2}\left(\frac{k_h}{k} - 1\right) + \frac{N^2}{k^2} = -\frac{N^2}{k_h^2}\left(\frac{k_h}{k} - 1 - \frac{k_h^2}{k^2}\right) \tag{C 9}$$

$$c_T^2\left(\frac{k_h}{k} - 1\right) - \frac{N_a^2}{k^2} + \frac{N^2}{k^2} = \frac{N_a^2 - N^2}{k_h^2}\left(\frac{k_h}{k} - 1 - \frac{k_h^2}{k^2}\right), \tag{C 10}$$

and the inequality $\left(\frac{k_h}{k} - 1 - \frac{k_h^2}{k^2}\right) < 0$.

Using Table 1 noting that $-\frac{N_1^2 + N_2^2}{k_h^2} = c_T^2 - \frac{N_{1a}^2 + N_{2a}^2}{k_h^2}$, we arrive at Theorem 1.

Appendix D. Derivation of the integral in section 4.1

Integrating $\phi^*\times$(4.1c)$+\psi^*\times$(4.1d) by parts over the domain noting $(U^{-1/2}\phi^*)_y\widetilde{q} + (U^{-1/2}\psi^*)_z\widetilde{q} = U^{-1/2}(\widehat{\kappa}_1^* - U^{-1}\widehat{\kappa}_2^*)\widetilde{q}$, and eliminating $\widetilde{q}$ and φ using

$$\varphi = \frac{(s^2\widehat{\kappa} + \widehat{\mathcal{G}})}{ik(U^2 - s^2)}, \qquad ik\varphi + \widehat{\kappa} = \frac{U^2\widehat{\kappa} + \widehat{\mathcal{G}}}{U^2 - s^2}, \tag{D 1}$$

$$U^{1/2}\widetilde{q} = -\frac{\overline{\rho}(a^2 + s^2)(U^2 - c_T^2)\widehat{\kappa}}{U^2 - s^2} - \frac{U^2\overline{\rho}\widehat{\mathcal{G}}}{U^2 - s^2}, \tag{D 2}$$

derived by (4.1a) and (4.1b), we arrive at

$$\begin{aligned} 0 = \Bigg\langle & k^2\overline{\rho}\frac{U^2 - a^2}{U}(|\phi|^2 + |\psi|^2) - \frac{G_y}{U\overline{\rho}_y}|\overline{\rho}_y\phi + \overline{\rho}_z\psi|^2 \\ & -\overline{\rho}\frac{a^2}{U}\left(|\widehat{\kappa}_1|^2 + \frac{|\widehat{\kappa}_2|^2}{U^2} - \frac{\widehat{\kappa}_2^*\widehat{\kappa}_1 + \widehat{\kappa}_2\widehat{\kappa}_1^*}{U}\right) \\ & -\overline{\rho}\frac{s^2U}{U^2 - s^2}\left(|\widehat{\kappa}_1|^2 + \frac{|\widehat{\kappa}_2|^2}{U^2} - \frac{\widehat{\kappa}_2^*\widehat{\kappa}_1 + \widehat{\kappa}_2\widehat{\kappa}_1^*}{U}\right) \\ & -\frac{U\overline{\rho}\{(\widehat{\kappa}_1^* - U^{-1}\widehat{\kappa}_2^*)\widehat{\mathcal{G}} + \widehat{\mathcal{G}}^*(\widehat{\kappa}_1 - U^{-1}\widehat{\kappa}_2)\}}{U^2 - s^2} - \frac{\overline{\rho}|\widehat{\mathcal{G}}|^2}{U(U^2 - s^2)}\Bigg\rangle. \end{aligned} \tag{D 3}$$

Let us extract the imaginary part of the integrand. The imaginary parts of the terms in the first to forth lines above can be found as

$$-k^2\overline{\rho}(1+\frac{a^2}{|U|^2})(|\phi|^2+|\psi|^2)-\frac{G_y}{|U|^2\overline{\rho}_y}|\overline{\rho}_y\phi+\overline{\rho}_z\psi|^2, \quad (\mathrm{D}\,4a)$$

$$-\overline{\rho}a^2\frac{|\widehat{\kappa}_1|^2}{|U|^2}-\overline{\rho}a^2\frac{(4U_r^2-|U|^2)|\widehat{\kappa}_2|^2}{|U|^6}+\overline{\rho}a^2\frac{2U_r(\widehat{\kappa}_2^*\widehat{\kappa}_1+\widehat{\kappa}_2\widehat{\kappa}_1^*)}{|U|^4}, \quad (\mathrm{D}\,4b)$$

$$-\overline{\rho}s^2\frac{(|U|^2+s^2)|\widehat{\kappa}_1|^2}{|U^2-s^2|^2}+\overline{\rho}s^2\frac{2U_r(\widehat{\kappa}_2^*\widehat{\kappa}_1+\widehat{\kappa}_2\widehat{\kappa}_1^*)}{|U^2-s^2|^2}+\overline{\rho}s^2\frac{(|U|^2+s^2-4U_r^2)|\widehat{\kappa}_2|^2}{|U|^2|U^2-s^2|^2}, \quad (\mathrm{D}\,4c)$$

$$-\overline{\rho}\frac{(|U|^2+s^2)(\widehat{\kappa}_1^*\widehat{\mathcal{G}}+\widehat{\mathcal{G}}^*\widehat{\kappa}_1)}{|U^2-s^2|^2}+\overline{\rho}\frac{2U_r(\widehat{\kappa}_2^*\widehat{\mathcal{G}}+\widehat{\mathcal{G}}^*\widehat{\kappa}_2)}{|U^2-s^2|^2}+\overline{\rho}\frac{(|U|^2+s^2-4U_r^2)|\widehat{\mathcal{G}}|^2}{|U|^2|U^2-s^2|^2}, \quad (\mathrm{D}\,4d)$$

respectively. Here U_r represents the real part of U.

The terms in (D 4b) can be transformed into

$$-\overline{\rho}\frac{a^2}{|U|^2}|\widehat{\kappa}-\frac{2U_r}{|U|^2}\widehat{\kappa}_2|^2+\overline{\rho}\frac{a^2}{|U|^4}|\widehat{\kappa}_2|^2, \quad (\mathrm{D}\,5)$$

while the summation of the terms shown in (D 4c) and (D 4d) becomes

$$\begin{aligned}&-\overline{\rho}s^2\frac{|U|^2+s^2}{|U^2-s^2|^2}\left|\widehat{\kappa}_1-\frac{2U_r}{|U|^2+s^2}\widehat{\kappa}_2+\frac{\widehat{g}}{s^2}\right|^2\\&+\overline{\rho}\frac{s^2|\widehat{\kappa}_2|^2}{|U^2-s^2|^2}\left(\frac{4U_r^2}{|U|^2+s^2}+\frac{|U|^2+s^2-4U_r^2}{|U|^2|U^2-s^2|^2}\right)\\&+\overline{\rho}\frac{|\widehat{\mathcal{G}}|^2}{|U^2-s^2|^2}\left(\frac{|U|^2+s^2}{s^2}+\frac{|U|^2+s^2-4U_r^2}{|U|^2}\right).\end{aligned} \quad (\mathrm{D}\,6)$$

Further applying the identities

$$\frac{4U_r^2}{|U|^2+s^2}+\frac{|U|^2+s^2-4U_r^2}{|U|^2}=\frac{|U^2-s^2|^2}{|U|^2(|U|^2+s^2)}, \quad (\mathrm{D}\,7a)$$

$$\frac{|U|^2+s^2}{s^2}+\frac{|U|^2+s^2-4U_r^2}{|U|^2}=\frac{|U^2-s^2|^2}{|U|^2s^2}, \quad (\mathrm{D}\,7b)$$

to (D 6), equation (4.3) in the main text follows.

Appendix E. Lower bound estimation of the buoyancy related term in section 4.2

In order to use the result in section 4.1, we need to link Q_1 and Q_2. Writing $\widehat{\kappa}_1 = U^{1/2}\kappa+U^{-1/2}\kappa_2$ with $\kappa_2=U^{-1/2}\widehat{\kappa}_2=(U_y\eta+U_z\zeta)/2$, we obtain the identity

$$\widehat{\kappa}_1-\frac{2U_r}{|U|^2+s^2}\widehat{\kappa}_2+\frac{\widehat{\mathcal{G}}}{s^2}=U^{1/2}(\kappa+\frac{\mathcal{G}}{s^2})+U^{-1/2}\kappa_2(1-\frac{2U_rU}{|U|^2+s^2}). \quad (\mathrm{E}\,1)$$

From the elemental inequality $f_1 f_2^* + f_1^* f_2 \le 2|f_1||f_2|$ for any complex values f_1, f_2, we can find the following estimates for the terms appeared in (4.4):

$$\left|\widehat{\kappa}_1 - \frac{2U_r}{|U|^2+s^2}\widehat{\kappa}_2 + \frac{\widehat{\mathcal{G}}}{s^2}\right|^2 \\ \ge |U|\left|\kappa + \frac{\mathcal{G}}{s^2}\right|^2 + \frac{|U^2-s^2|^2}{(|U|^2+s^2)^2}\frac{|\kappa_2|^2}{|U|} - 2\frac{|U^2-s^2|}{|U|^2+s^2}|\kappa_2|\left|\kappa + \frac{\mathcal{G}}{s^2}\right|, \tag{E 2}$$

$$\left|\widehat{\kappa}_1 - \frac{2U_r}{|U|^2}\widehat{\kappa}_2\right|^2 \ge |U||\kappa|^2 + \frac{|\kappa_2|^2}{|U|} - 2|\kappa_2||\kappa|. \tag{E 3}$$

Here the identity $|U^2-s^2|^2 = (|U|^2+s^2)^2 - 4U_r^2 s^2$ may be useful to derive the first inequality. The inequalities (E 2) and (E 3) can be used to show that

$$Q_1 \ge |U|Q_2 + \frac{\overline{\rho}|\widehat{\kappa}_2|^2}{|U|}\left(\frac{s^2}{|U|^2+s^2} + \frac{a^2}{|U|^2}\right) - 2\overline{\rho}|\kappa_2|\frac{s^2|\kappa + s^{-2}\mathcal{G}|}{|U^2-s^2|} - 2\overline{\rho}\frac{a^2}{|U|^2}|\kappa_2||\kappa|.$$

The above estimate can be combined with (4.3) to yield

$$0 \ge \left\langle |U|Q_2 + \frac{f_b}{|U|} - \frac{\overline{\rho}}{|U|}\frac{a_M^2}{c_i^2}|\kappa_2|^2 - 2\overline{\rho}|\kappa_2|\frac{s^2|\kappa + s^{-2}\mathcal{G}|}{|U^2-s^2|}\right\rangle. \tag{E 4}$$

Here it is convenient to write

$$\mathcal{A} \equiv \sqrt{\langle |U|Q_2\rangle}, \qquad \mathcal{B} \equiv \sqrt{\left\langle \frac{4\overline{\rho}}{|U|}|\kappa_2|^2\right\rangle}, \qquad \mathcal{F} \equiv \frac{1}{\mathcal{B}^2}\left\langle \frac{f_b}{|U|} - \frac{\overline{\rho}}{|U|}\frac{a_M^2}{c_i^2}|\kappa_2|^2\right\rangle. \tag{E 5}$$

Noting that $\langle 2\overline{\rho}|\kappa_2|\frac{s^2|\kappa+s^{-2}\mathcal{G}|}{|U^2-s^2|}\rangle \le \mathcal{AB}$ holds because of the Schwarz inequality, the inequality (E 4) can be written in the simple form

$$0 \ge \mathcal{A}^2 - \mathcal{AB} + \mathcal{B}^2\mathcal{F}. \tag{E 6}$$

Next we shall see how the above inequality can be used to estimate the buoyancy term. Let us consider

$$\frac{\widehat{f}_b}{\overline{\rho}} - 4J_m|\widehat{\kappa}_2|^2 = [\phi^*, \psi^*]\begin{bmatrix} \overline{u}_y^2(J_1 - J_m) & \overline{u}_y\overline{u}_z(J_{12}-J_m) \\ \overline{u}_y\overline{u}_z(J_{12}-J_m) & \overline{u}_z^2(J_2 - J_m)\end{bmatrix}\begin{bmatrix}\phi \\ \psi\end{bmatrix}, \tag{E 7}$$

which is similar to (4.7). The two eigenvalues λ_+, λ_- of the matrix in (E 7) are found as

$$\lambda_\pm = \frac{(\overline{u}_y^2+\overline{u}_z^2)(J-J_m) \pm \sqrt{(\overline{u}_y^2+\overline{u}_z^2)^2(J-J_m)^2 - 4\overline{u}_y^2\overline{u}_z^2 J_m(2J_{12}-J_1-J_2)}}{2}. \tag{E 8}$$

Therefore,

$$\widehat{f}_b - 4J_m\overline{\rho}|\widehat{\kappa}_2|^2 + \lambda_M\overline{\rho}(|\phi|^2 + |\psi|^2) \ge 0, \tag{E 9}$$

where $\lambda_M \equiv \max_\Omega |\lambda_-|$ is the quantity defined in (4.27). Integrating this inequality over Ω and using (4.22), we can deduce

$$\langle f_b\rangle \ge \langle 4J_m\overline{\rho}|\kappa_2|^2 - \lambda_M\overline{\rho}(|\eta|^2+|\zeta|^2)\rangle \ge 4J_m\langle\overline{\rho}|\kappa_2|^2\rangle - \frac{\lambda_M}{k^2}\langle Q_2\rangle. \tag{E 10}$$

This is essentially the estimate of the buoyancy term (4.26) but we still need to find the relation between $\langle\overline{\rho}|\kappa_2|^2\rangle$ and $\langle Q_2\rangle$. For this purpose we use (E 6).

We note that by the definition of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{F}$ given in (E 5),

$$\left(\mathcal{F} + \frac{a_M^2}{4c_i^2}\right)\mathcal{B}^2 = \left\langle \frac{f_b}{|U|} \right\rangle \geq 4J_m \left\langle \frac{\overline{\rho}|\kappa_2|^2}{|U|} \right\rangle - \frac{\lambda_M}{k^2} \left\langle \frac{Q_2}{|U|} \right\rangle \geq J_m \mathcal{B}^2 - \frac{\lambda_M}{k^2 c_i^2}\mathcal{A}^2. \quad \text{(E 11)}$$

Here to estimate f_b we have used (E 9). Together with (E 6), the latter inequality becomes

$$0 \geq \mu \left(\frac{\mathcal{A}}{\mathcal{B}}\right)^2 - \frac{\mathcal{A}}{\mathcal{B}} + J_m - \frac{a_M^2}{4c_i^2}, \quad \text{(E 12)}$$

where μ is defined in (4.29).

Hereafter we consider unstable eigenvalues that make μ positive. In this case (E 12) implies that the quantity $\mathcal{A}/\mathcal{B}$ can be bounded from above. In fact, since (E 12) can be rewritten as

$$0 \geq \mu \left(\frac{\mathcal{A}}{\mathcal{B}} - \frac{1}{2\mu}\right)^2 - \frac{1}{4\mu} + J_m - \frac{a_M^2}{4c_i^2}, \quad \text{(E 13)}$$

we have the estimate

$$\frac{\mathcal{A}}{\mathcal{B}} \leq \frac{1 + \sqrt{1 + \mu(\frac{a_M^2}{c_i^2} - 4J_m)}}{2\mu}. \quad \text{(E 14)}$$

The right side of this inequality is $\mathcal{C}$ defined in (4.28). Furthermore, the definitions of $\mathcal{A}$ and $\mathcal{B}$ (see (E 5)) imply

$$\frac{\mathcal{A}^2}{\mathcal{B}^2} \geq \frac{c_i^2 \langle Q_2 \rangle}{4\langle \overline{\rho}|\kappa_2|^2 \rangle} \quad \text{(E 15)}$$

and thus we have the estimation of the buoyancy term (4.26) from (E 14), (E 15) and (E 10).

Appendix F. Upper bound of c_i^2 for $J_m > \frac{1}{4}$ in Theorem 2

By assumption, $J_m > 0$. In view of (E 13), if $\mu > 0$ and $J_m > \frac{1}{4\mu}$ are satisfied

$$c_i^2 < \frac{a_M^2}{4J_m - \mu^{-1}}. \quad \text{(F 1)}$$

The conditions for this inequality to be valid can be transformed as follows.

$$J_m > \frac{1}{4\mu} \text{ and } \mu > 0 \iff \mu > \frac{1}{4J_m} \iff c_i^2(1 - \frac{1}{4J_m}) > k^{-2}\lambda_M. \quad \text{(F 2)}$$

Note that the rightmost condition cannot be satisfied when $1 - \frac{1}{4J_m} < 0$ because $\lambda_M \geq 0$. Thus the condition (F 2) is equivalent to

$$1 - \frac{1}{4J_m} > 0 \text{ and } c_i^2 > \frac{k^{-2}\lambda_M}{1 - \frac{1}{4J_m}} \equiv H_0. \quad \text{(F 3)}$$

If (F 3) is satisfied, we can use the inequality (F 1) which becomes

$$(4J_m - 1)c_i^4 - (4J_m k^{-2}\lambda_M + a_M^2)c_i^2 + k^{-2}\lambda_M a_M^2 < 0, \quad \text{(F 4)}$$

from which we can deduce

$$H_- < c_i^2 < H_+ \quad \text{(F 5)}$$

with

$$H_{\pm} = \frac{(4J_m k^{-2}\lambda_M + a_M^2) \pm \sqrt{(4J_m k^{-2}\lambda_M + a_M^2)^2 - 4(4J_m - 1)k^{-2}\lambda_M a_M^2}}{2(4J_m - 1)}. \quad \text{(F 6)}$$

Note that

$$H_{\pm} - H_0 = \frac{\pm\sqrt{(4J_m k^{-2}\lambda_M - a_M^2)^2 + 4k^{-2}\lambda_M a_M^2} - (4J_m k^{-2}\lambda_M - a_M^2)}{2(4J_m - 1)} \quad \text{(F 7)}$$

and thus $H_- < H_0 < H_+$.

Let $J_m > \frac{1}{4}$. Then H_+ gives the upper bound of c_i^2. We can prove this by contradiction. Suppose there exists an unstable eigenvalue satisfying $c_i^2 \geq H_+$. Then, since $H_0 < c_i^2$ the condition (F 3) is met, and thus we must have (F 5), which contradicts with the assumption.