

Supplement Material: Adjoint equations and concomitant boundary terms

Youcheng Xi

The LNSE (Linear Navier-Stokes Equations) can be written in the matrix form:

$$\begin{aligned} \mathbf{\Gamma} \frac{\partial \Phi}{\partial t} + \mathbf{A} \frac{\partial \Phi}{\partial x} + \mathbf{B} \frac{\partial \Phi}{\partial y} + \mathbf{C} \frac{\partial \Phi}{\partial z} + \mathbf{D} \Phi = \\ \mathbf{H}_{xx} \frac{\partial^2 \Phi}{\partial x^2} + \mathbf{H}_{xy} \frac{\partial^2 \Phi}{\partial x \partial y} + \mathbf{H}_{xz} \frac{\partial^2 \Phi}{\partial x \partial z} + \mathbf{H}_{yy} \frac{\partial^2 \Phi}{\partial y^2} + \mathbf{H}_{yz} \frac{\partial^2 \Phi}{\partial y \partial z} + \mathbf{H}_{zz} \frac{\partial^2 \Phi}{\partial z^2}. \end{aligned} \quad (1)$$

Now, we are solving the Temporal problem in the $x-y$ plane with the assumption

$$\Phi = \hat{\Phi}(x, y) e^{i(\beta z - \omega t)}. \quad (2)$$

For direct problem, the solving system could be expressed as

$$\begin{aligned} (\mathbf{D} + i\beta \mathbf{C} + \beta^2 \mathbf{H}_{zz}) \hat{\Phi} + (\mathbf{A} - i\beta \mathbf{H}_{xz}) \frac{\partial}{\partial x} \hat{\Phi} + (\mathbf{B} - i\beta \mathbf{H}_{yz}) \frac{\partial}{\partial y} \hat{\Phi} \\ - \mathbf{H}_{yy} \frac{\partial^2}{\partial y^2} \hat{\Phi} - \mathbf{H}_{xy} \frac{\partial^2}{\partial x \partial y} \hat{\Phi} - \mathbf{H}_{xx} \frac{\partial^2}{\partial x^2} \hat{\Phi} - i\omega \mathbf{\Gamma} \hat{\Phi} = 0. \end{aligned} \quad (3)$$

Based on the inner-product, an adjoint problem could be defined. Considering the inner-product of an adjoint vector q^\dagger with the original problem, we will have

$$\begin{aligned} \iint \left[q^\dagger \cdot (-i\omega \mathbf{\Gamma} + \mathbf{D} + i\beta \mathbf{C} + \beta^2 \mathbf{H}_{zz}) \hat{\Phi} \right] dx dy \\ = \iint \left[(-i\omega \mathbf{\Gamma}^T + \mathbf{D}^T + i\beta \mathbf{C}^T + \beta^2 \mathbf{H}_{zz}^T) q^\dagger \cdot \hat{\Phi} \right] dx dy, \end{aligned} \quad (4a)$$

$$\begin{aligned} \iint \left[q^\dagger \cdot (\mathbf{A} - i\beta \mathbf{H}_{xz}) \frac{\partial}{\partial x} \hat{\Phi} \right] dx dy = \iint \frac{\partial}{\partial x} \left[q^\dagger \cdot (\mathbf{A} - i\beta \mathbf{H}_{xz}) \hat{\Phi} \right] dx dy \\ - \iint \left[(\mathbf{A}^T - i\beta \mathbf{H}_{xz}^T) \frac{\partial q^\dagger}{\partial x} \cdot \hat{\Phi} \right] dx dy \\ - \iint \left[\frac{\partial (\mathbf{A}^T - i\beta \mathbf{H}_{xz}^T)}{\partial x} q^\dagger \cdot \hat{\Phi} \right] dx dy, \end{aligned} \quad (4b)$$

$$\begin{aligned}
\iint \left[q^\dagger \cdot (\mathbf{B} - i\beta \mathbf{H}_{yz}) \frac{\partial}{\partial y} \hat{\Phi} \right] dx dy &= \iint \frac{\partial}{\partial y} \left[q^\dagger \cdot (\mathbf{B} - i\beta \mathbf{H}_{yz}) \hat{\Phi} \right] dx dy \quad (4c) \\
&- \iint \left[(\mathbf{B}^T - i\beta \mathbf{H}_{yz}^T) \frac{\partial q^\dagger}{\partial y} \cdot \hat{\Phi} \right] dx dy \\
&- \iint \left[\frac{\partial (\mathbf{B}^T - i\beta \mathbf{H}_{yz}^T)}{\partial y} q^\dagger \cdot \hat{\Phi} \right] dx dy,
\end{aligned}$$

$$\begin{aligned}
\iint \left[q^\dagger \cdot \mathbf{H}_{yy} \frac{\partial^2}{\partial y^2} \hat{\Phi} \right] dx dy &= \iint \frac{\partial}{\partial y} \left(\left[\mathbf{H}_{yy}^T q^\dagger \cdot \frac{\partial \hat{\Phi}}{\partial y} \right] - \left[\frac{\partial \mathbf{H}_{yy}^T q^\dagger}{\partial y} \cdot \hat{\Phi} \right] \right) dx dy \quad (4d) \\
&+ \iint \left[\left(\frac{\partial^2 \mathbf{H}_{yy}^T}{\partial y^2} + 2 \frac{\partial \mathbf{H}_{yy}^T}{\partial y} \frac{\partial}{\partial y} + \mathbf{H}_{yy}^T \frac{\partial^2}{\partial y^2} \right) \cdot \hat{\Phi} \right] dx dy,
\end{aligned}$$

$$\begin{aligned}
\iint \left[q^\dagger \cdot \mathbf{H}_{xx} \frac{\partial^2}{\partial x^2} \hat{\Phi} \right] dx dy &= \iint \frac{\partial}{\partial x} \left(\left[\mathbf{H}_{xx}^T q^\dagger \cdot \frac{\partial \hat{\Phi}}{\partial x} \right] - \left[\frac{\partial \mathbf{H}_{xx}^T q^\dagger}{\partial x} \cdot \hat{\Phi} \right] \right) dx dy \quad (4e) \\
&+ \iint \left[\left(\frac{\partial^2 \mathbf{H}_{xx}^T}{\partial x^2} + 2 \frac{\partial \mathbf{H}_{xx}^T}{\partial x} \frac{\partial}{\partial x} + \mathbf{H}_{xx}^T \frac{\partial^2}{\partial x^2} \right) \cdot \hat{\Phi} \right] dx dy,
\end{aligned}$$

$$\begin{aligned}
&\iint \left[q^\dagger \cdot \mathbf{H}_{xy} \frac{\partial^2}{\partial x \partial y} \hat{\Phi} \right] dx dy \quad (4f) \\
&= \frac{1}{2} \iint \left[q^\dagger \cdot \mathbf{H}_{xy} \frac{\partial^2}{\partial x \partial y} \hat{\Phi} \right] dx dy + \frac{1}{2} \iint \left[q^\dagger \cdot \mathbf{H}_{xy} \frac{\partial^2}{\partial x \partial y} \hat{\Phi} \right] dx dy \\
&= \frac{1}{2} \iint \frac{\partial}{\partial x} \left(\left[\mathbf{H}_{xy}^T q^\dagger \cdot \frac{\partial \hat{\Phi}}{\partial y} \right] - \left[\frac{\partial \mathbf{H}_{xy}^T q^\dagger}{\partial y} \cdot \hat{\Phi} \right] \right) dx dy \\
&+ \frac{1}{2} \iint \frac{\partial}{\partial y} \left(\left[\mathbf{H}_{xy}^T q^\dagger \cdot \frac{\partial \hat{\Phi}}{\partial x} \right] - \left[\frac{\partial \mathbf{H}_{xy}^T q^\dagger}{\partial x} \cdot \hat{\Phi} \right] \right) dx dy \\
&+ \iint \left[\left(\frac{\partial^2 \mathbf{H}_{xy}^T}{\partial x \partial y} + \frac{\partial \mathbf{H}_{xy}^T}{\partial x} \frac{\partial}{\partial y} + \frac{\partial \mathbf{H}_{xy}^T}{\partial y} \frac{\partial}{\partial x} + \mathbf{H}_{xy}^T \frac{\partial^2}{\partial x \partial y} \right) \cdot \hat{\Phi} \right] dx dy,
\end{aligned}$$

And the terms marked red represent non-homogeneous terms during the integration by parts. Using the integration detailed in section 1 and moving all the terms to the right,

we can get that

$$\begin{aligned}
B.C. = & - \int_{\Gamma} \left[q^{\dagger} \cdot (\mathbf{A} - i\beta \mathbf{H}_{xz}) \hat{\Phi} \right] dy + \int_{\Gamma} \left[q^{\dagger} \cdot (\mathbf{B} - i\beta \mathbf{H}_{yz}) \hat{\Phi} \right] dx \\
& + \int_{\Gamma} \left(\left[\mathbf{H}_{xx}^T q^{\dagger} \cdot \frac{\partial \hat{\Phi}}{\partial x} \right] - \left[\frac{\partial \mathbf{H}_{xx}^T q^{\dagger}}{\partial x} \cdot \hat{\Phi} \right] \right) dy \\
& - \int_{\Gamma} \left(\left[\mathbf{H}_{yy}^T q^{\dagger} \cdot \frac{\partial \hat{\Phi}}{\partial y} \right] - \left[\frac{\partial \mathbf{H}_{yy}^T q^{\dagger}}{\partial y} \cdot \hat{\Phi} \right] \right) dx \\
& + \frac{1}{2} \int_{\Gamma} \left(\left[\mathbf{H}_{xy}^T q^{\dagger} \cdot \frac{\partial \hat{\Phi}}{\partial y} \right] - \left[\frac{\partial \mathbf{H}_{xy}^T q^{\dagger}}{\partial y} \cdot \hat{\Phi} \right] \right) dy \\
& - \frac{1}{2} \int_{\Gamma} \left(\left[\mathbf{H}_{xy}^T q^{\dagger} \cdot \frac{\partial \hat{\Phi}}{\partial x} \right] - \left[\frac{\partial \mathbf{H}_{xy}^T q^{\dagger}}{\partial x} \cdot \hat{\Phi} \right] \right) dx.
\end{aligned} \tag{5}$$

Taking only the perturbations along wall surface for instance, the line integration along Γ is simplified along Γ_1 . Thus, taking the detailed form of boundary conditions for adjoint problems, the remain $B.C.$ terms can be expressed as

$$\int_{\Gamma} \left[q^{\dagger} \cdot (\mathbf{A} - i\beta \mathbf{H}_{xz}) \hat{\Phi} \right] dy = \int_{\Gamma_1} \rho^{\dagger} \rho_0 u_w dy \tag{6a}$$

$$\int_{\Gamma} \left[q^{\dagger} \cdot (\mathbf{B} - i\beta \mathbf{H}_{yz}) \hat{\Phi} \right] dx = \int_{\Gamma_1} \rho^{\dagger} \rho_0 v_w dx \tag{6b}$$

$$\begin{aligned}
& \int_{\Gamma} \left[\frac{\partial \mathbf{H}_{xx}^T q^{\dagger}}{\partial x} \cdot \hat{\Phi} \right] dy \\
& = \int_{\Gamma_1} \left(\frac{4\mu}{3Re} \frac{\partial u^{\dagger}}{\partial x} u_w + \frac{\mu}{Re} \frac{\partial v^{\dagger}}{\partial x} v_w + \frac{\mu}{Re} \frac{\partial w^{\dagger}}{\partial x} w_w + \frac{\mu}{RePr} \frac{\partial T^{\dagger}}{\partial x} T_w \right) dy
\end{aligned} \tag{6c}$$

$$\begin{aligned}
& \int_{\Gamma} \left[\frac{\partial \mathbf{H}_{yy}^T q^{\dagger}}{\partial y} \cdot \hat{\Phi} \right] dx \\
& = \int_{\Gamma_1} \left(\frac{\mu}{Re} \frac{\partial u^{\dagger}}{\partial y} u_w + \frac{4\mu}{3Re} \frac{\partial v^{\dagger}}{\partial y} v_w + \frac{\mu}{Re} \frac{\partial w^{\dagger}}{\partial y} w_w + \frac{\mu}{RePr} \frac{\partial T^{\dagger}}{\partial y} T_w \right) dx
\end{aligned} \tag{6d}$$

$$\int_{\Gamma} \left[\frac{\partial \mathbf{H}_{xy}^T q^{\dagger}}{\partial y} \cdot \hat{\Phi} \right] dy = \int_{\Gamma} \left[\frac{\mu}{3Re} \frac{\partial v^{\dagger}}{\partial y} u_w + \frac{\mu}{3Re} \frac{\partial u^{\dagger}}{\partial y} v_w \right] dy \tag{6e}$$

$$\int_{\Gamma} \left[\frac{\partial \mathbf{H}_{xy}^T q^{\dagger}}{\partial x} \cdot \hat{\Phi} \right] dx = \int_{\Gamma} \left[\frac{\mu}{3Re} \frac{\partial v^{\dagger}}{\partial x} u_w + \frac{\mu}{3Re} \frac{\partial u^{\dagger}}{\partial x} v_w \right] dx \tag{6f}$$

Therefore, the *B.C.* term can be expressed as

$$\begin{aligned}
 B.C. = & - \int_{\Gamma_1} \rho^\dagger \rho_0 u_w dy + \int_{\Gamma_1} \rho^\dagger \rho_0 v_w dx \\
 & - \int_{\Gamma_1} \left(\frac{4\mu}{3Re} \frac{\partial u^\dagger}{\partial x} u_w + \frac{\mu}{Re} \frac{\partial v^\dagger}{\partial x} v_w + \frac{\mu}{Re} \frac{\partial w^\dagger}{\partial x} w_w + \frac{\mu}{RePr} \frac{\partial T^\dagger}{\partial x} T_w \right) dy \\
 & + \int_{\Gamma_1} \left(\frac{\mu}{Re} \frac{\partial u^\dagger}{\partial y} u_w + \frac{4\mu}{3Re} \frac{\partial v^\dagger}{\partial y} v_w + \frac{\mu}{Re} \frac{\partial w^\dagger}{\partial y} w_w + \frac{\mu}{RePr} \frac{\partial T^\dagger}{\partial y} T_w \right) dx \\
 & - \frac{1}{2} \int_{\Gamma_1} \left[\frac{\mu}{3Re} \frac{\partial v^\dagger}{\partial y} u_w + \frac{\mu}{3Re} \frac{\partial u^\dagger}{\partial y} v_w \right] dy + \frac{1}{2} \int_{\Gamma_1} \left[\frac{\mu}{3Re} \frac{\partial v^\dagger}{\partial x} u_w + \frac{\mu}{3Re} \frac{\partial u^\dagger}{\partial x} v_w \right] dx
 \end{aligned} \tag{7}$$

From the derivation of this term, one can find that this formula is available for any two-dimensional geometry and extension to full three-dimensional cases is straightforward by using the Gauss' law.

1 Integration of Green Type

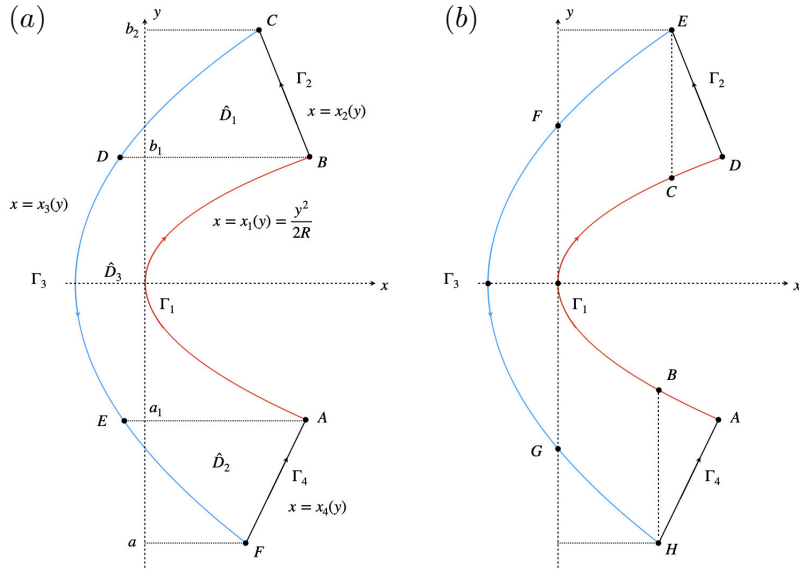


Fig. 1: Schematic diagram of integration. The boundaries of the domain is labeled by $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4 with the arrows indicating the direction. (a) and (b) represent the same domain with different divided strategies.

Let's consider integrations over the whole domain shown in figure 1(a). At first, the integration domain can be divided along y direction into three parts marked as,

$$\iint_{all} dx dy = \iint_{\hat{D}_1} dx dy + \iint_{\hat{D}_2} dx dy + \iint_{\hat{D}_3} dx dy, \tag{8}$$

where \hat{D}_1, \hat{D}_2 and \hat{D}_3 are shown in figure 1(a). And these regions can be expressed as

$$\begin{aligned} \text{in } \hat{D}_1 : \quad & x_3(y) \leq x \leq x_2(y), b_1 \leq y \leq b_2 \\ \text{in } \hat{D}_3 : \quad & x_3(y) \leq x \leq x_1(y), a_1 \leq y \leq b_1 \\ \text{in } \hat{D}_2 : \quad & x_3(y) \leq x \leq x_4(y), a \leq y \leq a_1 \end{aligned} \quad (9)$$

Assuming that function $F(x, y)$ is smooth enough over the whole domain, including the boundary lines. Considering the integrations of $\partial F(x, y)/\partial x$ over the whole domain,

$$\iint_{all} \frac{\partial F(x, y)}{\partial x} dx dy = \iint_{\hat{D}_1} \frac{\partial F(x, y)}{\partial x} dx dy + \iint_{\hat{D}_2} \frac{\partial F(x, y)}{\partial x} dx dy + \iint_{\hat{D}_3} \frac{\partial F(x, y)}{\partial x} dx dy, \quad (10)$$

we have

$$\begin{aligned} \iint_{\hat{D}_1} \frac{\partial F(x, y)}{\partial x} dx dy &= \int_{b_1}^{b_2} \left[dy \int_{x_3(y)}^{x_2(y)} \frac{\partial F(x, y)}{\partial x} dx \right] \\ &= \int_{b_1}^{b_2} [F(x_2(y), y) - F(x_3(y), y)] dy = \int_{\widehat{BC}} F dy + \int_{\widehat{CD}} F dy, \end{aligned} \quad (11)$$

$$\begin{aligned} \iint_{\hat{D}_3} \frac{\partial F(x, y)}{\partial x} dx dy &= \int_{a_1}^{b_1} \left[dy \int_{x_3(y)}^{x_1(y)} \frac{\partial F(x, y)}{\partial x} dx \right] \\ &= \int_{a_1}^{b_1} [F(x_1(y), y) - F(x_3(y), y)] dy = \int_{\widehat{AB}} F dy + \int_{\widehat{DE}} F dy, \end{aligned} \quad (12)$$

$$\begin{aligned} \iint_{\hat{D}_2} \frac{\partial F(x, y)}{\partial x} dx dy &= \int_a^{a_1} \left[dy \int_{x_3(y)}^{x_4(y)} \frac{\partial F(x, y)}{\partial x} dx \right] \\ &= \int_a^{a_1} [F(x_4(y), y) - F(x_3(y), y)] dy = \int_{\widehat{FA}} F dy + \int_{\widehat{EF}} F dy. \end{aligned} \quad (13)$$

Thus for any smooth function $F(x, y)$,

$$\iint_{all} \frac{\partial F(x, y)}{\partial x} dx dy = \int_{\Gamma_1} F dy + \int_{\Gamma_2} F dy + \int_{\Gamma_3} F dy + \int_{\Gamma_4} F dy. \quad (14)$$

Using the similar process with divided the domain along x direction as presented in figure 1(b), we can find that

$$- \iint_{all} \frac{\partial F(x, y)}{\partial y} dx dy = \int_{\Gamma_1} F dx + \int_{\Gamma_2} F dx + \int_{\Gamma_3} F dx + \int_{\Gamma_4} F dx \quad (15)$$