# A continuum model to study fluid dynamics within oscillating elastic nanotubes
# Supplementary Material

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## A Derivation of the coupled equations of motion via the Minimal Action's Principle

The Hamilton's Principle for an open system at constant temperature, is given by

$$\delta S + \delta W + \delta C = 0 \ . \tag{A.1}$$

Expressions for the variation of each term are given in the following subsections.

## Variation of the kinetic energy of the fluid

It is given by

$$\delta \int_t T_f dt = \delta \int_t \int_V \frac{1}{2} \rho \left| \mathbf{v_{fluid}} \right|^2 dV\, dt \ . \tag{A.2}$$

In Cartesian coordinates, fluid velocity vector $\boldsymbol{v_{fluid}}$ is given as follows:

$$\mathbf{v_{fluid}} = \left( 0, \frac{\partial u}{\partial t} + \frac{v \frac{\partial u}{\partial z}}{\sqrt{1 + \left(\frac{\partial u}{\partial z}\right)^2}}, \frac{v}{\sqrt{1 + \left(\frac{\partial u}{\partial z}\right)^2}} \right) \ . \tag{A.3}$$

In the small deformation limit, we expand the term $\frac{1}{\sqrt{1+\left(\frac{\partial u}{\partial z}\right)^2}}$, as follows:

$$\frac{1}{\sqrt{1 + \left(\frac{\partial u}{\partial z}\right)^2}} = 1 - \frac{1}{2}\left(\frac{\partial u}{\partial z}\right)^2 + \frac{3}{8}\left(\frac{\partial u}{\partial z}\right)^4 + ... \approx 1 \tag{A.4}$$

and, subsequently, fluid velocity vector is simplified to

$$\mathbf{v_{fluid}} \approx \left( 0, \frac{\partial u}{\partial t} + v \frac{\partial u}{\partial z}, v \right) \tag{A.5}$$

and, accordingly, the magnitude of $\boldsymbol{v_{fluid}}$ is given by

$$\left| \mathbf{v_{fluid}} \right|^2 = v^2 + \left(\frac{\partial u}{\partial t}\right)^2 + 2v \frac{\partial u}{\partial t}\frac{\partial u}{\partial z} + v^2 \left(\frac{\partial u}{\partial z}\right)^2 \ . \tag{A.6}$$

Afterwards, $|\boldsymbol{v_{fluid}}|$ from (A.6) is incorporated into the Kinetic Energy in (A.3). The variation of the kinetic energy of the fluid leads to the following expression:

$$\delta \int_t T_f dt = \int_t \int_V \left( -\rho \frac{\partial^2 u}{\partial t^2} - \rho v^2 \frac{\partial^2 u}{\partial z^2} - 2\rho v \frac{\partial^2 u}{\partial t \partial z} - \rho \frac{\partial v}{\partial t} \frac{\partial u}{\partial z} - \rho \frac{\partial v}{\partial z} \frac{\partial u}{\partial t} \right) \delta u$$
$$+ \left( -\rho \frac{\partial v}{\partial t} \left( 1 + \left( \frac{\partial u}{\partial z} \right)^2 \right) - 2\rho v \frac{\partial u}{\partial z} \frac{\partial^2 u}{\partial z \partial t} - \rho \left( \frac{\partial^2 u}{\partial t^2} \frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial z \partial t} \right) \right) \delta z_{fluid} \, dV \, dt \ . \tag{A.7}$$

## Variation of the kinetic energy of the tube

It is given by

$$\delta \int_t T_t dt = \delta \int_t \int_V \frac{1}{2} \rho \left| \mathbf{v_{tube}} \right|^2 dV \, dt \ . \tag{A.8}$$

In Cartesian coordinates,

$$\mathbf{v_{tube}} = \left( 0, \frac{\partial u}{\partial t} + v \frac{\frac{\partial u}{\partial z}}{\sqrt{1 + \left(\frac{\partial u}{\partial z}\right)^2}}, v \frac{1}{\sqrt{1 + \left(\frac{\partial u}{\partial z}\right)^2}} \right) \tag{A.9}$$

and the magnitude of tube velocity vector is given by

$$\left| \mathbf{v_{tube}} \right|^2 = \left( \frac{\partial u}{\partial t} \right)^2 \ . \tag{A.10}$$

Subsequently, substituting the magnitude of tube velocity from (A.10) into the kinetic energy of tube in (A.9). The computation of the variation leads to the following expression:

$$\delta \int_t T_t dt = \int_t \int_V \left( -\rho_t \frac{\partial^2 u}{\partial t^2} \right) \delta u \, dV \, dt \ . \tag{A.11}$$

## Variation of the potential energy of the tube

It is given by the following expression:

$$\delta \int_t V_t dt = \delta \int_t \int_V \frac{1}{2} E y^2 \left( \frac{\partial^2 u}{\partial z^2} \right)^2 dV \, dt \ , \tag{A.12}$$

and its variation is computed as follows:

$$\delta \int_t V_t dt = \delta \int_t \int_V E y^2 \frac{\partial^4 u}{\partial z^4} \delta u \, dV \, dt \ . \tag{A.13}$$

## Variation of the external work

It is given by the expression previously stated in Section 2 of the main article, as

$$\delta W = \int_t \int_S \boldsymbol{F_{ext}} \cdot \delta \mathbf{r_{fluid}} \, dS \, dt = \int_t \int_S (-p\mathbf{1} + \tau) \cdot \boldsymbol{n} \cdot \delta \boldsymbol{r_{fluid}} \, dS \, dt \ . \tag{A.14}$$

## Variation of the constrains

The conservation of fluid mass has been incorporated as a restriction in the following form:

$$\delta C = \delta \int_t \int_V \Lambda \nabla \cdot \mathbf{r_{fluid}} dV\, dt \ . \tag{A.15}$$

From the properties of the differential operators, it is followed that

$$\nabla \cdot \Lambda \mathbf{r_{fluid}} = \mathbf{r_{fluid}} \cdot \nabla \Lambda + \Lambda \nabla \cdot \mathbf{r_{fluid}} \tag{A.16}$$

and, by a rearrangement, the following expression is obtained:

$$\Lambda \nabla \cdot \mathbf{r_{fluid}} = \nabla \cdot \Lambda \mathbf{r_{fluid}} - \mathbf{r_{fluid}} \cdot \nabla \Lambda \ . \tag{A.17}$$

Substituting (A.17) into (A.15), leads to

$$\int_V \Lambda \nabla \cdot \mathbf{r_{fluid}} dV = \int_V \nabla \cdot \Lambda \mathbf{r_{fluid}} dV - \int_V \mathbf{r_{fluid}} \cdot \nabla \Lambda dV \ . \tag{A.18}$$

By the divergence theorem, the following expression is obtained:

$$\int_V \nabla \cdot \Lambda \mathbf{r_{fluid}}\, dV = \int_S \Lambda \mathbf{r_{fluid}} \cdot \mathbf{n}\, dS \ , \tag{A.19}$$

which is incorporated in the constrain, as follows:

$$\int_V \Lambda \nabla \cdot \mathbf{r_{fluid}} dV = \int_S \Lambda \mathbf{r_{fluid}} \cdot \mathbf{n}\, dS - \int_V \mathbf{r_{fluid}} \cdot \nabla \Lambda dV \ . \tag{A.20}$$

By considering that the fluid motion only occurs along the axial direction of the tube as stated in the main article, is given as below:

$$\mathbf{r_{fluid}} = \mathbf{r_{tube}} + z_{fluid}\, \mathbf{q_{tan}} \ , \tag{A.21}$$

which corresponds to the statement in the body of the article, as

$$\mathbf{v_{fluid}} = \mathbf{v_{tube}} + v\, \mathbf{q_{tan}} \ . \tag{A.22}$$

Within the limit of small deformation, the differential operators are simplified to the following expressions:

$$\nabla \cdot \mathbf{r_{fluid}} = \frac{\partial z_{fluid}}{\partial z} \ , \tag{A.23}$$

$$\mathbf{r_{fluid}} \cdot \mathbf{e_z} = z_{fluid} \ , \tag{A.24}$$

$$\mathbf{r_{fluid}} \cdot \nabla \Lambda = z_{fluid} \left( \mathbf{e_z} \cdot \nabla \Lambda \right) \ . \tag{A.25}$$

Afterwards, the considerations in (A.20), (A.23) and (A.24) are incorporated in the constraint in (A.15), and the variation is computed, leading to the following result:

$$\delta C = \int_t \int_S \Lambda \delta z_{fluid}\, dS\, dt - \int_t \int_V \nabla \Lambda \cdot \boldsymbol{e_z} \delta z_{fluid}\, dV\, dt \ . \tag{A.26}$$

Finally, the expressions in (A.7), (A.11), (A.13), (A.14) and (A.26) are incorporated in the Hamilton's principle stated in (A.1), the following expression is obtained:

$$
\begin{aligned}
&\int_t \int_V \left( -(\rho+\rho_t)\frac{\partial^2 u}{\partial t^2} - \rho v^2 \frac{\partial^2 u}{\partial z^2} - 2\rho v \frac{\partial^2 u}{\partial t \partial z} - \rho \frac{\partial v}{\partial t}\frac{\partial u}{\partial z} - \rho \frac{\partial v}{\partial z}\frac{\partial u}{\partial t} - E y^2 \frac{\partial^4 u}{\partial z^4} \right) \delta u \, dV \, dt \\
&+ \int_t \int_V \left( -\rho \frac{\partial v}{\partial t} - 2\rho v \frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z \partial t} - \rho \left( \frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial z \partial t} \right) - \nabla \Lambda \cdot \boldsymbol{e_z} \right) \delta z_{fluid} \, dV \, dt \\
&+ \int_t \int_S ((-p\mathbf{1} + \tau)\cdot \boldsymbol{n} + \Lambda \boldsymbol{n}) \cdot \boldsymbol{e_z} \delta z_{fluid} \, dS \, dt = 0 \ ,
\end{aligned} \tag{A.27}
$$

where the sub-indexes in volume regions $V_1$, $V_2$ and the surfaces $S_1$, $S_2$ where omitted for the sake of simplicity in the expression; however, each term in the integrals should be performed on its corresponding region. Equation (A.27) leads to one equation of motion for the variation of each of the field variables (which are $u$ and $z_{fluid}$).

For the variation of tube displacement, $\delta u$, we obtain the following expression:

$$
\int_V \left( -\rho \frac{\partial^2 u}{\partial t^2} - \rho v^2 \frac{\partial^2 u}{\partial z^2} - 2\rho v \frac{\partial^2 u}{\partial t \partial z} - \rho \frac{\partial v}{\partial t}\frac{\partial u}{\partial z} - \rho \frac{\partial v}{\partial z}\frac{\partial u}{\partial t} - \rho_t \frac{\partial^2 u}{\partial t^2} - E y^2 \frac{\partial^4 u}{\partial z^4} \right) dV = 0 \ . \tag{A.28}
$$

For the variation of fluid displacement $\delta z_{fluid}$, we obtain the following expression:

$$
\begin{aligned}
&\int_V \left( -\rho \frac{\partial v}{\partial t} \left( 1 + \left( \frac{\partial u}{\partial z} \right)^2 \right) - 2\rho v \frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z \partial t} - \rho \left( \frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial z \partial t} \right) - \nabla \Lambda \cdot \boldsymbol{e_z} \right) dV \\
&+ \int_S ((-p\mathbf{1} + \tau)\cdot \boldsymbol{n} + \Lambda \boldsymbol{n}) \cdot \boldsymbol{e_z} dS = 0 \ .
\end{aligned} \tag{A.29}
$$

Integrals in (A.29) are applied on different integration regions: the first one is a volume integral, whereas the second one is a surface integral. Therefore, each integral vanishes independently, leading to the following expression for the volume integral:

$$
\int_V \left( -\rho \frac{\partial v}{\partial t} \left( 1 + \left( \frac{\partial u}{\partial z} \right)^2 \right) - 2\rho v \frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z \partial t} - \rho \left( \frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial z \partial t} \right) - \nabla \Lambda \cdot \boldsymbol{e_z} \right) dV = 0 \ , \tag{A.30}
$$

and this is the expression given for the surface integral:

$$
\int_S ((-p\mathbf{1} + \tau)\cdot \boldsymbol{n} + \Lambda \boldsymbol{n}) \cdot \boldsymbol{e_z} \delta z_{fluid} dS = 0 \ . \tag{A.31}
$$

From (A.31), we obtain the value of the Lagrangian multiplier $\Lambda$, as

$$
\Lambda \boldsymbol{n} = (p\mathbf{1} - \tau) \cdot \boldsymbol{n} \ . \tag{A.32}
$$

Therefore, the physical meaning of the Lagrangian multiplier $\Lambda$ is the net stress applied along the plane in which fluid flow occurs. It is possible to see that the volume integral in $\delta C$ in (A.26) incorporates a force in the equation of motion, given by the spatial derivative of $\Lambda$, as follows:

$$
\nabla \Lambda \cdot \boldsymbol{e_z} = \frac{\partial p}{\partial z} - (\nabla \cdot \tau) \cdot \boldsymbol{e_z} \ , \tag{A.33}
$$

where the stress tensor of a Newtonian fluid and its divergence must be given in terms of the dynamic coordinates $(r', \theta', z')$, leading to the following expression:

$$
\begin{aligned}
(\nabla \cdot \tau) \cdot \boldsymbol{e_{z'}} \quad = \quad & \mu \left( \frac{\partial^2 v_{z'}}{\partial r'^2} + \frac{1}{r'} \frac{\partial v_{z'}}{\partial r'} + \frac{1}{r'^2} \frac{\partial^2 v_{z'}}{\partial \theta'^2} - \frac{\frac{\partial^2 u}{\partial z'^2} \left( \sin(\theta') \frac{\partial v_{z'}}{\partial r'} + \frac{\cos(\theta')}{r'} \frac{\partial v_{z'}}{\partial \theta'} \right)}{\left(1 + \left(\frac{\partial u}{\partial z}\right)^2\right)^{3/2} - r' \sin(\theta') \frac{\partial^2 u}{\partial z^2}} \right. \\
& \left. - \frac{\left(\frac{\partial^2 u}{\partial z^2}\right)^2 v_{z'}}{\left(\left(1 + \left(\frac{\partial u}{\partial z}\right)^2\right)^{3/2} - r' \sin(\theta') \frac{\partial^2 u}{\partial z^2}\right)^2} \right) . \qquad \text{(A.34)}
\end{aligned}
$$

When the small deformation limit is considered, along with a tube radius much smaller than the radius of curvature of the tube, and the angular dependence of the flow velocity is neglected, then the last three terms in (A.34) are zero, which leads to the following approximated expression:

$$
(\nabla \cdot \tau) \cdot \boldsymbol{e_{z'}} = \mu \left( \frac{\partial^2 v_{z'}}{\partial r'^2} + \frac{1}{r'} \frac{\partial v_{z'}}{\partial r'} \right) . \qquad \text{(A.35)}
$$

and the prime notation in (A.35) omitted as in the rest of this section. Incorporating $\Lambda$ from (A.32) into (A.30), the following expression is obtained:

$$
\int_V \left( -\rho \frac{\partial v}{\partial t} - 2\rho v \frac{\partial u}{\partial z} \frac{\partial^2 u}{\partial z \partial t} - \rho \left( \frac{\partial^2 u}{\partial t^2} \frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial z \partial t} \right) - \frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 v_{z'}}{\partial r'^2} + \frac{1}{r'} \frac{\partial v_{z'}}{\partial r'} \right) \right) dV = 0 . \qquad \text{(A.36)}
$$

# B Analytical solution of flow velocity influenced by tube vibration

For the case of small flow velocity respect to the velocity of propagation of elastic waves along the tube, the equations of motion are rewritten below:

$$
EI \frac{\partial^4 u}{\partial z^4} + \left(\rho A_f + \rho_t A_t\right) \frac{\partial^2 u}{\partial t^2} = 0 , \qquad \text{(B.1)}
$$

$$
\rho \frac{\partial v}{\partial t} + \rho g(t) v + \rho h(t) + \frac{\partial p}{\partial z} - \mu \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right) = 0 , \qquad \text{(B.2)}
$$

where $E$ corresponds to Young modulus of the tube, $I$ is the second moment of inertia, which for a cylindrical shell is given by

$$
I = \frac{\pi}{4} \left(R_o^4 - R^4\right) , \qquad \text{(B.3)}
$$

where $R_o$ and $R$ are the outer and inner radius, respectively.

Equation (B.1) is solved when initial and boundary conditions are given, and leads to a solution of the form

$$
u(z,t) = \sum_{n=1}^{N} A_n \sin(k_n z - \omega_n t) , \qquad \text{(B.4)}
$$

where $A_n$ is the amplitude of each plane wave, $k_n$ is the spatial modulation and $\omega_n$ is the corresponding frequency, which is obtained by the dispersion relation associated to (B.1), as

$$
\omega(k) = \pm k^2 \sqrt{\frac{EI}{\rho A_f + \rho_t A_t}} . \qquad \text{(B.5)}
$$

Then, the solution $u(z,t)$ is incorporated in $g(t)$ and $h(t)$. In the following treatment, the solution of (B.2) for flow velocity is provided for arbitrary functions $g(t)$ and $h(t)$.

Equation (B.2) is a linear non-homogeneous partial differential equation with time-dependent coefficients —particularly, $\rho g(t)$—. The non-homogeneous term is only dependent on time as well —particularly, $\rho h(t)$—. Therefore, we write the general solution of such differential equation in the following form:

$$v(r,t) = v_{homo}(r,t) + v_{part}(t) \ , \tag{B.6}$$

where $v_{homo}(r,t)$ is the solution of the associated homogeneous differential equation, whereas $v_{part}(t)$ is a particular solution of the inhomogeneous term. Equation for $v_{homo}(r,t)$ is

$$\frac{\partial v_{homo}}{\partial t} + g(t)v_{homo} - \frac{\mu}{\rho}\left(\frac{\partial^2 v_{homo}}{\partial r^2} + \frac{1}{r}\frac{\partial v_{homo}}{\partial r}\right) = 0 \ . \tag{B.7}$$

By performing separation of variables, $v_{homo}(r,t) = R(r)T(t)$ and substituting in (B.7), we have

$$\frac{1}{T}\frac{\partial T}{\partial t} + g(t) = \frac{1}{R}\frac{\mu}{\rho}\left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r}\right) = -i\lambda \ , \tag{B.8}$$

where $-i\lambda$ is the constant of separation, it has been expressed in such a way in order to simplify further treatment. In all the integrals involved, we will omit the primed indexes for integration over time, for the sake of simplicity during the derivation. Separation of variables allows one to solve two independent ordinary differential equations from (B.8). First, the equation for $T(t)$ results:

$$\frac{dT}{dt} + (i\lambda + g(t))\,T = 0 \ , \tag{B.9}$$

whose solution is given by

$$T = T_0 e^{-i\lambda t} e^{-\int_{t_0}^{t} g(t)dt} \ . \tag{B.10}$$

The equation for $R(r)$ is given by

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + \frac{i\lambda\rho}{\mu}r^2 R = 0 \ , \tag{B.11}$$

with the following solution:

$$R(r) = R_1 J_0\left(\sqrt{\frac{i\lambda\rho}{\mu}}r\right) + R_2 Y_0\left(-\sqrt{\frac{i\lambda\rho}{\mu}}r\right) \ . \tag{B.12}$$

Incorporating expressions from (B.10) and (B.12) in $v_{homo}$ and after renaming the constant coefficients, the solution of (B.7) is given by

$$v_{homo}(r,t) = \left[C_1 J_0\left(\sqrt{\frac{\lambda\rho}{\mu}}r\right) + C_2 Y_0\left(-\sqrt{\frac{\lambda\rho}{\mu}}r\right)\right] e^{-i\lambda t} e^{-\int_0^t g(t)dt} \ . \tag{B.13}$$

Afterwards, differential equation for $v_{part}(t)$ is

$$\frac{dv_{part}}{dt} + g(t)v_{part} + \frac{1}{\rho}\frac{\partial p}{\partial z} + h(t) = 0 \ . \tag{B.14}$$

Equation (B.14) is a first-order ordinary differential equation; hence, we rewrite (B.14) as a differential form, as follows:

$$M(v_{part},t)dv_{part} + N(v_{part},t)dt = 0 \ , \tag{B.15}$$

with

$$M(v_{part}, t) = 1 \ , \tag{B.16}$$

$$N(v_{part}, t) = g(t)v_{part} + \frac{1}{\rho}\frac{\partial p}{\partial z} + h(t) \ . \tag{B.17}$$

An exact differential form for the function $F(v_{part}, t)$ is given by

$$dF = \frac{\partial F}{\partial v_{part}} dv_{part} + \frac{\partial F}{\partial t} dt \tag{B.18}$$

with equal cross derivatives, as stated below:

$$\frac{\partial M}{\partial v_{part}} = \frac{\partial N}{\partial t} \ . \tag{B.19}$$

However, (B.16) is not an exact differential form, because the cross derivatives are unequal, as

$$\frac{\partial M}{\partial t} = 0 \ , \tag{B.20}$$

$$\frac{\partial N}{\partial v_{part}} = g(t) \ . \tag{B.21}$$

An integrating factor $\mu(t)$ is incorporated in (B.16), to obtain the following exact differential form:

$$dF = \mu(t) \left[ M(v,t)dv + N(v,t)dt \right] = 0 \ , \tag{B.22}$$

where $\mu(t)$ is given by

$$\begin{aligned} \mu(t) &= \exp\left( \int \frac{\frac{\partial N}{\partial v_{part}} - \frac{\partial M}{\partial t}}{M} dt \right) \\ &= \exp\left( \int g(t)dt \right) \ . \end{aligned} \tag{B.23}$$

The solution of the differential form is given by

$$F(v,t) = constant \ , \tag{B.24}$$

where the partial derivatives of $F(v,t)$ correspond to

$$\frac{\partial F}{\partial v_{part}} = e^{\int g(t)dt} \ , \tag{B.25}$$

$$\frac{\partial F}{\partial t} = e^{\int g(t)dt} \left( g(t)v_{part} + \frac{1}{\rho}\frac{\partial p}{\partial z} + h(t) \right) \ . \tag{B.26}$$

By partial integration of (B.25) and (B.26), the solution of (B.14) is shown below:

$$F = v_{part}(t)e^{\int g(t)dt} + \int e^{\int g(t)dt} \left( \frac{1}{\rho}\frac{\partial p}{\partial z} + h(t) \right) dt = C_0 \ . \tag{B.27}$$

A rearrangement of terms allows one to obtain the following explicit expression for $v_{part}$, as

$$v_{part}(t) = e^{-\int g(t)dt} \left( C_0 - \int e^{\int g(t)dt} \left( \frac{1}{\rho} \frac{\partial p}{\partial z} + h(t) \right) dt \right) \ . \tag{B.28}$$

The expressions for $v_{homog}$ in (B.13) and $v_{part}$ in (B.28) are incorporated into (B.2), leading to

$$\begin{aligned} v(r,t) \quad = \quad & e^{-\int_{t_0}^{t} g(t')dt'} \left( C_0 + e^{-i\lambda t} C_1 J_0 \left( \sqrt{\frac{i\lambda\rho}{\mu}} r \right) + e^{-i\lambda t} C_2 Y_0 \left( -\sqrt{\frac{i\lambda\rho}{\mu}} r \right) \right. \\ & \left. - \int_{t_0}^{t} e^{\int_{t_0}^{t'} g(t'')dt''} \left( \frac{1}{\rho} \frac{\partial p}{\partial z} + h(t') \right) dt' \right) \ , \end{aligned} \tag{B.29}$$

where $t_0$ is an arbitrary lower time for integration. For practical purposes, we will compute such integrals by considering $t_0 = 0$.

Equation (B.29) is not general in the sense that considers that a single value of $\lambda$ has been provided. However, some boundary conditions require several values of the eigenvalue. Thus, a general solution of (B.2) can be expressed as a linear combination of expressions like the one in (B.29). Moreover, the eigenvalue $\lambda$ can spread any real value, i. e., $\lambda \ \epsilon \ (-\infty, \infty)$, so the linear combination is generalised to an integral, as follows:

$$\begin{aligned} v(r,t) \quad = \quad & e^{-\int_{0}^{t} g(t')dt'} \left( \int_{-\infty}^{\infty} e^{-i\lambda t} \left( C_1(\lambda) J_0 \left( \sqrt{\frac{i\lambda\rho}{\mu}} r \right) + C_2(\lambda) Y_0 \left( -\sqrt{\frac{i\lambda\rho}{\mu}} r \right) \right) d\lambda \right. \\ & \left. - \int_{0}^{t} e^{\int_{0}^{t'} g(t'')dt''} \left( \frac{1}{\rho} \frac{\partial p}{\partial z} + h(t') \right) dt' \right) \ . \end{aligned} \tag{B.30}$$

where the term $C_0$ has been omitted since it is incorporated in $C_1(\lambda)$ as

$$C_1(\lambda) e^{-i\lambda t} J_0 \left( \sqrt{\frac{i\lambda\rho}{\mu}} r \right) \Bigg|_{\lambda=0} = C_0 \ . \tag{B.31}$$

Equation (B.30) is capable to account for any boundary condition. Particularly, we study the classical conditions of Hagen-Poiseuille flow, i. e., finite flow in the center of the tube, given by

$$v(r,t) \Big|_{r=0} = \text{finite} \ , \tag{B.32}$$

and the no-slip condition at the tube walls, which is given by

$$v(r,t) \Big|_{r=R} = 0 \ . \tag{B.33}$$

Incorporation of finite flow stated in (B.32) leads to the following result:

$$C_2 = 0 \ , \tag{B.34}$$

since the Neumann function diverges at $r = 0$. Besides, the no-slip condition stated in (B.33) is substituted into (B.30), and leads to the following result:

$$e^{-\int_{0}^{t} g(t')dt'} \left( \int_{-\infty}^{\infty} e^{-i\lambda t} C_1(\lambda) J_0 \left( \sqrt{\frac{i\lambda\rho}{\mu}} R \right) d\lambda - \int_{0}^{t} e^{\int_{0}^{t'} g(t'')dt''} \left( \frac{1}{\rho} \frac{\partial p}{\partial z} + h(t') \right) dt' \right) = 0 \ . \tag{B.35}$$

In order to solve (B.35) for the coefficients $C_1(\lambda)$, we rewrite the time-dependent integral in terms of the Fourier identity, as follows:

$$\int_0^t e^{\int_0^{t'} g(t'')dt''} \left( \frac{1}{\rho} \frac{\partial p}{\partial z} + h(t') \right) dt' = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{i\lambda} \int_{-\infty}^{\infty} e^{\int_0^t g(t')dt'} \left( \frac{1}{\rho} \frac{\partial p}{\partial z} + h(t) \right) e^{i\lambda t} dt \, e^{-i\lambda t} d\lambda \quad \text{(B.36)}$$

Substituting (B.36) into the no-slip condition in (B.33), we obtain the following expression:

$$\int_{-\infty}^{\infty} e^{-i\lambda t} C_1(\lambda) J_0 \left( \sqrt{\frac{i\lambda\rho}{\mu}} R \right) d\lambda + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{i\lambda} \int_{-\infty}^{\infty} e^{\int_0^t g(t')dt'} \left( \frac{1}{\rho} \frac{\partial p}{\partial z} + h(t) \right) e^{i\lambda t} dt \, e^{-i\lambda t} d\lambda = 0 \quad \text{(B.37)}$$

A rearrange of the integral on $\lambda$ in (B.37) leads to the following result:

$$\int_{-\infty}^{\infty} \left( C_1(\lambda) J_0 \left( \sqrt{\frac{i\lambda\rho}{\mu}} R \right) + \frac{1}{2\pi i\lambda} \int_{-\infty}^{\infty} e^{\int_0^t g(t')dt'} \left( \frac{1}{\rho} \frac{\partial p}{\partial z} + h(t) \right) e^{i\lambda t} dt \right) e^{-i\lambda t} d\lambda = 0 \, . \quad \text{(B.38)}$$

The integral kernel in (B.38) must vanish, leading to the following expression for $C_1(\lambda)$:

$$C_1(\lambda) = -\frac{1}{2\pi i\lambda J_0 \left( \sqrt{\frac{i\lambda\rho}{\mu}} R \right)} \int_{-\infty}^{\infty} e^{\int_0^t g(t')dt'} \left( \frac{1}{\rho} \frac{\partial p}{\partial z} + h(t) \right) e^{i\lambda t} dt \, . \quad \text{(B.39)}$$

Finally, by incorporating $C_1(\lambda)$ in (B.39) and $C_2(\lambda)$ in (B.34) in the general solution stated in (B.30), the following result is obtained for the flow velocity with no-slip at the tube walls:

$$v(r,t) = \frac{e^{-\int_0^t g(t')dt'}}{2\pi} \int_{-\infty}^{\infty} \frac{1}{i\rho\lambda} \left( 1 - \frac{J_0 \left( \sqrt{\frac{i\lambda\rho}{\mu}} r \right)}{J_0 \left( \sqrt{\frac{i\lambda\rho}{\mu}} R \right)} \right) \int_{-\infty}^{\infty} e^{\int_0^t g(t')dt'} \left( \frac{\partial p}{\partial z} + \rho \, h(t) \right) e^{i\lambda t} dt \, e^{-i\lambda t} d\lambda \quad \text{(B.40)}$$

# C Details of the solution of fluid dynamics influenced by a tube moving in a single vibration mode

Equation (B.40) allows one to solve fluid dynamics if the condition of tube motion is previously given and computed in $g(t)$ and $h(t)$, as defined in the body of article. Tube dynamics, as stated in (2.31) is solved along with boundary conditions.

A specific experimental setting of the tube would determine the way in which edges are fixed in an experiment (Arash & Wang 2012). Experimental literature on elastic nano-tubes shows three common geometrical conditions for the tube edges (Krishnan *et al.* 1998), as shown below:

- Pinned edge. It means that the displacement of the tube edge is zero, and that there is no curvature at that point. Physically, this implies that no elastic strain is imposed at the tube edge. Mathematically, for a tube edge located at $z = z_0$, this is written as:

$$u\Big|_{z=z_0} = 0 \qquad \text{and} \qquad \frac{\partial^2 u}{\partial z^2}\Big|_{z=z_0} = 0 \, . \quad \text{(C.1)}$$

- Clamped edge. It means that the displacement of the tube edge is zero, and that the tube at that point is constrained to be horizontal. Mathematically, for a tube edge located at $z = z_0$, this is written as:

$$u\Big|_{z=z_0} = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial z}\Big|_{z=z_0} = 0 \, . \quad \text{(C.2)}$$

- Free edge. It means that the displacement of the tube edge is not fixed, the only constrain is that there is no curvature at that point and on its neighborhood. Mathematically, for a tube edge located at $z = z_0$, this is written in the following way:

$$\left.\frac{\partial^2 u}{\partial z^2}\right|_{z=z_0} = 0 \qquad \text{and} \qquad \left.\frac{\partial^3 u}{\partial z^3}\right|_{z=z_0} = 0 \; . \tag{C.3}$$

For a finite-size tube, which has two edges, any combination of these three possibilities should be, in principle, experimentally possible. This gives 6 sets of boundary conditions that discretise differently the dispersion relation, namely, pinned-pinned, clamped-clamped, pinned-clamped, pinned-free and clamped-free. Each of these sets imply four conditions on $u$ and/or its spatial derivatives and leads to different vibration modes.

Fourier transform of (2.31) leads to

$$EI\frac{d^4\hat{u}}{dz^4} - (\rho A_f + \rho_t A_t)\omega^2\hat{u} = 0 \; , \tag{C.4}$$

where $\hat{u}(z,\omega)$ denotes the Fourier transform of $u(z,t)$.

The general solution of (C.4) is given by

$$\hat{u}(z,\omega) = C_1 e^{ikz} + C_2 e^{-ikz} + C_3 e^{kz} + C_4 e^{-kz} \; , \tag{C.5}$$

where $k$ is given by

$$k = \left(\frac{(\rho A_f + \rho_t A_t)\omega^2}{EI}\right)^{\frac{1}{4}} \; . \tag{C.6}$$

In order to determine the particular solution of (C.4) for each set of boundary conditions, (C.1)-(C.3) are incorporated in the general solution in (C.5), leading to a $4\times 4$ system of algebraic homogeneous equations for $C_1$, $C_2$, $C_3$ and $C_4$. A homogeneous system leads to non-trivial solutions only if the determinant of its coefficients vanishes, as stated below:

$$D_{BC} = 0 \; , \tag{C.7}$$

where the suffix $BC$ accounts for each set of boundary conditions. For each case, the expression of the determinant is given below:

- Pinned-pinned

$$D_{PP} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ -k^2 & -k^2 & k^2 & k^2 \\ e^{ikL} & e^{-ikL} & e^{kL} & e^{-kL} \\ -k^2e^{ikL} & -k^2e^{-ikL} & k^2e^{kL} & k^2e^{-kL} \end{vmatrix} = 16ik^4\sin(kL)\sinh(kL) \; . \tag{C.8}$$

- Clamped-clamped

$$D_{CC} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ ik & -ik & k & -k \\ e^{ikL} & e^{-ikL} & e^{kL} & e^{-kL} \\ ike^{ikL} & -ike^{-ikL} & ke^{kL} & -ke^{-kL} \end{vmatrix} = 8ik^2(\cos(kL)\cosh(kL) - 1) \; . \tag{C.9}$$

- Pinned-clamped

$$\begin{aligned} D_{PC} &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ ik & -ik & k & -k \\ e^{ikL} & e^{-ikL} & e^{kL} & e^{-kL} \\ -k^2e^{ikL} & -k^2e^{-ikL} & k^2e^{kL} & k^2e^{-kL} \end{vmatrix} \\ &= 8ik^3(\cosh(kL)\sin(kL) - \cos(kL)\sinh(kL)) \; . \end{aligned} \tag{C.10}$$

- Pinned-free

$$\begin{aligned} D_{PF} &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ -k^2 & -k^2 & k^2 & k^2 \\ -k^2 e^{ikL} & -k^2 e^{-ikL} & k^2 e^{kL} & k^2 e^{-kL} \\ -ik^3 e^{ikL} & ik^3 e^{-ikL} & k^3 e^{kL} & -k^3 e^{-kL} \end{vmatrix} \\ &= -8ik^7 (\cosh(kL)\sin(kL) - \cos(kL)\sinh(kL)) \ . \end{aligned} \tag{C.11}$$

- Clamped-free

$$D_{FC} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ ik & -ik & k & -k \\ -k^2 e^{ikL} & -k^2 e^{-ikL} & k^2 e^{kL} & k^2 e^{-kL} \\ -ik^3 e^{ikL} & ik^3 e^{-ikL} & k^3 e^{kL} & -k^3 e^{-kL} \end{vmatrix} = -8ik^6 (\cos(kL)\cosh(kL) + 1) \ . \tag{C.12}$$

The condition for non-trivial solutions, as stated in (C.8)-(C.12), is only accomplished for certain values of $k$, leading to discretised values $k_n$, which are summarized in table (1).

Table 1: Discretisation of $k = k_n$ induced by the different sets of boundary conditions. Values shown for $k_n$ are asymptotic approximated solutions for (C.8)-(C.12).

| Set of boundary conditions | $k_n L$ |
|---|---|
| Pinned-pinned | $n\pi$ |
| Clamped-clamped | $(n + 1/2)\,\pi$ |
| Free-free | $(n + 1/2)\,\pi$ |
| Pinned-clamped | $(n + 1/4)\,\pi$ |
| Pinned-free | $(n + 1/4)\,\pi$ |
| Clamped-free | $(n - 1/2)\,\pi$ |

The discretisation of $k = k_n$ implies also the discretisation of the frequency $\omega = \omega_n$, since $k$ and $\omega$ are related by (C.6).

After discretisation of $k_n$ and $\omega_n$, each $4 \times 4$ system of equations is simplified to a $4 \times 3$ system of equations in which one of the variables is left as a degree of freedom.

Such treatment leads to non-trivial solutions of the following form:

$$\hat{u}_n(z, \omega) = f_n(z) \left( C_{1,n}\delta(\omega - \omega_n) + D_{1,n}\delta(\omega + \omega_n) \right) \ , \tag{C.13}$$

where $f_n(z)$ is a spatial function obtained for each set of B. C. By performing the inverse Fourier transform of (C.13), the following expression is obtained for $u_n(z,t)$, as follows:

$$u_n(z, t) = U_0 f_n(z) \sin(\omega_n t + \varphi) \ . \tag{C.14}$$

The explicit expression for $f_n(z)$ for the different sets of boundary conditions is provided in the following list.

- Pinned-pinned

$$f_n(z) = \sin(k_n z) \tag{C.15}$$

- Clamped-clamped

$$f_n(z) = \sin(k_n z) - \sinh(k_n z) + \frac{(-1)^n - \sinh(k_n L)}{\cosh(k_n L)} \left(\cos(k_n z) - \cosh(k_n z)\right) \tag{C.16}$$

- Pinned-clamped

$$f_n(z) = \sin(k_n z) - \frac{(-1)^n}{\sqrt{2}} \frac{\sinh(k_n z)}{\sinh(k_n L)} \tag{C.17}$$

- Pinned-free

$$f_n(z) = \sin(k_n z) + \frac{(-1)^n}{\sqrt{2}} \frac{\sinh(k_n z)}{\sinh(k_n L)} \tag{C.18}$$

- Clamped-free

$$f_n(z) = \sin(k_n z) - \sinh(k_n z) + \frac{(-1)^n - \sinh(k_n L)}{\cosh(k_n L)} \left(\cos(k_n z) - \cosh(k_n z)\right) \tag{C.19}$$

The solution $u_n(z,t)$ from (C.14) is incorporated into the Coriolis and pulling forces denoted by $g(t)$ and $h(t)$. The phase $\phi = 0$ is considered for simplicity, leading to the following result:

$$g(t) = \frac{U_0^2 A\omega}{L^2} \sin(2\omega t) \ , \tag{C.20}$$

$$h(t) = \frac{U_0^2 B\omega^2}{L} \cos(2\omega t) \ , \tag{C.21}$$

where $A$ and $B$ are factors that depend on the specific boundary conditions, as shown below:

- Pinned-pinned

$$A = \frac{k_n^2}{2} \tag{C.22}$$

$$B = 0 \tag{C.23}$$

- Clamped-clamped

$$A = \frac{2k_n \left(4(-1)^n \cosh(k_n) + 2k_n(\cosh(2k_n) + 1) - 4\sinh(2k_n)\right)}{8\cosh^2(k_n)} \tag{C.24}$$

$$B = 0 \tag{C.25}$$

- Pinned-clamped

$$A = \frac{k_n(2 + (4k_n - 2)\cosh(2k_n) - 2\sinh(2k_n))}{16\sinh^2(k_n)} \tag{C.26}$$

$$B = 0 \tag{C.27}$$

- Pinned-free

$$A = \frac{k_n((6+4\pi\beta_n)\cosh(2k_n)+6(\sinh(2k_n)-1))}{16\sinh^2(k_n)} \tag{C.28}$$

$$B = 1 \tag{C.29}$$

- Clamped-free

$$\begin{aligned} A \quad = \quad & -\frac{2k_n(\sinh(k_n)-3(-1)^n)}{4\cosh(k_n)} - \frac{k_n(-1)^n(1+i)\cosh(2in\pi-k_n)}{4\cosh^2(k_n)} - \frac{2k_n^2}{4\cosh^2(k_n)} \\ & + \frac{k_n(1-i)(-1)^n\cosh(k_n(1+2i))}{4\cosh^2(k_n)} + \frac{k_n(k_n\cosh(2k_n)+5\sinh(2k_n))}{4\cosh^2(k_n)} \end{aligned} \tag{C.30}$$

$$B = 2 \tag{C.31}$$

# D High-frequency terms in flow velocity influenced by a tube vibrating in a single mode under a constant pressure gradient

The radially-averaged flow velocity, $\langle v\rangle$, is given by

$$\begin{aligned} \langle v\rangle \quad = \quad & K_0 + K_{2\omega,c}\cos(2\omega_n t) + K_{2\omega,s}\sin(2\omega_n t) + K_{4\omega,c}\cos(4\omega_n t) + K_{4\omega,s}\sin(4\omega_n t) \\ & + K_{6\omega,c}\cos(6\omega_n t) + K_{6\omega,s}\sin(6\omega_n t) \ , \end{aligned} \tag{D.1}$$

where $K_0$, $K_{2\omega,c}$, $K_{2\omega,s}$, $K_{4\omega,c}$, $K_{4\omega,s}$, $K_{6\omega,c}$ and $K_{6\omega,s}$ are given, respectively, by

$$K_0 = -\frac{\partial p}{\partial z}\frac{R^2}{8\mu} + \mathcal{O}(\varepsilon^4) \ , \tag{D.2}$$

$$K_{2\omega,c} = -\frac{\partial p}{\partial z}\frac{A\varepsilon^2R^2}{16\mu} + \left(-\frac{\partial p}{\partial z}\frac{A\varepsilon^2R^2}{16\mu} + \frac{\rho\omega_n^2LB\varepsilon^2R^2}{8\mu}\right)\mathrm{Re}f_{bes}\left(\frac{2\rho\omega_nR^2}{\mu}\right) + \mathcal{O}(\varepsilon^6) \ , \tag{D.3}$$

$$K_{2\omega,s} = \left(-\frac{\partial p}{\partial z}\frac{A\varepsilon^2R^2}{16\mu} + \frac{\rho\omega_n^2LB\varepsilon^2R^2}{8\mu}\right)\mathrm{Im}f_{bes}\left(\frac{2\rho\omega_nR^2}{\mu}\right) + \mathcal{O}(\varepsilon^6) \ , \tag{D.4}$$

$$\begin{aligned} K_{4\omega,c} \quad = \quad & -\frac{\partial p}{\partial z}\frac{A^2\varepsilon^4R^2}{128\mu} + \left(-\frac{\partial p}{\partial z}\frac{A^2\varepsilon^4R^2}{64\mu} + \frac{\rho\omega_n^2L\,A\,B\varepsilon^4R^2}{32\mu}\right)\mathrm{Re}f_{bes}\left(\frac{2\rho\omega_nR^2}{\mu}\right) \\ & + \left(\frac{\partial p}{\partial z}\frac{A^2\varepsilon^4R^2}{128\mu} - \frac{\rho\omega_n^2L\,A\,B\varepsilon^4R^2}{32\mu}\right)\mathrm{Re}f_{bes}\left(\frac{4\rho\omega_nR^2}{\mu}\right) + \mathcal{O}(\varepsilon^8) \ , \end{aligned} \tag{D.5}$$

$$\begin{aligned} K_{4\omega,s} \quad = \quad & \left(-\frac{\partial p}{\partial z}\frac{A^2\varepsilon^4R^2}{64\mu} + \frac{\rho\omega_n^2L\,A\,B\varepsilon^4R^2}{32\mu}\right)\mathrm{Im}f_{bes}\left(\frac{2\rho\omega_nR^2}{\mu}\right) \\ & + \left(\frac{\partial p}{\partial z}\frac{A^2\varepsilon^4R^2}{128\mu} - \frac{\rho\omega_n^2L\,A\,B\varepsilon^4R^2}{32\mu}\right)\mathrm{Im}f_{bes}\left(\frac{4\rho\omega_nR^2}{\mu}\right) + \mathcal{O}(\varepsilon^8) \ , \end{aligned} \tag{D.6}$$

$$\begin{aligned} K_{6\omega,c} \quad = \quad & -\frac{\partial p}{\partial z}\frac{A^3\varepsilon^6R^2}{1536\mu} + \left(-\frac{\partial p}{\partial z}\frac{A^3\varepsilon^6R^2}{512\mu} + \frac{\rho\omega_n^2L\,A^2B\varepsilon^6R^2}{256\mu}\right)\mathrm{Re}f_{bes}\left(\frac{2\rho\omega_nR^2}{\mu}\right) \\ & + \left(\frac{\partial p}{\partial z}\frac{A^3\varepsilon^6R^2}{512\mu} - \frac{\rho\omega_n^2L\,A^2B\varepsilon^6R^2}{128\mu}\right)\mathrm{Re}f_{bes}\left(\frac{4\rho\omega_nR^2}{\mu}\right) \\ & + \left(-\frac{\partial p}{\partial z}\frac{A^3\varepsilon^6R^2}{1536\mu} + \frac{\rho\omega_n^2L\,A^2B\varepsilon^6R^2}{256\mu}\right)\mathrm{Re}f_{bes}\left(\frac{6\rho\omega_nR^2}{\mu}\right) + \mathcal{O}(\varepsilon^{10}) \ , \end{aligned} \tag{D.7}$$

$$
\begin{aligned}
K_{6\omega,s} \quad = \quad & \left(-\frac{\partial p}{\partial z}\frac{A^3\varepsilon^6 R^2}{512\mu} + \frac{\rho\omega_n^2 L\, A^2 B\varepsilon^6 R^2}{256\mu}\right)\mathrm{Im}f_{bes}\left(\frac{2\rho\omega_n R^2}{\mu}\right) \\
& + \left(\frac{\partial p}{\partial z}\frac{A^3\varepsilon^6 R^2}{512\mu} - \frac{\rho\omega_n^2 L\, A^2 B\varepsilon^6 R^2}{128\mu}\right)\mathrm{Im}f_{bes}\left(\frac{4\rho\omega_n R^2}{\mu}\right) \\
& + \left(-\frac{\partial p}{\partial z}\frac{A^3\varepsilon^6 R^2}{1536\mu} + \frac{\rho\omega_n^2 L\, A^2 B\varepsilon^6 R^2}{256\mu}\right)\mathrm{Im}f_{bes}\left(\frac{6\rho\omega_n R^2}{\mu}\right) + \mathcal{O}(\varepsilon^{10})\ ,
\end{aligned} \tag{D.8}
$$

with $f_{bes}$ given by

$$
f_{bes}(x) = \frac{8}{ix}\left(1 - \frac{2J_1\sqrt{ix}}{\sqrt{ix}\,J_0\sqrt{ix}}\right)\ , \tag{D.9}
$$

and Re $f_{bes}$ and Im $f_{bes}$ account for its real and imaginary parts, respectively.