Supplementary material

7. Infrared and ultraviolet singularities

7.1. *Infrared singularity*

In the region IR, in red in Fig. 9, let us change variables using

$$k_1 = k(1+y), \qquad k_2 = kx\,, \tag{7.1}$$

which gives the new integration area shown in the left panel of Fig. 7, with $x, y = O(\epsilon)$, $\epsilon \ll 1$. Now, using a power-series expansion the factor f^0_{12}, we notice that the dominant matrix elements are those given by the resonances (Ia) and (IIa), the Induced Diffusion contributions, and we obtain

$$\begin{aligned} R^0_{12} f^0_{12} - & R^1_{02} f^1_{02} - R^2_{01} f^2_{01} \\ & \simeq R^k_{k(1+y),kx}\left[n_{k(1+y)} n_{kx} - n_k (n_{k(1+y)} + n_{kx})\right] - R^{k(1+y)}_{k,kx}\left[n_k n_{kx} - n_{k(1+y)}(n_k + n_{kx})\right] \\ & \simeq n_{kx} \frac{\partial n_k}{\partial k} k y \left[R^k_{k(1+y),kx} + R^{k(1+y)}_{k,kx}\right] \\ & \simeq -a k^{-2a} x^{-a} y \left[R^k_{k(1+y),kx} + R^{k(1+y)}_{k,kx}\right], \end{aligned} \tag{7.2}$$

where R^2_{01} has been neglected, and the so-called *first cancellation* has taken place due to the fact that $\lim_{k\to 0} n_{k(1+y)} = n_k$. To evaluate the terms $R^k_{k(1+y),kx}$ and $R^{k(1+y)}_{k,kx}$ we notice that the two leading-order contributions are given by the conditions (Ia) and (IIa) where, respectively,

$$m^\star_1 \simeq m(1+\sqrt{x}), \qquad m^\star_2 \simeq -m(\sqrt(x) + \frac{1}{2}(x+y))\,, \tag{7.3}$$

$$m^\star_1 \simeq m(1-\sqrt{x}), \qquad m^\star_2 \simeq -m(\sqrt(x) - \frac{1}{2}(x+y))\,. \tag{7.4}$$

Thus, we obtain

$$|V^k_{k(1+y),kx}|^2 \simeq 4k^3 m^{-1}\left[\frac{y^2}{x^2}\sqrt{x} + \frac{y^2}{2x^2}(x+y) - y\right], \tag{7.5}$$

$$|V^{k(1+y)}_{k,kx}|^2 \simeq 4k^3 m^{-1}\left[\frac{y^2}{x^2}\sqrt{x} + \frac{y^2}{2x^2}(x+y) + y\right]. \tag{7.6}$$

Furthermore, we have

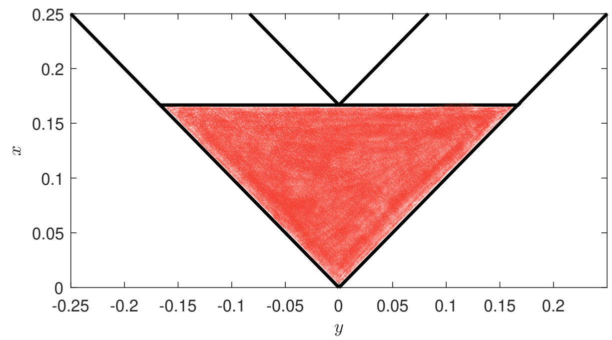
$$|{g^0_{12}}'| \simeq |{g^1_{02}}'| \simeq m^2/2k\,, \qquad \Delta_{012} \simeq \frac{k^2}{\sqrt{x^2-y^2}}\,. \tag{7.7}$$

Using Eqs. (7.5)-(7.7), we finally obtain

$$R^k_{k(1+y),kx} + R^{k(1+y)}_{k,kx} \simeq 2k^3 m x\left[2\frac{y^2}{x^2}\sqrt{x} + \frac{y}{x^2}(y^2 - x^2)\right] / \sqrt{x^2-y^2}\,. \tag{7.8}$$

Thus, Eq. (7.2) yields a contribution $\mathcal{I}_C(k)$ to the total $\mathcal{I}(k)$ (due to integration in region C):

$$\begin{aligned} \mathcal{I}_{\mathrm{IR}}(k) &= 8\pi k \int_{\mathrm{IR}} dx\, dy\, \left[-a k^{-2a} x^{-a} y \left(R^k_{k(1+y),kx} + R^{k(1+y)}_{k,kx}\right)\right] \\ &= -16\pi a k^{-2a+4} m \int_0^{x_{\mathrm{IR}}} dx \int_{-x}^{x} dy\, x^{-a-1} \frac{2y^3\sqrt{x} + y^2(y^2-x^2)}{\sqrt{x^2-y^2}}\,. \end{aligned} \tag{7.9}$$



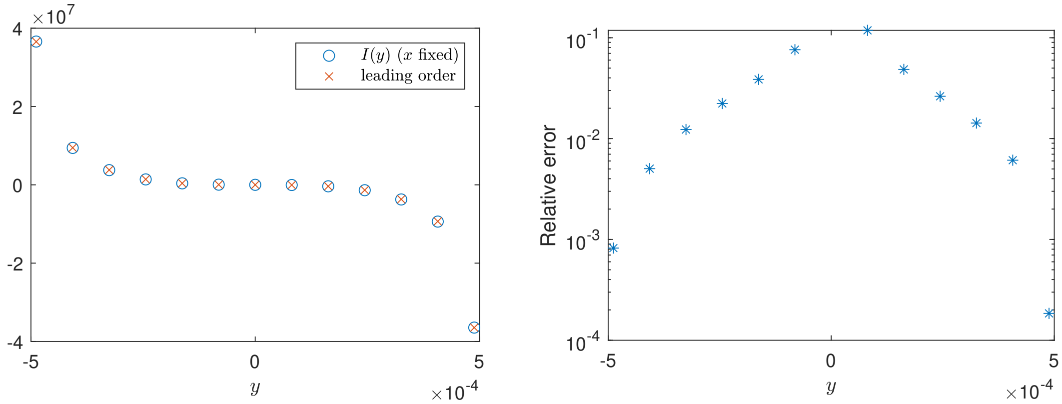


Figure 7: Top: Sub-region IR, containing the singularity $k_2 \to 0$. The region is represented after performing the change of variables $(k_1, k_2) \to (y, x)$. Bottom left: Some values of the integrand of (7.9) during the double loop of integration in the x and y variables. The circles are the full integrand integrated numerically, and the crosses are the leading order of the integrand in (7.9). In the plot, x is fixed and y varies from $-x$ to x. Note the magnitude of the integrand as the singular boundary is approached. Bottom right: relative error of the points in the left panel, in logaritmic scale.

The first term of the integrand is dominating but it is an odd function of y in a domain symmetric with respect to the origin, so its contribution is vanishing: this is the *second cancellation*, implying:

$$\begin{aligned} I_{\mathrm{IR}}(k) &\sim -16\pi a k^{-2a+4} m \int_0^{x_{\mathrm{IR}}} dx \int_{-x}^{x} dy\; x^{-a-1} \frac{y^2(y^2-x^2)}{\sqrt{x^2-y^2}} \\ &= 2\pi^2 \frac{a}{4-a} m k^{-a} {k_{\mathrm{IR}}}^{-a+4}, \end{aligned} \tag{7.10}$$

where k_{IR} is the (small) height of the red region in Fig. 7. For convergence of the integral we have to impose that $a < 4$, and we notice that the contribution is positive.

The leading order of the integrand of Eq. (7.9) is tested numerically in the right panel of Fig. 7, leaving no doubt about its correctness.

7.2. *Ultraviolet singularity*

Let us focus on the region UV, applying the following change of variables:

$$k_1 = \frac{k}{x}, \qquad k_2 = \frac{k}{x}(1 + y - x), \tag{7.11}$$

for $x \ll 1$ and $y \in (0, x)$. $1/x$ provides the largeness of k_1 and k_2, while $y = 0$ corresponds to the singular line $k_2 = k_1 - k$, thereby allowing singular power expansions around the point $y = 0$. Finally, $y = x$ corresponds to the line $k_2 = k_1$. In the new variables, the region UV is represented

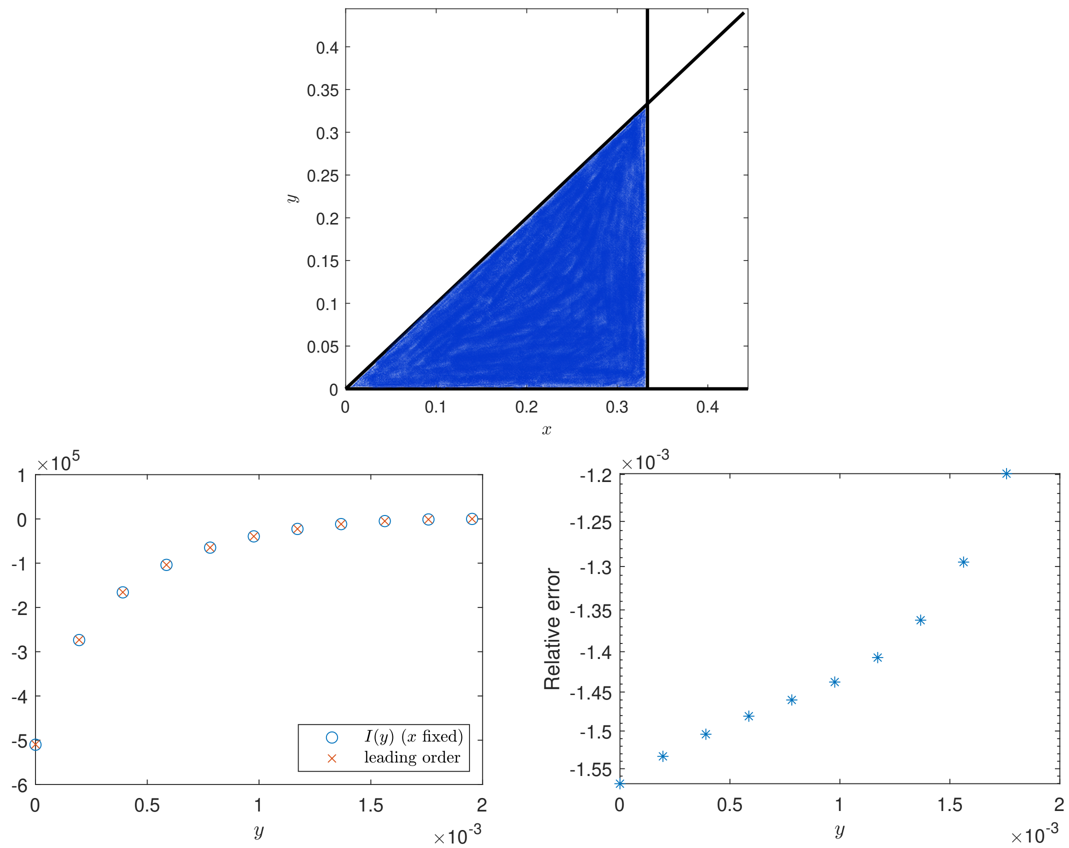


Figure 8: Top: Sub-region D, in the variables defined by Eq. (7.11). Bottom left: Some values of the integrand of (7.18) during the double loop of integration in the x and y variables. In the plot, x is fixed and y varies from $-x$ to x. Note the magnitude of the integrand as the singular boundary is approached. Bottom right: relative error of the points in the left panel, in logaritmic scale.

in Fig. 8 (for a value $k_{\mathrm{UV}} = 3k$). We have that the integrand of the UV contribution is given by

$$\begin{aligned} R^0_{12} f^0_{12} - R^1_{02} f^1_{02} - R^2_{01} f^2_{01} &\\ \simeq & - R^{k/x}_{k,k(1+y-x)/x} [n_k n_{k(1+y-x)/x} - n_{k/x}(n_k + n_{k(1+y-x)/x})] \\ & - R^{k(1+y-x)/x}_{k,k/x} [n_k n_{k/x} - n_{k(1+y-x)/x}(n_k + n_{k/x})] \\ \simeq & - a(x-y)k^{-2a}x^a \left[R^{k/x}_{k,k(1+y-x)/x} - R^{k(1+y-x)/x}_{k,k/x} \right] , \end{aligned} \tag{7.12}$$

where R^0_{12} has been neglected (subleading), and the *first cancellation* has taken place in the terms f^1_{02} and f^2_{01}, expanded to first order in x and y. Now, the two leading order contributions are given by the matrix elements satisfying the resonance conditions (IIb) and (IIIb), again the Induced Diffusion contributions, yielding respectively

$$m^\star_1 \simeq -m \left(\frac{1}{\sqrt{x}} - \frac{y}{2x} \right) , \qquad m^\star_2 \simeq -m \left(\frac{1}{\sqrt{x}} + 1 - \frac{y}{2x} \right) , \tag{7.13}$$

$$m^\star_1 \simeq -m \left(\frac{1}{\sqrt{x}} + \frac{y}{2x} \right) , \qquad m^\star_2 \simeq -m \left(\frac{1}{\sqrt{x}} - 1 + \frac{y}{2x} \right) , \tag{7.14}$$

which imply the leading order expressions

$$|V_{02}^{1}|^2/|g_{13}^{2}{}'| \simeq \frac{2k^2m(x-y)^2}{x^{9/2}} + \frac{2k^2m(2x^3-4x^2y+3xy^2-y^3)}{x^5}\,, \tag{7.15}$$

$$|V_{01}^{2}|^2/|g_{01}^{2}{}'| \simeq \frac{2k^2m(x-y)^2}{x^{9/2}} - \frac{2k^2m(2x^3-4x^2y+3xy^2-y^3)}{x^5}\,. \tag{7.16}$$

Plugging into the expression (7.2), with

$$\Delta_{012} \simeq \frac{k^2}{x^2}\sqrt{(2x-y)y}\,, \tag{7.17}$$

we finally obtain the following expression for the integrand:

$$\begin{aligned}
-a(x-y)k^{-2a}x^{a}&\left[R^{k/x}_{k,k(1+y-x)/x} - R^{k(1+y-x)/x}_{k,k/x}\right] \\
&= -a(x-y)k^{-2a}x^{a}\frac{k^3}{x^2}\left[\frac{4k^2m}{x^5}(2x^3-4x^2y+3xy^2-y^3)\right]\frac{x^2}{k^2\sqrt{(2x-y)y}} \\
&= -4ak^{-2a+5}mx^{a-8}\left[(x-y)^4+x^2(x-y)^2\right]/\sqrt{(2x-y)y}\,.
\end{aligned} \tag{7.18}$$

This can be integrated analytically, giving the following result:

$$\begin{aligned}
\mathcal{I}_{\mathrm{UV}}(k) &= -32\pi a k^{-2a+4}m\int_0^{x_{\mathrm{UV}}}dx\int_0^{x}dy\,\frac{k^2}{x^3}\,x^{a-8}\left[(x-y)^4+x^2(x-y)^2\right]/\sqrt{(2x-y)y} \\
&\simeq -14\pi^2\frac{a}{a-3}k^{-a+1}m\,k_{\mathrm{UV}}^{3-a}\,.
\end{aligned} \tag{7.19}$$

For convergence of the integral we must have that $a > 3$, which along with the infrared condition results into the convergence segment $3 < a < 4$. Evidence of the correctness of the convergence condition $-3 < a < 4$ is given in the figures in the next Section, where the convergent integral is computed numerically.

8. Numerical computation of integrable singularities

The five regions represented in Fig. 9, identified by the values of the two parameters k_{IR} and k_{UV}, are treated separately. Just for numerical purpose, it is easier to consider these new regions, and then stich the results together to reconstruct the contributions of each of the regions in the main text. We recall that k_{IR} defines the height of the red region in Fig. 7 in the k_2 direction, and k_{UV} defines the lower boundary of the blue region in the k_1 direction. The singularities are integrated with the following procedure: the divergent leading order is subtracted to the integrand, giving a convergent difference that is integrated numerically. Subsequently, the exact analytical result of the integral of the subtracted part is added, ensuring convergence. The leading order in the regions $\mathcal{R}_1$ and $\mathcal{R}_5$ is given in the previous section, while the regions $\mathcal{R}_2$ and $\mathcal{R}_3$ are singular because of the Δ_{012} denominator at the boundary, i.e. the vanishing area of triangles formed by triads of collinear horizontal wave vectors.

In the code, the five terms are decomposed and integrated in the following form:

- Region $\mathcal{R}_1$:

$$\mathcal{I}_1 = \mathcal{L} + k^2\int_0^{k_{\mathrm{IR}}/k}dx'\int_{-x}^{x}dy\,\left[J(x,y)-L(x,y)\right]\,, \tag{8.1}$$

where

$$J(x,y) = R^{k}_{k(1+y),kx}f^{k}_{k(1+y),kx} - R^{k(1+y)}_{k,kx}f^{k(1+y)}_{k,kx} - R^{kx}_{k,k(1+y)}f^{kx}_{k,k(1+y)}\,,$$

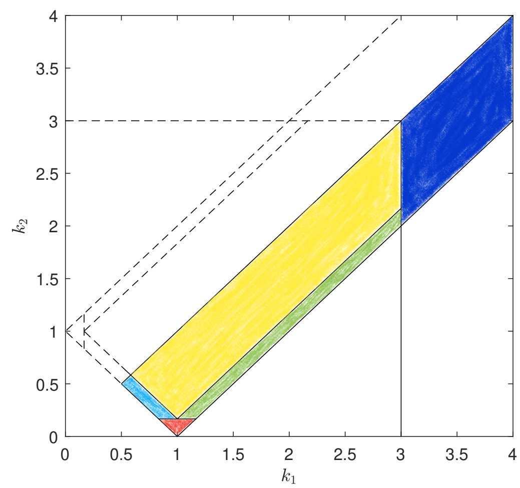


Figure 9: The five subregions of the kinematic box, where integration is performed (in the figure, $k = 1$): region $\mathcal{R}_1$ (or IR in the main text) in red, region $\mathcal{R}_2$ in light blue, region $\mathcal{R}_3$ in green, region $\mathcal{R}_4$ in yellow, and region $\mathcal{R}_5$ (or UV in the main text) in blue. $\mathcal{R}_1$ contains the singularity for $k_2 \to 0$; $\mathcal{R}_2$ and $\mathcal{R}_3$ contain the singularities due to a vanishing *area of the interaction triangle with sides* k, k_1 *and* k_2, which is at denominator in the integrand; $\mathcal{R}_4$ contains no singularities of the integrand; $\mathcal{R}_5$ is integrated over after a suitable change of coordinates that map the singularity of the integrand approaching infinity into the origin in the new coordinate system.

$$L(x,y) = -2ak^{-2a+3}mx^{-a-1}\frac{2y^3\sqrt{x}+y^2(y^2-x^2)}{\sqrt{x^2-y^2}}\,, \qquad \mathcal{L} = \frac{\pi}{4}\frac{a}{4-a}mk^{-a+1}k_{\mathrm{IR}}^{-a+4}\,.$$

- Region $\mathcal{R}_2$:

$$\mathcal{I}_2 = \int_{k/2}^{k-k_{\mathrm{IR}}} dk_1 \left[\mathcal{L}(k_1) + \int_0^{k_{\mathrm{IR}}} dx\,(J(x,k_1) - L(x,k_1))\right], \tag{8.2}$$

where

$$J(x,k_1) = \frac{T^k_{k_1,k-k1+x} - T^{k_1}_{k,k-k_1+x} - T^{k-k_1+x}_{k,k_1}}{\Delta_{k,k_1,k-k_1+x}}\,,$$

$$L(x,k_1) = \frac{T^k_{k_1,k-k_1} - T^{k_1}_{k,k-k_1} - T^{k-k_1}_{k,k_1}}{\sqrt{2kk_1(k-k_1)x}}\,, \qquad \mathcal{L}(k_1) = \left(T^k_{k_1,k-k_1} - T^{k_1}_{k,k-k_1} - T^{k-k_1}_{k,k_1}\right)\frac{\sqrt{2k_{\mathrm{IR}}}}{\sqrt{kk_1(k-k_1)}}\,,$$

and the following definition was used:

$$T^0_{12} = kk_1k_2|V^0_{12}|^2 f^0_{12}/|{g^0_{12}}'|\,.$$

- Region $\mathcal{R}_3$:

$$\mathcal{I}_3 = \int_{k+k_{\mathrm{IR}}}^{k_{\mathrm{UV}}} dk_1 \left[\mathcal{L}(k_1) + \int_0^{k_{\mathrm{IR}}} dx\,(J(x,k_1) - L(x,k_1))\right], \tag{8.3}$$

where

$$J(x,k_1) = \frac{T^k_{k_1,k1-k+x} - T^{k_1}_{k,k_1-k+x} - T^{k_1-k+x}_{k,k_1}}{\Delta_{k,k_1,k_1-k+x}}\,,$$

$$L(x,k_1) = \frac{T^k_{k_1,k_1-k} - T^{k_1}_{k,k_1-k} - T^{k_1-k}_{k,k_1}}{\sqrt{2kk_1(k_1-k)x}}\,, \qquad \mathcal{L}(k_1) = \left(T^k_{k_1,k_1-k} - T^{k_1}_{k,k_1-k} - T^{k_1-k}_{k,k_1}\right)\frac{\sqrt{2k_{\mathrm{IR}}}}{\sqrt{kk_1(k_1-k)}}\,,$$

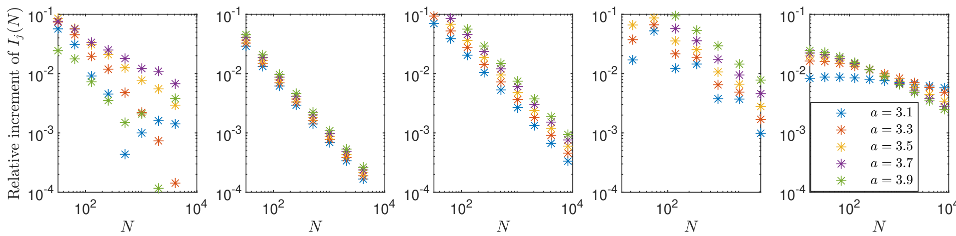


Figure 10: Test of convergence in each subregion of integration, as the number of gridpoints (N) increases. From left to right, the plots refer to regions from $\mathcal{R}_1$ to $\mathcal{R}_5$, respectively, and the plotted quantity is the absolute value of the relative increment of each contribution, as N doubles (in log-log scale). All contributions appear to converge up to variations of at most 1%, for the largest numbers of points here considered.

- In region $\mathcal{R}_4$ the integrand is finite and integration is straightforward:

$$\mathcal{I}_4 = \int_{\mathcal{R}_4} dk_{12}\, (R^0_{12} f^0_{12} - R^1_{02} f^1_{02} - R^2_{01} f^2_{01})\,. \tag{8.4}$$

- Region $\mathcal{R}_5$:

$$\mathcal{I}_5 = \mathcal{L} + \int_0^{k/k_{\rm UV}} dx \int_0^x dy\, [J(x,y) - L(x,y)]\ , \tag{8.5}$$

where

$$J(x,y) = \left[R^k_{k/x,k(1+y-x)/x} f^k_{k/x,k(1+y-x)/x} - R^{k/x}_{k,k(1+y-x)/x} f^{k/x}_{k,k(1+y-x)/x} - R^{k(1+y-x)/x}_{k,k/x} f^{k(1+y-x)/x}_{k,k/x}\right] \frac{k^2}{x^3}\,,$$

$$L(x,y) = -4ak^{-2a+5} m x^{a-8} \left[(x-y)^4 + x^2(x-y)^2\right] / \sqrt{(2x-y)y}\,, \qquad \mathcal{L} = -\frac{7}{4}\pi \frac{a}{a-3} k^{-a+2} m\, k_{\rm UV}^{3-a}\,.$$

9. Numerical convergence and independence from the cuts

In Fig. 10 such numerical convergence is shown independently for each of the five regions. In region UV the two leading order contributions alone are integrated in Eq. (8.5), since there are subleading contributions whose integrand is divergent as well, hindering convergence. In the following, letting the position of the cut ($k_{\rm UV}$) vary, we will show that such term can indeed be neglected. The width of the regions around $k_2 = 0$ is determined by the parameter $k_{\rm IR}$, while the cut at large k's is performed at $k_1 = k_{\rm UV}$. For the result to be general, it must be independent of the choice of $k_{\rm IR}$ and $k_{\rm UV}$, as long as they are finite numbers, $k_{\rm IR}$ being sufficiently small and $k_{\rm UV}$ sufficiently large. This has been checked. In Fig. 11 we show how convergence is reached as $k_{\rm UV}$ increases, as the neglected contribution in UV vanishes. Independence of the result upon variations of $k_{\rm IR}$ is even more robust (not shown here).

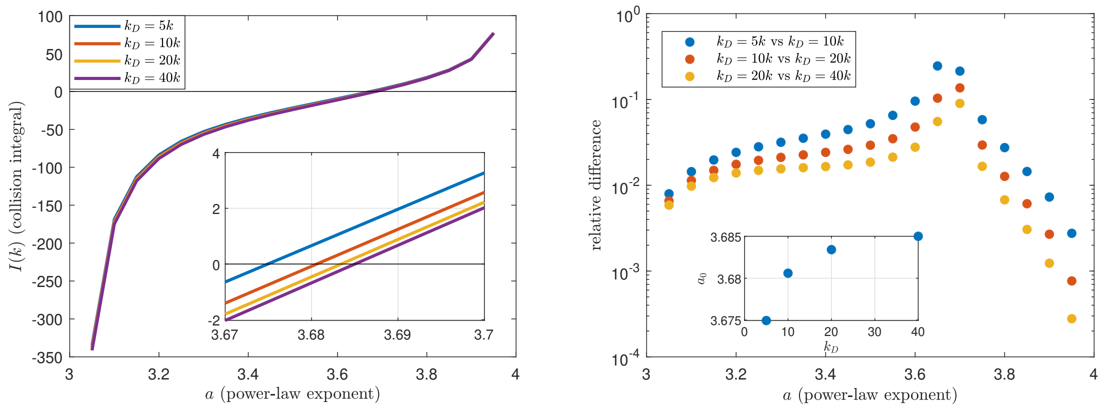


Figure 11: Left: Result of the integral as a function of the exponent a, as the cut at $k_1 = k_{\mathrm{UV}}$ is sent toward infinity. The inset shows a zoomed region around the point where the integral vanishes. Right: Relative difference between the curves in the left panel, in logaritmic scale, giving evidence of convergence. In the inset, convergence of the stationary solution exponent (zero-crossing point in the left panel) is shown.