Supplementary Material - Jetting in finite-amplitude, free, capillary-gravity waves

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(Received xx; revised xx; accepted xx)

1. Governing equations and boundary conditions

In cylindrical axisymmetric coordinates, the governing equation with boundary and initial conditions are

$$\frac{\partial^2\hat{\phi}}{\partial\hat{r}^2}+\frac{1}{\hat{r}}\frac{\partial\hat{\phi}}{\partial\hat{r}}+\frac{\partial^2\hat{\phi}}{\partial\hat{z}^2}=0, \tag{1.1}$$

$$\left(\frac{\partial^2\hat{\phi}}{\partial\hat{t}^2}+g\frac{\partial\phi}{\partial\hat{z}}+\left[\frac{\partial}{\partial\hat{t}}+\frac{1}{2}\left(\frac{\partial\hat{\phi}}{\partial\hat{r}}\right)\frac{\partial}{\partial\hat{r}}+\frac{1}{2}\left(\frac{\partial\hat{\phi}}{\partial\hat{z}}\right)\frac{\partial}{\partial\hat{z}}\right]\left|\boldsymbol{\nabla}\hat{\phi}\right|^2\right)_{\hat{z}=\hat{\eta}}$$
$$-\frac{T}{\rho}\left\{\left[\frac{\partial}{\partial\hat{t}}+\left(\frac{\partial\hat{\phi}}{\partial\hat{r}}\right)_{\hat{z}=\hat{\eta}}\frac{\partial}{\partial\hat{r}}\right]\left(\frac{\frac{\partial^2\hat{\eta}}{\partial\hat{r}^2}}{\left\{1+\left(\frac{\partial\hat{\eta}}{\partial\hat{r}}\right)^2\right\}^{3/2}}+\frac{1}{\hat{r}}\frac{\frac{\partial\hat{\eta}}{\partial\hat{r}}}{\left\{1+\left(\frac{\partial\hat{\eta}}{\partial\hat{r}}\right)^2\right\}^{1/2}}\right)\right\}=0, \tag{1.2}$$

$$\left(\frac{\partial\hat{\phi}}{\partial\hat{t}}+\frac{1}{2}\left|\boldsymbol{\nabla}\hat{\phi}\right|^2+g\hat{z}\right)_{\hat{z}=\hat{\eta}}-\frac{T}{\rho}\left(\frac{\frac{\partial^2\hat{\eta}}{\partial\hat{r}^2}}{\left\{1+\left(\frac{\partial\hat{\eta}}{\partial\hat{r}}\right)^2\right\}^{3/2}}+\frac{1}{\hat{r}}\frac{\frac{\partial\hat{\eta}}{\partial\hat{r}}}{\left\{1+\left(\frac{\partial\hat{\eta}}{\partial\hat{r}}\right)^2\right\}^{1/2}}\right)=0, \tag{1.3}$$

$$\int_0^{\hat{R}_0}\hat{r}\hat{\eta}(\hat{r},\hat{t})d\hat{r}=0 \tag{1.4}$$

$$\left(\frac{\partial\hat{\phi}}{\partial\hat{r}}\right)_{\hat{r}=0}=\left(\frac{\partial\hat{\phi}}{\partial\hat{r}}\right)_{\hat{r}=\hat{R}_0}=0,\quad\left(\frac{\partial\hat{\eta}}{\partial\hat{r}}\right)_{\hat{r}=0}=\left(\frac{\partial\hat{\eta}}{\partial\hat{r}}\right)_{\hat{r}=\hat{R}_0}=0, \tag{1.5}$$

$$\hat{\eta}(\hat{r},0)=a_0\mathrm{J}_0\left(l_q\frac{\hat{r}}{\hat{R}_0}\right),\ \frac{\partial\hat{\eta}}{\partial\hat{t}}(\hat{r},0)=0,\ \hat{\phi}(\hat{r},\hat{z},0)=0. \tag{1.6}$$

We non-dimensionalize these equations using

$$r=\frac{l_q\hat{r}}{\hat{R}_0},\quad z=\frac{l_q\hat{z}}{\hat{R}_0},\quad t=\left(\frac{gl_q}{\hat{R}_0}\right)^{1/2}\hat{t},\quad\eta=\frac{l_q\hat{\eta}}{\hat{R}_0}\quad\text{and }\phi=\left(\frac{l_q^3}{g\hat{R}_0^3}\right)^{1/2}\hat{\phi}, \tag{1.7}$$

This leads to the following non-dimensional equations.

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}=0, \tag{1.8}$$

$$\left(\frac{\partial^2\phi}{\partial t^2}+\frac{\partial\phi}{\partial z}+\left\{\frac{\partial}{\partial t}+\frac{1}{2}\left(\frac{\partial\phi}{\partial r}\right)\frac{\partial}{\partial r}+\frac{1}{2}\left(\frac{\partial\phi}{\partial z}\right)\frac{\partial}{\partial z}\right\}|\nabla\phi|^2\right)_{z=\eta}$$
$$-\sigma\left\{\left[\frac{\partial}{\partial t}+\left(\frac{\partial\phi}{\partial r}\right)_{z=\eta}\frac{\partial}{\partial r}\right]\left(\frac{\frac{\partial^2\eta}{\partial r^2}}{\left\{1+\left(\frac{\partial\eta}{\partial r}\right)^2\right\}^{3/2}}+\frac{1}{r}\frac{\frac{\partial\eta}{\partial r}}{\left\{1+\left(\frac{\partial\eta}{\partial r}\right)^2\right\}^{1/2}}\right)\right\}=0, \tag{1.9}$$

$$\left(\frac{\partial\phi}{\partial t}+\frac{1}{2}|\nabla\phi|^2+z\right)_{z=\eta}-\sigma\left[\frac{\frac{\partial^2\eta}{\partial r^2}}{\left\{1+\left(\frac{\partial\eta}{\partial r}\right)^2\right\}^{3/2}}+\frac{1}{r}\frac{\frac{\partial\eta}{\partial r}}{\left\{1+\left(\frac{\partial\eta}{\partial r}\right)^2\right\}^{1/2}}\right]=0, \tag{1.10}$$

$$\int_0^{l_q} r\eta(r,t)dr=0, \tag{1.11}$$

$$\left(\frac{\partial\phi}{\partial r}\right)_{r=0}=\left(\frac{\partial\phi}{\partial r}\right)_{r=l_q}=0,\quad \left(\frac{\partial\eta}{\partial r}\right)_{r=0}=\left(\frac{\partial\eta}{\partial r}\right)_{r=l_q}=0. \tag{1.12}$$

$$\eta(r,0)=\epsilon \mathrm{J}_0(r),\quad \frac{\partial\eta}{\partial t}(r,0)=0,\quad \phi(r,z,0)=0, \tag{1.13}$$

with two non-dimensional numbers viz. $\epsilon\equiv a_0\dfrac{l_q}{\hat{R}_0}$ and $\sigma\equiv\dfrac{T\left(\dfrac{l_q}{\hat{R}_0}\right)^2}{\rho g}$. The former is a measure of nonlinearity and is referred to in the literature as the steepness parameter. σ measures relative strength of surface tension to gravity and is the inverse of Bond number. Note that while ϵ appears only in the initial condition, σ appears in the boundary conditions. In further analysis, we treat ϵ as a small parameter and $\sigma=\mathcal{O}(1)$ expanding ϕ, η and t in equations 1.8-1.12 as

$$\phi(r,z,t)=0+\epsilon\phi_1(r,z,t)+\epsilon^2\phi_2(r,z,t)+\epsilon^3\phi_3(r,z,t)+\mathcal{O}(\epsilon^4) \tag{1.14}$$
$$\eta(r,t)=0+\epsilon\eta_1(r,t)+\epsilon^2\eta_2(r,t)+\epsilon^3\eta_3(r,t)+\mathcal{O}(\epsilon^4) \tag{1.15}$$
$$\tau=t\left[1+\epsilon^2\Omega_2+\mathcal{O}(\epsilon^3)\right] \tag{1.16}$$

The expansion of t in 1.16 is necessary in order to take into account the nonlinear dependence of the oscillation frequency on the amplitude ϵ of the Bessel mode. Substituting 1.14-1.16 into 1.8-1.13 and using Taylor series expansions about $z=0$, we may generically write at any $\mathcal{O}(\epsilon^i)$ $(i = 1, 2, 3, \ldots)$

$$\frac{\partial^2 \phi_i}{\partial r^2} + \frac{1}{r}\frac{\partial \phi_i}{\partial r} + \frac{\partial^2 \phi_i}{\partial z^2} = 0, \tag{1.17}$$

$$\left(\frac{\partial^2 \phi_i}{\partial \tau^2} + \frac{\partial \phi_i}{\partial z}\right)_{z=0} - \sigma\left[\left(\frac{\partial^3 \eta_i}{\partial \tau \partial r^2}\right) + \frac{1}{r}\left(\frac{\partial^2 \eta_i}{\partial \tau \partial r}\right)\right] = G_i(r,\tau) \tag{1.18}$$

$$\left(\frac{\partial \phi_i}{\partial \tau}\right)_{z=0} + \eta_i - \sigma\left[\left(\frac{\partial^2 \eta_i}{\partial r^2}\right) + \frac{1}{r}\left(\frac{\partial \eta_i}{\partial r}\right)\right] = F_i(r,\tau) \tag{1.19}$$

$$\int_0^{l_q} r\eta_i(r,t)dr = 0, \tag{1.20}$$

$$\left(\frac{\partial \phi_i}{\partial r}\right)_{r=0} = \left(\frac{\partial \phi_i}{\partial r}\right)_{r=l_q} = 0, \quad \left(\frac{\partial \eta_i}{\partial r}\right)_{r=0} = \left(\frac{\partial \eta_i}{\partial r}\right)_{r=l_q} = 0 \tag{1.21}$$

$$\eta_i(r,0) = \mathrm{J}_0(r)\delta_{1i}, \quad \frac{\partial \eta_i}{\partial \tau}(r,0) = \phi_i(r,z,0) = 0, \tag{1.22}$$

where δ_{ij} is the Kronecker delta. As is usual in perturbation methods, the only difference at different orders in 1.17-1.22 are the expressions F_i and G_i on the right hand side. As expected, $F_1(r,\tau) = G_1(r,\tau) = 0$ while $F_2(r,\tau), G_2(r,\tau), F_3(r,\tau), G_3(r,\tau)$ have lengthy expressions provided in Appendix B of the paper. Our task is to now solve equations 1.17-1.22 upto $i = 2$ (second order). Note that for solving upto $i = 2$ completely, it is necessary to also obtain the equations at $i = 3$. Observe that $\phi = p(\tau)\mathrm{J}_0(\alpha r)\exp[\alpha z]$ satisfies the axisymmetric Laplace equation 1.17 for any (real) α and $p(\tau)$. In addition, if we choose $\eta(r,\tau) = a(\tau)\mathrm{J}_0\left(l_j \frac{r}{l_q}\right)$ and $\phi(r,z,\tau) = p(\tau)\mathrm{J}_0\left(l_j \frac{r}{l_q}\right)\exp\left(l_j \frac{z}{l_q}\right)$, alongwith the definition $\mathrm{J}_1(l_j) = 0 \forall j \in \mathbb{Z}^+$ leading to the identity $\int_0^{l_q} a(\tau)\mathrm{J}_0(l_j \frac{r}{l_q}) r dr = 0$, this ensures that 1.20 and 1.21 are also satisfied. Thus the general solution for ϕ_i and η_i satisfying 1.17 and conditions 1.20 and 1.21 may be posed as an eigenfunction expansion (Dini series) as

$$\phi_i(r,z,\tau) = \sum_{j=0}^{\infty} p_i^{(j)}(\tau)\mathrm{J}_0(\alpha_j r)\exp(\alpha_j z), \tag{1.23}$$

$$\eta_i(r,\tau) = \sum_{j=0}^{\infty} a_i^{(j)}(\tau)\mathrm{J}_0(\alpha_j r) \quad \text{with} \quad \alpha_j \equiv \frac{l_j}{l_q}. \tag{1.24}$$

Our task is now reduced to solve for $p_i^{(j)}(\tau)$ and $a_i^{(j)}(\tau)$ using 1.18, 1.19 and 1.22 (all other equations are automatically satisfied by 1.23 and 1.24). An important point worth noting is that in both 1.23 and 1.24, the first term is a function of time only. This is to take into account any nonlinear corrections to the base state. In all subsequent algebra, it should be remembered that q is a positive non-zero integer, fixed by initial conditions. Substituting 1.23 and 1.24 into 1.18 and 1.19 with $\alpha_j \equiv \frac{l_j}{l_q}$, we obtain

$$\sum_{j=0}^{\infty}\left[\frac{d^2 p_i^{(j)}}{d\tau^2} + \alpha_j p_i^{(j)}(\tau) + \sigma\alpha_j^2\left(\frac{da_i^{(j)}}{d\tau}\right)\right]\mathrm{J}_0(\alpha_j r) = G_i(r,\tau) \tag{1.25}$$

$$\sum_{j=0}^{\infty}\left[\frac{dp_i^{(j)}}{d\tau} + \left(1 + \sigma\alpha_j^2\right)a_i^{(j)}(\tau)\right]\mathrm{J}_0(\alpha_j r) = F_i(r,\tau) \tag{1.26}$$

In order to obtain equations governing $p_i^{(j)}(\tau)$ and $a_i^{(j)}(\tau)$, we multiply equations 1.25 and 1.26 with $r\mathrm{J}_0(\alpha_m r)$ and integrate using the first of the following two orthogonality relations (see Chapter 2, equation 12.4 in Bagrov *et al.* (2012)), taking into account that by definition $\mathrm{J}_1(l_q) = 0$,

$$\int_0^{l_q} r\mathrm{J}_0(\alpha_m r)\,\mathrm{J}_0(\alpha_j r)\,dr = \delta_{mj}\frac{l_q^2\mathrm{J}_0^2(l_m)}{2}, \tag{1.27}$$

$$\int_0^{l_q} r\mathrm{J}_1(\alpha_m r)\,\mathrm{J}_1(\alpha_j r)\,dr = \delta_{mj}\frac{l_q^2\left\{\mathrm{J}_1'(l_m)\right\}^2}{2} = \delta_{mj}\frac{l_q^2\mathrm{J}_0^2(l_m)}{2}. \tag{1.28}$$

Using this, we obtain from 1.25 and 1.26 for $m = 0, 1, 2\ldots$

$$\frac{d^2p_i^{(m)}}{d\tau^2} + \alpha_m p_i^{(m)}(\tau) + \sigma\alpha_m^2\left(\frac{da_i^{(m)}}{d\tau}\right) = \mathbb{G}_i^{(m)}(\tau) \tag{1.29}$$

$$\frac{dp_i^{(m)}}{d\tau} + \left(1 + \sigma\alpha_m^2\right)a_i^{(m)}(\tau) = \mathbb{F}_i^{(m)}(\tau) \tag{1.30}$$

with

$$\mathbb{G}_i^{(m)}(\tau) \equiv \frac{2}{l_q^2\mathrm{J}_0^2(l_m)}\int_0^{l_q} dr\; r\mathrm{J}_0(\alpha_m r)\,G_i(r,\tau) \tag{1.31}$$

$$\mathbb{F}_i^{(m)}(\tau) \equiv \frac{2}{l_q^2\mathrm{J}_0^2(l_m)}\int_0^{l_q} dr\; r\mathrm{J}_0(\alpha_m r)\,F_i(r,\tau) \tag{1.32}$$

Equations 1.29 and 1.30 further may be decoupled as

$$\frac{d^2p_i^{(m)}}{d\tau^2} + \omega_m^2 p_i^{(m)}(\tau) = \left(1 + \sigma\alpha_m^2\right)\mathbb{G}_i^{(m)}(\tau) - \sigma\alpha_m^2\frac{d\mathbb{F}_i^{(m)}}{d\tau} \tag{1.33}$$

$$a_i^{(m)}(\tau) = \frac{1}{1 + \sigma\alpha_m^2}\left(\mathbb{F}_i^{(m)}(\tau) - \frac{dp_i^{(m)}}{d\tau}\right) \tag{1.34}$$

with $\omega_m^2 \equiv \alpha_m\left(1 + \sigma\alpha_m^2\right)$ being the deep water linear dispersion relation for capillary-gravity waves written in scaled variables.

Equation 1.33 may now be solved for $p_i^{(m)}(\tau)$ and the resultant expression determines $a_i^{(m)}(\tau)$ using 1.34. We also need initial conditions which are obtained from 1.22. With $m = 0, 1, 2, 3\ldots$ and $i = 1, 2, 3\ldots$, we obtain

$$p_i^{(m)}(0) = 0, \quad \frac{dp_i}{d\tau}^{(m)}(0) = \mathbb{F}_i^{(m)}(0) - \left(1 + \sigma\alpha_m^2\right)\delta_{mq}\delta_{i1} \tag{1.35}$$

$$a_i^{(m)}(0) = \delta_{mq}\delta_{i1}, \quad \frac{da_i^{(m)}}{d\tau}(0) = 0 \tag{1.36}$$

We now solve 1.33 and 1.34 at various orders using the initial conditions provided in 1.35 and 1.36.

1.1. $O(\epsilon)$ - *Linear solution*

With $F_1(r,\tau) = G_1(r,\tau) = 0$ implying $\mathbb{F}_1^{(m)}(\tau) = 0$ and $\mathbb{G}_1^{(m)}(\tau) = 0$, equation 1.33 is a homogenous equation. We obtain

$$\frac{d^2p_1^{(m)}}{d\tau^2} + \omega_m^2\, p_1^{(m)}(\tau) = 0,\; a_i^{(m)}(\tau) = -\frac{1}{1 + \sigma\alpha_m^2}\left(\frac{dp_i^{(m)}}{d\tau}\right). \tag{1.37}$$

The solutions at this order using initial conditions 1.35 and 1.36 are

$$p_1^{(m)}(\tau) = -\delta_{mq}\left(\frac{1+\sigma\alpha_m^2}{\omega_m}\right)\sin(\omega_m\tau), \quad a_1^{(m)}(\tau) = \delta_{mq}\cos(\omega_m\tau), \tag{1.38}$$

implying from 1.23 and 1.24 that

$$\phi_1(r,z,\tau) = -\omega_q\sin(\omega_q\tau)\mathrm{J}_0(r)\exp\left[z\right], \quad \eta_1(r,\tau) = \cos(\omega_q\tau)\mathrm{J}_0(r). \tag{1.39}$$

1.2. $O(\epsilon^2)$ - *Nonlinear correction*

Using 1.39 and expressions for $F_2(r,\tau)$ and $G_2(r,\tau)$ provided in Appendix B, we obtain

$$F_2(r,\tau) = \frac{\omega_q^2}{4}\left[\left\{\mathrm{J}_0^2(r) - \mathrm{J}_1^2(r)\right\} + \left\{3\mathrm{J}_0^2(r) + \mathrm{J}_1^2(r)\right\}\cos(2\omega_q\tau)\right], \tag{1.40}$$

$$G_2(r,\tau) = -\frac{\omega_q}{2}\left[\left(3\omega_q^2 - 1\right)\mathrm{J}_0^2(r) + \left(2\omega_q^2 - \sigma\right)\mathrm{J}_1^2(r)\right]\sin(2\omega_q\tau). \tag{1.41}$$

At second order ($i = 2$), 1.33 and 1.34 lead to

$$\frac{d^2p_2^{(m)}}{d\tau^2} + \omega_m^2 p_2^{(m)}(\tau) = \left(1+\sigma\alpha_m^2\right)\mathbb{G}_2^{(m)}(\tau) - \sigma\alpha_m^2\frac{d\mathbb{F}_2^{(m)}}{d\tau} \tag{1.42}$$

$$a_2^{(m)}(\tau) = \frac{1}{1+\sigma\alpha_m^2}\left(\mathbb{F}_2^{(m)}(\tau) - \frac{dp_2^{(m)}}{d\tau}\right). \tag{1.43}$$

Equation 1.42 is to be solved with the initial condition

$$p_2^{(m)}(0) = 0, \quad \frac{dp_2^{(m)}}{d\tau}(0) = \mathbb{F}_2^{(m)}(0) \tag{1.44}$$

In 1.42 and 1.43, $q \in \mathbb{Z}^+$ while $m = 0, 1, 2, 3\ldots$. Note that we have omitted the initial conditions on $a_2^{(m)}(\tau)$ and its first derivative in 1.44 as $a_2^{(m)}(\tau)$ is determined by equation 1.43, once $p_2^{(m)}(\tau)$ is known. Our task is now to solve 1.42 using 1.44. For this we need expressions for $\mathbb{F}_2^{(m)}(\tau)$ and $\mathbb{G}_2^{(m)}(\tau)$. As shorthand notation, we define

$$\mathrm{I}_{\nu_1\text{ - }m_1,\nu_2\text{ - }m_2,\ldots} \equiv \int_0^1 d\tilde{r}\;\tilde{r}\;\mathrm{J}_{\nu 1}\left(l_{m_1}\tilde{r}\right)\mathrm{J}_{\nu 2}\left(l_{m_2}\tilde{r}\right)\ldots \tag{1.45}$$

where $\nu_i, m_i = 0, 1, 2, \ldots$. Using 1.31, 1.32, 1.40 and 1.41, we obtain

$$\mathbb{F}_2^{(m)}(\tau) = \frac{\omega_q^2}{2\mathrm{J}_0^2(l_m)}\left[(\mathrm{I}_{0\text{ - }m,0\text{ - }q,0\text{ - }q} - \mathrm{I}_{0\text{ - }m,1\text{ - }q,1\text{ - }q}) + (3\mathrm{I}_{0\text{ - }m,0\text{ - }q,0\text{ - }q} + \mathrm{I}_{0\text{ - }m,1\text{ - }q,1\text{ - }q})\cos(2\omega_q\tau)\right] \tag{1.46}$$

$$\mathbb{G}_2^{(m)}(\tau) = -\frac{\omega_q}{\mathrm{J}_0^2(l_m)}\left[\left(3\omega_q^2 - 1\right)\mathrm{I}_{0\text{ - }m,0\text{ - }q,0\text{ - }q} + \left(2\omega_q^2 - \sigma\right)\mathrm{I}_{0\text{ - }m,1\text{ - }q,1\text{ - }q}\right]\sin(2\omega_q t) \tag{1.47}$$

and $$\frac{d\mathbb{F}_2^{(m)}}{d\tau} = -\frac{\omega_q^3}{\mathrm{J}_0^2(l_m)}\left(3\mathrm{I}_{0\text{ - }m,0\text{ - }q,0\text{ - }q} + \mathrm{I}_{0\text{ - }m,1\text{ - }q,1\text{ - }q}\right)\sin(2\omega_q\tau) \tag{1.48}$$

Using 1.47 and 1.48, equation 1.42 may be written as

$$\frac{d^2p_2^{(m)}}{d\tau^2} + \omega_m^2 p_2^{(m)}(\tau) = \chi_2^{(m)}\sin\left(2\omega_q\tau\right) \tag{1.49}$$

with

$$\chi_2^{(m)} \equiv -\frac{\omega_q}{\mathrm{J}_0^2(l_m)}\left[\left\{2+\left(3-\alpha_m^2\right)\sigma\right\}\mathrm{I}_{0\,\text{-}\,m,0\,\text{-}\,q,0\,\text{-}\,q}+\left\{2+\left(1+\alpha_m^2\right)\sigma\right\}\mathrm{I}_{0\,\text{-}\,m,1\,\text{-}\,q,1\,\text{-}\,q}\right] \tag{1.50}$$

It is evident that the solution to 1.49 is algebraically growing when the resonance condition $\omega_m = 2\omega_q$ is satisfied. We will presently assume that this is not the case. For $m = 0$ the general solution to 1.49 is

$$p_2^{(0)}(\tau) = c_2^{(0)}\tau + d_2^{(0)} - \frac{\chi_2^{(0)}}{4\omega_q^2}\sin\left(2\omega_q\tau\right) \tag{1.51}$$

Using the first initial condition in 1.44, we obtain $d_2^{(0)} = 0$. We also obtain from 1.50,

$$\chi_2^{(0)} = -\omega_q\left[(2+3\sigma)\,\mathrm{I}_{0\,\text{-}\,0,0\,\text{-}\,q,0\,\text{-}\,q} + (2+\sigma)\,\mathrm{I}_{0\,\text{-}\,0,1\,\text{-}\,q,1\,\text{-}\,q}\right] \tag{1.52}$$

We have the following identities

$$\mathrm{I}_{0\,\text{-}\,0,0\,\text{-}\,q,0\,\text{-}\,q} = \frac{\mathrm{J}_0^2(l_q)}{2}, \quad \mathrm{I}_{0\,\text{-}\,0,1\,\text{-}\,q,1\,\text{-}\,q} = \frac{\mathrm{J}_0^2(l_q)}{2} \tag{1.53}$$

Combining 1.52 and 1.53, we obtain

$$\chi_2^{(0)} = -2\omega_q\mathrm{J}_0^2(l_q)\,(1+\sigma) \tag{1.54}$$

and we can now rewrite 1.51 as

$$p_2^{(0)}(\tau) = c_2^{(0)}\tau + \frac{\mathrm{J}_0^2(l_q)}{2\omega_q}\,(1+\sigma)\sin\left(2\omega_q\tau\right) \tag{1.55}$$

In order to determine $c_2^{(0)}$, we use the second initial condition in 1.44. For this we need $\mathbb{F}_2^{(0)}(0)$ from 1.46. This is

$$\mathbb{F}_2^{(0)}(0) = 2\omega_q^2\mathrm{I}_{0\,\text{-}\,0,0\,\text{-}\,q,0\,\text{-}\,q} = \omega_q^2\mathrm{J}_0^2(l_q) \tag{1.56}$$

Combining the second initial condition in 1.44 with 1.55 and 1.56, we obtain $c_2^{(0)} = 0$, thus implying from 1.55

$$p_2^{(0)}(\tau) = \frac{\mathrm{J}_0^2(l_q)}{2}\sqrt{1+\sigma}\sin\left(2\omega_q\tau\right). \tag{1.57}$$

For $m > 0$, the general solution to 1.49 may be written as

$$p_2^{(m)}(\tau) = c_2^{(m)}\sin(\omega_m\tau) + d_2^{(m)}\cos(\omega_m\tau) + \frac{\chi_2^{(m)}}{\omega_m^2 - 4\omega_q^2}\sin(2\omega_q\tau) \tag{1.58}$$

Using initial conditions 1.44 we obtain

$$p_2^{(m)}(\tau) = \left\{\frac{\mathbb{F}_2^{(m)}(0)}{\omega_m} - \frac{2\left(\frac{\omega_q}{\omega_m}\right)\chi_2^{(m)}}{\omega_m^2-4\omega_q^2}\right\}\sin(\omega_m\tau) + \frac{\chi_2^{(m)}}{\omega_m^2-4\omega_q^2}\sin(2\omega_q\tau) \tag{1.59}$$

Simplifying the first term inside brackets in 1.59, we obtain

$$\frac{\mathbb{F}_2^{(m)}(0)\left(\omega_m^2-4\omega_q^2\right)-2\omega_q\chi_2^{(m)}}{\left(\omega_m^2-4\omega_q^2\right)\omega_m}$$
$$=\frac{2\,(1+\sigma)}{\mathrm{J}_0^2(l_m)\left(\omega_m^2-4\omega_q^2\right)\omega_m}\left[\left\{\,(\alpha_m-2)+\sigma\left(\alpha_m^3-\alpha_m^2-1\right)\,\right\}\mathrm{I}_{0\,\text{-}\,m,0\,\text{-}\,q,0\,\text{-}\,q}\right.$$
$$\left.+\left\{2+\sigma\left(1+\alpha_m^2\right)\,\right\}\mathrm{I}_{0\,\text{-}\,m,1\,\text{-}\,q,1\,\text{-}\,q}\right] \tag{1.60}$$

Using 1.60 expression 1.59 may be written as

$$p_2^{(m)}(\tau)=\frac{2\,(1+\sigma)}{\mathrm{J}_0^2(l_m)\left(\omega_m^2-4\omega_q^2\right)\omega_m}\left[\left\{\,(\alpha_m-2)+\sigma\left(\alpha_m^3-\alpha_m^2-1\right)\,\right\}\mathrm{I}_{0\,\text{-}\,m,0\,\text{-}\,q,0\,\text{-}\,q}\right.$$
$$\left.+\left\{2+\sigma\left(1+\alpha_m^2\right)\,\right\}\mathrm{I}_{0\,\text{-}\,m,1\,\text{-}\,q,1\,\text{-}\,q}\right]\sin\left(\omega_m\tau\right)$$
$$-\frac{\omega_q}{\mathrm{J}_0^2(l_m)\left(\omega_m^2-4\omega_q^2\right)}\left[\left\{2+\left(3-\alpha_m^2\right)\sigma\right\}\mathrm{I}_{0\,\text{-}\,m,0\,\text{-}\,q,0\,\text{-}\,q}\right.$$
$$\left.+\left\{2+\sigma\left(1+\alpha_m^2\right)\,\right\}\mathrm{I}_{0\,\text{-}\,m,1\,\text{-}\,q,1\,\text{-}\,q}\right]\sin\left(2\omega_q\tau\right) \tag{1.61}$$

Using 1.43, we also obtain

$$a_2^{(0)}(\tau)=\frac{\omega_q^2}{2}\Bigg[\left(\mathrm{I}_{0\,\text{-}\,0,0\,\text{-}\,q,0\,\text{-}\,q}-\mathrm{I}_{0\,\text{-}\,0,1\,\text{-}\,q,1\,\text{-}\,q}\right)$$
$$+\left(3\mathrm{I}_{0\,\text{-}\,0,0\,\text{-}\,q,0\,\text{-}\,q}+\mathrm{I}_{0\,\text{-}\,0,1\,\text{-}\,q,1\,\text{-}\,q}\right)\cos(2\omega_q\tau)\Bigg]-\omega_q^2\mathrm{J}_0^2(l_q)\cos(2\omega_q\tau)$$
$$=0 \tag{1.62}$$

where the last step follows from the identities in 1.53. For $m>0$, we need $\frac{dp_2^{(m)}}{d\tau}$ from 1.61 and obtain,

$$\frac{dp_2^{(m)}}{d\tau}=\frac{2\omega_q^2}{\mathrm{J}_0^2(l_m)\left(\omega_m^2-4\omega_q^2\right)}\left[\left\{\,(\alpha_m-2)+\sigma\left(\alpha_m^3-\alpha_m^2-1\right)\,\right\}\mathrm{I}_{0\,\text{-}\,m,0\,\text{-}\,q,0\,\text{-}\,q}\right.$$
$$\left.+\left\{2+\sigma\left(1+\alpha_m^2\right)\,\right\}\mathrm{I}_{0\,\text{-}\,m,1\,\text{-}\,q,1\,\text{-}\,q}\right]\cos\left(\omega_m\tau\right)$$
$$-\frac{2\omega_q^2}{\mathrm{J}_0^2(l_m)\left(\omega_m^2-4\omega_q^2\right)}\left[\left\{2+\left(3-\alpha_m^2\right)\sigma\right\}\mathrm{I}_{0\,\text{-}\,m,0\,\text{-}\,q,0\,\text{-}\,q}\right.$$
$$\left.+\left\{2+\sigma\left(1+\alpha_m^2\right)\,\right\}\mathrm{I}_{0\,\text{-}\,m,1\,\text{-}\,q,1\,\text{-}\,q}\right]\cos\left(2\omega_q\tau\right) \tag{1.63}$$

Combining 1.46 and 1.63, we obtain from 1.43

$$a_2^{(m)}(\tau) = \frac{\omega_q^2}{2\mathrm{J}_0^2(l_m)\left(1+\sigma\alpha_m^2\right)}\Bigg[\left(\mathrm{I}_{0\,\text{-}\,m,0\,\text{-}\,q,0\,\text{-}\,q} - \mathrm{I}_{0\,\text{-}\,m,1\,\text{-}\,q,1\,\text{-}\,q}\right)$$
$$+ \frac{1}{\omega_m^2 - 4\omega_q^2}\left\{\left((3\alpha_m - 4) + \sigma\left(3\alpha_m^3 - 4\alpha_m^2\right)\right)\mathrm{I}_{0\,\text{-}\,m,0\,\text{-}\,q,0\,\text{-}\,q}\right.$$
$$\left. + \left((\alpha_m + 4) + \sigma\left(\alpha_m^3 + 4\alpha_m^2\right)\right)\mathrm{I}_{0\,\text{-}\,m,1\,\text{-}\,q,1\,\text{-}\,q}\right\}\cos\left(2\omega_q\tau\right)$$
$$- \frac{4}{\omega_m^2 - 4\omega_q^2}\left\{\left((\alpha_m - 2) + \sigma\left(\alpha_m^3 - \alpha_m^2 - 1\right)\right)\mathrm{I}_{0\,\text{-}\,m,0\,\text{-}\,q,0\,\text{-}\,q}\right.$$
$$\left. + \left(2 + \sigma\left(1+\alpha_m^2\right)\right)\mathrm{I}_{0\,\text{-}\,m,1\,\text{-}\,q,1\,\text{-}\,q}\right\}\cos\left(\omega_m\tau\right)\Bigg] \tag{1.64}$$

At $\mathcal{O}(\epsilon^2)$, the solutions ϕ_2 and η_2 are

$$\phi_2(r,z,\tau) = \frac{\sqrt{1+\sigma}}{2}\mathrm{J}_0^2(l_q)\sin(2\omega_q\tau)$$
$$+\sqrt{1+\sigma}\sum_{k=1}^{\infty}\left[\xi_1^{(k)}\sin(\omega_k\tau) + \xi_2^{(k)}\sin(2\omega_q\tau)\right]\mathrm{J}_0(\alpha_k r)e^{\alpha_k z} \tag{1.65}$$

where

$$\xi_1^{(k)} \equiv \frac{2\omega_q}{\mathrm{J}_0^2(l_k)\omega_k\left(\omega_k^2 - 4\omega_q^2\right)}\left\{\left[(\alpha_k - 2) + \left(\alpha_k^3 - \alpha_k^2 - 1\right)\sigma\right]\mathrm{I}_{0\,\text{-}\,k,0\,\text{-}\,q,0\,\text{-}\,q}\right.$$
$$\left. + \left[2 + \left(1+\alpha_k^2\right)\sigma\right]\mathrm{I}_{0\,\text{-}\,k,1\,\text{-}\,q,1\,\text{-}\,q}\right\} \tag{1.66}$$

and

$$\xi_2^{(k)} \equiv -\frac{\left\{\left[2 + \left(3-\alpha_k^2\right)\sigma\right]\mathrm{I}_{0\,\text{-}\,k,0\,\text{-}\,q,0\,\text{-}\,q} + \left[2 + \left(1+\alpha_k^2\right)\sigma\right]\mathrm{I}_{0\,\text{-}\,k,1\,\text{-}\,q,1\,\text{-}\,q}\right\}}{\mathrm{J}_0^2(l_k)\left(\omega_k^2 - 4\omega_q^2\right)} \tag{1.67}$$

while

$$\eta_2(r,\tau) = \sum_{k=1}^{\infty}\left[\zeta_1^{(k)}\cos(\omega_k\tau) + \zeta_2^{(k)}\cos(2\omega_q\tau) + \zeta_3^{(k)}\right]\mathrm{J}_0(\alpha_k r) \tag{1.68}$$

where

$$\zeta_1^{(k)} \equiv -\frac{2\left(1+\sigma\right)}{\mathrm{J}_0^2(l_k)\left(1+\alpha_k^2\sigma\right)\left(\omega_k^2 - 4\omega_q^2\right)}\left\{\left[(\alpha_k - 2) + \left(\alpha_k^3 - \alpha_k^2 - 1\right)\sigma\right]\mathrm{I}_{0\,\text{-}\,k,0\,\text{-}\,q,0\,\text{-}\,q}\right.$$
$$\left. + \left[2 + \left(1+\alpha_k^2\right)\sigma\right]\mathrm{I}_{0\,\text{-}\,k,1\,\text{-}\,q,1\,\text{-}\,q}\right\} \tag{1.69}$$

$$\zeta_2^{(k)} \equiv \frac{1+\sigma}{2\mathrm{J}_0^2(l_k)\left(1+\alpha_k^2\sigma\right)\left(\omega_k^2-4\omega_q^2\right)}\left\{\left[(3\alpha_k-4)+\left(3\alpha_k^3-4\alpha_k^2\right)\sigma\right]\mathrm{I}_{0\text{ - }k,0\text{ - }q,0\text{ - }q}\right.$$
$$\left.+\left[(4+\alpha_k)+\left(\alpha_k^3+4\alpha_k^2\right)\sigma\right]\mathrm{I}_{0\text{ - }k,1\text{ - }q,1\text{ - }q}\right\} \tag{1.70}$$

and

$$\zeta_3^{(k)} \equiv \frac{1+\sigma}{2\mathrm{J}_0^2(l_k)\left(1+\alpha_k^2\sigma\right)}\left\{\mathrm{I}_{0\text{ - }k,0\text{ - }q,0\text{ - }q}-\mathrm{I}_{0\text{ - }k,1\text{ - }q,1\text{ - }q}\right\} \tag{1.71}$$

1.3. $\mathcal{O}(\epsilon^3)$ - *Nonlinear correction to frequency*

Using expressions for $F_3(r,\tau)$ and $G_3(r,\tau)$ provided in Appendix B in the paper alongwith expressions for η_1,η_2 and ϕ_1,ϕ_2 obtained earlier, we find

$$F_3(r,\tau) = A(r)\cos(\omega_q\tau)+B(r)\cos(3\omega_q\tau)$$
$$+\sum_{k=1}^{\infty}\left(C_k(r)\cos\left[(\omega_k+\omega_q)\tau\right]+D_k(r)\cos\left[(\omega_k-\omega_q)\tau\right]\right) \tag{1.72}$$
$$G_3(r,\tau) = P(r)\sin(\omega_q\tau)+Q(r)\sin(3\omega_q\tau)$$
$$+\sum_{k=1}^{\infty}\left(R_k(r)\sin\left[(\omega_k+\omega_q)\tau\right]+S_k(r)\sin\left[(\omega_k-\omega_q)\tau\right]\right) \tag{1.73}$$

where,

$$A(r) \equiv \frac{\sqrt{1+\sigma}}{2}\left[\sum_{k=1}^{\infty}\left[\left\{-\alpha_k\xi_2^{(k)}+\zeta_2^{(k)}+2\zeta_3^{(k)}\right\}\omega_q\mathrm{J}_0(r)\mathrm{J}_0(\alpha_k r)\right.\right.$$
$$\left.+\alpha_k\omega_q\xi_2^{(k)}\mathrm{J}_1(r)\mathrm{J}_1(\alpha_k r)\right]+\frac{\omega_q}{4}\mathrm{J}_0^3(r)-\frac{(1-2\sigma)}{2\omega_q}\mathrm{J}_0(r)\mathrm{J}_1^2(r)$$
$$\left.-\frac{3\sigma}{4\omega_q}\mathrm{J}_1^2(r)\mathrm{J}_2(r)\right]+\Omega_2\left(1+\sigma\right)\mathrm{J}_0(r) \tag{1.74}$$

$$B(r) \equiv \frac{\sqrt{1+\sigma}}{2}\left[\sum_{k=1}^{\infty}\left[\left\{-3\alpha_k\omega_q\xi_2^{(k)}+\omega_q\zeta_2^{(k)}\right\}\mathrm{J}_0(r)\mathrm{J}_0(\alpha_k r)\right.\right.$$
$$\left.-\left\{\alpha_k\omega_q\xi_2^{(k)}\right\}\mathrm{J}_1(r)\mathrm{J}_1(\alpha_k r)\right]+\frac{3\omega_q}{4}\mathrm{J}_0^3(r)+\frac{(1+2\sigma)}{2\omega_q}\mathrm{J}_0(r)\mathrm{J}_1^2(r)$$
$$\left.-\frac{\sigma}{4\omega_q}\mathrm{J}_1^2(r)\mathrm{J}_2(r)\right] \tag{1.75}$$

$$C_k(r) \equiv \frac{\sqrt{1+\sigma}}{2}\left[\left\{-\alpha_k\left(\omega_k+\omega_q\right)\xi_1^{(k)}+\omega_q\zeta_1^{(k)}\right\}\mathrm{J}_0(r)\mathrm{J}_0(\alpha_k r)\right.$$
$$\left.-\left\{\alpha_k\omega_q\xi_1^{(k)}\right\}\mathrm{J}_1(r)\mathrm{J}_1(\alpha_k r)\right] \tag{1.76}$$

and

$$D_k(r) \equiv \frac{\sqrt{1+\sigma}}{2}\left[\left\{-\alpha_k\left(\omega_k-\omega_q\right)\xi_1^{(m)}+\omega_q\zeta_1^{(k)}\right\}\mathrm{J}_0(r)\mathrm{J}_0(\alpha_k r)\right.$$
$$\left.+\left\{\alpha_k\omega_q\xi_1^{(k)}\right\}\mathrm{J}_1(r)\mathrm{J}_1(\alpha_k r)\right] \tag{1.77}$$

while

$$P(r) \equiv \frac{\sqrt{1+\sigma}}{2}\left[\sum_{k=1}^{\infty}\left[\left\{\left[2\alpha_k\left(1+\sigma\right)-\alpha_k^2\right]\xi_2^{(k)}+\sigma\zeta_2^{(k)}-2\sigma\zeta_3^{(m)}\right\}\mathrm{J}_0(r)\mathrm{J}_0(\alpha_k r)\right.\right.$$
$$\left.-\left\{\alpha_k\left(2+3\sigma\right)\xi_2^{(k)}+\alpha_k^3\sigma\zeta_2^{(k)}-2\alpha_k^3\sigma\zeta_3^{(k)}\right\}\mathrm{J}_1(r)\mathrm{J}_1(\alpha_k r)\right]$$
$$\left.-\frac{(2+3\sigma)}{4}\mathrm{J}_0^3(r)+\frac{(1-3\sigma)}{4}\mathrm{J}_0(r)\mathrm{J}_1^2(r)+\frac{3\left(1+2\sigma\right)}{4}\mathrm{J}_1^2(r)\mathrm{J}_2(r)\right]$$
$$-\Omega_2\left(2+\sigma\right)\sqrt{1+\sigma}\mathrm{J}_0(r) \tag{1.78}$$

with δ_{1j} representing the Kronecker delta.

$$Q(r) \equiv \frac{\sqrt{1+\sigma}}{2}\left[\sum_{k=1}^{\infty}\left[\left\{\left[10\alpha_k\left(1+\sigma\right)-\alpha_k^2\right]\xi_2^{(k)}-\sigma\zeta_2^{(k)}\right\}\mathrm{J}_0(r)\mathrm{J}_0(\alpha_k r)\right.\right.$$
$$\left.+\left\{\alpha_k\left(6+5\sigma\right)\xi_2^{(k)}+\alpha_k^3\sigma\zeta_2^{(k)}\right\}\mathrm{J}_1(r)\mathrm{J}_1(\alpha_k r)\right]-\frac{(10+11\sigma)}{4}\mathrm{J}_0^3(r)$$
$$\left.-\frac{(11+15\sigma)}{4}\mathrm{J}_0(r)\mathrm{J}_1^2(r)-\frac{(1-2\sigma)}{4}\mathrm{J}_1^2(r)\mathrm{J}_2(r)\right] \tag{1.79}$$

$$R_k(r) \equiv \frac{\sqrt{1+\sigma}}{2}\left[\left\{\alpha_k\left[\omega_k^2+2\omega_q\omega_k+2\omega_q^2-\alpha_k\right]\xi_1^{(k)}-\sigma\zeta_1^{(k)}\right\}\mathrm{J}_0(r)\mathrm{J}_0(\alpha_k r)\right.$$
$$\left.+\left\{\alpha_k\left[2\omega_q\omega_k+(2+\sigma)\right]\xi_1^{(k)}+\alpha_k^3\sigma\zeta_1^{(k)}\right\}\mathrm{J}_1(r)\mathrm{J}_1(\alpha_k r)\right] \tag{1.80}$$

and

$$S_k(r) \equiv \frac{\sqrt{1+\sigma}}{2}\left[\left\{\alpha_k\left[\omega_k^2-2\omega_q\omega_k+2\omega_q^2-\alpha_k\right]\xi_1^{(k)}+\sigma\zeta_1^{(k)}\right\}\mathrm{J}_0(r)\mathrm{J}_0(\alpha_k r)\right.$$
$$\left.+\left\{\alpha_k\left[-2\omega_q\omega_k+(2+\sigma)\right]\xi_1^{(k)}-\alpha_k^3\sigma\zeta_1^{(k)}\right\}\mathrm{J}_1(r)\mathrm{J}_1(\alpha_k r)\right] \tag{1.81}$$

Using 1.72 and 1.73 we obtain,

$$\mathbb{F}_3^{(m)}(\tau) = A^{(m)}\cos(\omega_q\tau) + B^{(m)}\cos(3\omega_q\tau) + \sum_{k=1}^{\infty}\left(C_k^{(m)}\cos\left[\left(\omega_k+\omega_q\right)\tau\right] + D_k^{(m)} cos\left[\left(\omega_k-\omega_q\right)\tau\right]\right) \tag{1.82}$$

$$\mathbb{G}_3^{(m)}(\tau) = P^{(m)}\sin(\omega_q\tau) + Q^{(m)}\sin(3\omega_q\tau) + \sum_{k=1}^{\infty}\left(R_k^{(m)}\sin\left[\left(\omega_k+\omega_q\right)\tau\right] + S_k^{(m)}\sin\left[\left(\omega_k-\omega_q\right)\tau\right]\right) \tag{1.83}$$

where $\left[A^{(m)}, B^{(m)}, C_k^{(m)}, D_k^{(m)}, P^{(m)}, Q^{(m)}, R_k^{(m)}, S_k^{(m)}\right]$ are defined as

$$\int_0^1 d\tilde{r}\tilde{r}\mathrm{J}_0(l_m\tilde{r})\left[A(\tilde{r}), B(\tilde{r}), C_k(\tilde{r}), D_k(\tilde{r}), P(\tilde{r}), Q(\tilde{r}), R_k(\tilde{r}), S_k(\tilde{r})\right], \quad \tilde{r} \equiv \frac{r}{l_q}. \tag{1.84}$$

The expressions for $A^{(m)}$, $B^{(m)}$, $C_k^{(m)}$, $D_k^{(m)}$ and $P^{(m)}$, $Q^{(m)}$, $R_k^{(m)}$, $S_k^{(m)}$ are provided below. Analogous to second order, we now have

$$\frac{d^2p_3^{(m)}}{d\tau^2} + \omega_m^2 p_3^{(m)}(\tau) = \left(1+\sigma\alpha_m^2\right)\mathbb{G}_3^{(m)}(\tau) - \sigma\alpha_m^2\frac{d\mathbb{F}_3^{(m)}}{d\tau} \tag{1.85}$$

$$a_3^{(m)}(\tau) = \frac{1}{1+\sigma\alpha_m^2}\left[\mathbb{F}_3^{(m)}(\tau) - \frac{dp_3^{(m)}}{d\tau}\right]. \tag{1.86}$$

with the initial conditions

$$p_3^{(m)}(0) = 0, \qquad \frac{dp_3^{(m)}}{d\tau}(0) = \mathbb{F}_3^{(m)}(0) \tag{1.87}$$

with $q \neq 0$, $m = 0, 1, 2, 3\ldots$.

A number of interesting features are apparent from equation 1.85 alongwith expressions 1.82 and 1.83. At third order, we now have the *third* harmonic resonance condition viz. $3\omega_q = \omega_m$, analogous to the second harmonic case earlier. In addition, we also have the possibility of additional resonances viz. $\omega_k \pm \omega_q = \omega_m$. We do not investigate the consequences of any of these in the current study. Instead, we only look for resonant forcing of the primary mode when $m = q$ in 1.85. This is seen to arise from the coefficient of $\sin(\omega_q\tau)$ on the right hand side of 1.86. After some algebra we find that the coefficient of $\sin(\omega_q\tau)$ for the case $m = q$ is $(1+\sigma)\,P^{(q)} + \sigma\omega_q A^{(q)}$. Setting this to zero, we eliminate the possibility of resonant forcing of the primary mode and also obtain the value of Ω_2, the nonlinear correction to the frequency. This is,

$$\Omega_2 = \frac{1}{2\mathrm{J}_0^2(l_q)}\left[\sum_{l=1}^{\infty}\left[\left\{\left[\alpha_l\left(2+\sigma\right)-\alpha_l^2\right]\xi_2^{(l)} + 2\sigma\zeta_2^{(l)}\right\}\mathrm{I}_{0\text{ - }l,0\text{ - }q,0\text{ - }q} - \left\{2\left(1+\sigma\right)\alpha_l\xi_2^{(l)} + \alpha_l^3\sigma\zeta_2^{(l)} - 2\alpha_l^3\sigma\zeta_3^{(l)}\right\}\mathrm{I}_{0\text{ - }q,1\text{ - }q,1\text{ - }l}\right] - \frac{(1+\sigma)}{2}\mathrm{I}_{0\text{ - }q,0\text{ - }q,0\text{ - }q,0\text{ - }q} + \frac{\left(1-4\sigma+\sigma^2\right)}{4\left(1+\sigma\right)}\mathrm{I}_{0\text{ - }q,0\text{ - }q,1\text{ - }q,1\text{ - }q} + \frac{3\left(1+3\sigma+\sigma^2\right)}{4\left(1+\sigma\right)}\mathrm{I}_{0\text{ - }q,1\text{ - }q,1\text{ - }q,2\text{ - }q}\right] \tag{1.88}$$

The expressions for $\left[A^{(m)}, B^{(m)}, C_k^{(m)}, D_k^{(m)}, P^{(m)}, Q^{(m)}, R_k^{(m)}, S_k^{(m)}\right]$ in 1.84 are

$$A^{(m)} \equiv \frac{\sqrt{1+\sigma}}{\mathrm{J}_0^2(l_m)} \left[\sum_{k=1}^{\infty} \left[\left\{ -\alpha_k \xi_2^{(k)} + \zeta_2^{(k)} + 2\zeta_3^{(k)} \right\} \omega_q \mathrm{I}_{0\text{ - }q,0\text{ - }m,0\text{ - }k} + \alpha_k \omega_q \xi_2^{(k)} \mathrm{I}_{0\text{ - }m,1\text{ - }q,1\text{ - }k} \right] + \frac{\omega_q}{4} \mathrm{I}_{0\text{ - }q,0\text{ - }q,0\text{ - }q,0\text{ - }m} - \frac{(1-2\sigma)}{2\omega_q} \mathrm{I}_{0\text{ - }q,0\text{ - }m,1\text{ - }q,1\text{ - }q} - \frac{3\sigma}{4\omega_q} \mathrm{I}_{0\text{ - }m,1\text{ - }q,1\text{ - }q,2\text{ - }q} \right] + \delta_{qm}\Omega_2 (1+\sigma) \tag{1.89}$$

$$B^{(m)} \equiv \frac{\sqrt{1+\sigma}}{\mathrm{J}_0^2(l_m)} \left[2\sum_{k=1}^{\infty} \left[\left\{ -3\alpha_k \omega_q \xi_2^{(k)} + \omega_q \zeta_2^{(k)} \right\} \mathrm{I}_{0\text{ - }q,0\text{ - }m,0\text{ - }k} - \left\{ \alpha_k \omega_q \xi_2^{(k)} \right\} \mathrm{I}_{0\text{ - }m,1\text{ - }q,1\text{ - }k} \right] + \frac{3\omega_q}{4} \mathrm{I}_{0\text{ - }q,0\text{ - }q,0\text{ - }q,0\text{ - }m} + \frac{(1+2\sigma)}{2\omega_q} \mathrm{I}_{0\text{ - }q,0\text{ - }m,1\text{ - }q,1\text{ - }q} - \frac{\sigma}{4\omega_q} \mathrm{I}_{0\text{ - }m,1\text{ - }q,1\text{ - }q,2\text{ - }q} \right] \tag{1.90}$$

$$C_k^{(m)} \equiv \frac{\sqrt{1+\sigma}}{\mathrm{J}_0^2(l_m)} \left[\left\{ -\alpha_k (\omega_k + \omega_q) \xi_1^{(k)} + \omega_q \zeta_1^{(k)} \right\} \mathrm{I}_{0\text{ - }q,0\text{ - }m,0\text{ - }k} - \left\{ \alpha_k \omega_q \xi_1^{(k)} \right\} \mathrm{I}_{0\text{ - }m,1\text{ - }q,1\text{ - }k} \right] \tag{1.91}$$

and

$$D_k^{(m)} \equiv \frac{\sqrt{1+\sigma}}{\mathrm{J}_0^2(l_m)} \left[\left\{ -\alpha_k (\omega_k - \omega_q) \xi_1^{(m)} + \omega_q \zeta_1^{(k)} \right\} \mathrm{I}_{0\text{ - }q,0\text{ - }m,0\text{ - }k} + \left\{ \alpha_k \omega_q \xi_1^{(k)} \right\} \mathrm{I}_{0\text{ - }m,1\text{ - }q,1\text{ - }k} \right] \tag{1.92}$$

while

$$P^{(m)} \equiv \frac{\sqrt{1+\sigma}}{\mathrm{J}_0^2(l_m)} \left[\sum_{k=1}^{\infty} \left[\left\{ \left[2\alpha_k (1+\sigma) - \alpha_k^2\right] \xi_2^{(k)} + \sigma\zeta_2^{(k)} - 2\sigma\zeta_3^{(m)} \right\} \mathrm{I}_{0\text{ - }q,0\text{ - }m,0\text{ - }k} - \left\{ \alpha_k (2+3\sigma) \xi_2^{(k)} + \alpha_k^3 \sigma \zeta_2^{(k)} - 2\alpha_k^3 \sigma \zeta_3^{(k)} \right\} \mathrm{I}_{0\text{ - }m,1\text{ - }q,1\text{ - }k} \right] - \frac{(2+3\sigma)}{4} \mathrm{I}_{0\text{ - }q,0\text{ - }q,0\text{ - }q,0\text{ - }m} + \frac{(1-3\sigma)}{4} \mathrm{I}_{0\text{ - }q,0\text{ - }m,1\text{ - }q,1\text{ - }q} + \frac{3(1+2\sigma)}{4} \mathrm{I}_{0\text{ - }m,1\text{ - }q,1\text{ - }q,2\text{ - }q} \right] - \delta_{qm}\Omega_2 (2+\sigma)\sqrt{1+\sigma} \tag{1.93}$$

with δ_{qm} representing the Kronecker delta.

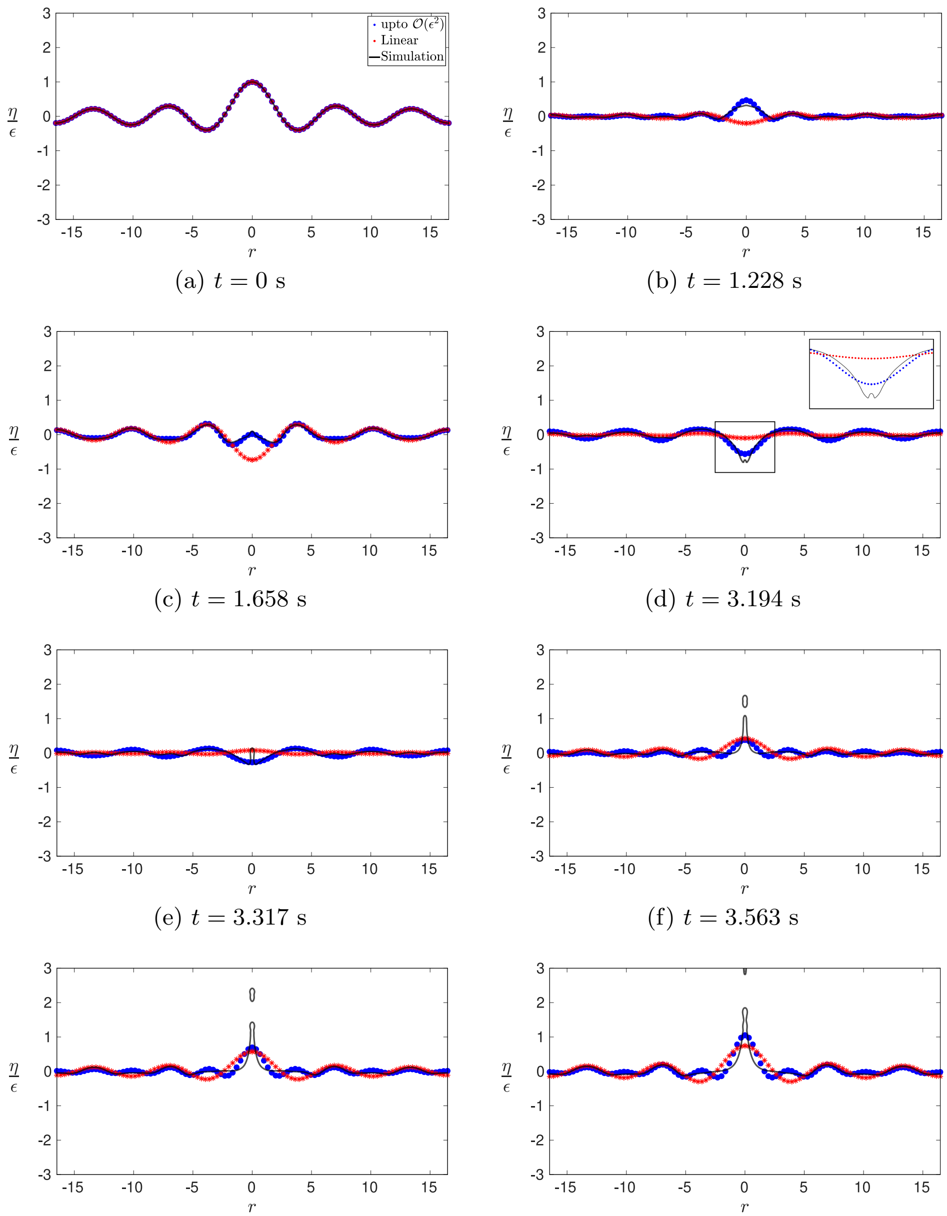
$$Q^{(m)} \equiv \frac{\sqrt{1+\sigma}}{\mathrm{J}_0^2(l_m)} \left[\sum_{k=1}^{\infty} \left[\left\{ \left[10\alpha_k\left(1+\sigma\right) - \alpha_k^2\right] \xi_2^{(k)} - \sigma\zeta_2^{(k)} \right\} \mathrm{I}_{0\text{ - }q,0\text{ - }m,0\text{ - }k} \right.\right.$$
$$\left. + \left\{ \alpha_k\left(6+5\sigma\right)\xi_2^{(k)} + \alpha_k^3\sigma\zeta_2^{(k)} \right\} \mathrm{I}_{0\text{ - }m,1\text{ - }q,1\text{ - }k} \right] - \frac{(10+11\sigma)}{4}\mathrm{I}_{0\text{ - }q,0\text{ - }q,0\text{ - }q,0\text{ - }m}$$
$$\left. -\frac{(11+15\sigma)}{4}\mathrm{I}_{0\text{ - }q,0\text{ - }m,1\text{ - }q,1\text{ - }q} - \frac{(1-2\sigma)}{4}\mathrm{I}_{0\text{ - }m,1\text{ - }q,1\text{ - }q,2\text{ - }q} \right] \tag{1.94}$$

$$R_k^{(m)} \equiv \frac{\sqrt{1+\sigma}}{\mathrm{J}_0^2(l_m)} \left[\left\{ \alpha_k\left[\omega_k^2 + 2\omega_q\omega_k + 2\omega_q^2 - \alpha_k\right] \xi_1^{(k)} - \sigma\zeta_1^{(k)} \right\} \mathrm{I}_{0\text{ - }q,0\text{ - }m,0\text{ - }k} \right.$$
$$\left. + \left\{ \alpha_k\left[2\omega_q\omega_k + (2+\sigma)\right]\xi_1^{(k)} + \alpha_k^3\sigma\zeta_1^{(k)} \right\} \mathrm{I}_{0\text{ - }m,1\text{ - }q,1\text{ - }k} \right] \tag{1.95}$$

and

$$S_k^{(m)} \equiv \frac{\sqrt{1+\sigma}}{\mathrm{J}_0^2(l_m)} \left[\left\{ \alpha_k\left[\omega_k^2 - 2\omega_q\omega_k + 2\omega_q^2 - \alpha_k\right] \xi_1^{(k)} + \sigma\zeta_1^{(k)} \right\} \mathrm{I}_{0\text{ - }q,0\text{ - }m,0\text{ - }k} \right.$$
$$\left. + \left\{ \alpha_k\left[-2\omega_q\omega_k + (2+\sigma)\right]\xi_1^{(k)} - \alpha_k^3\sigma\zeta_1^{(k)} \right\} \mathrm{I}_{0\text{ - }m,1\text{ - }q,1\text{ - }k} \right] \tag{1.96}$$

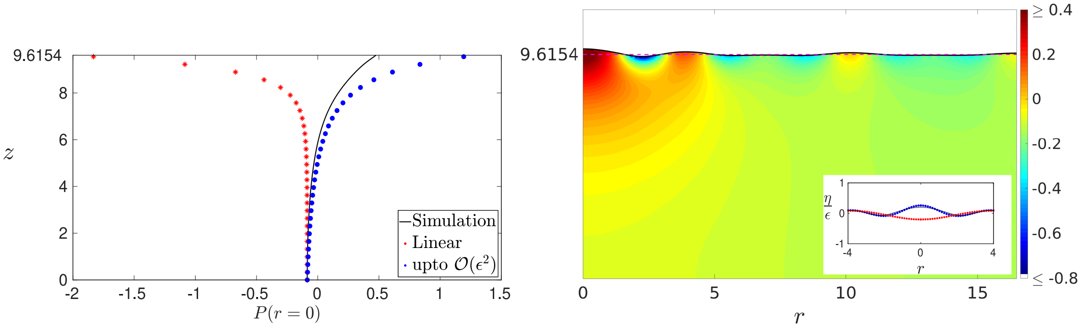
2. Comparison of interface shape at $\epsilon = 1.7$



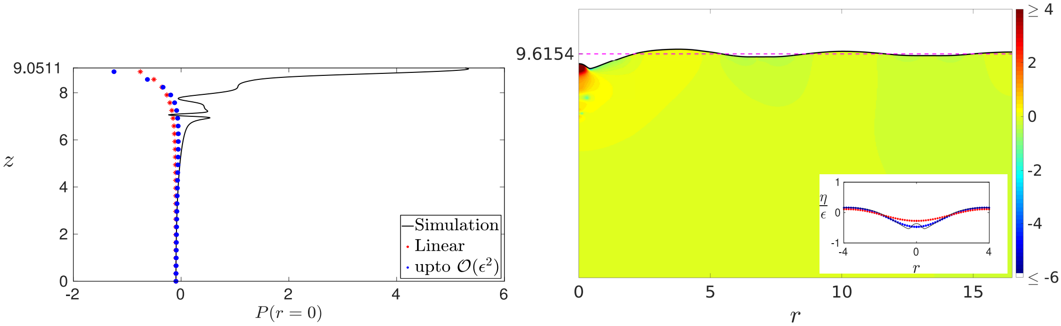
(a) $t = 0$ s
(b) $t = 1.228$ s
(c) $t = 1.658$ s
(d) $t = 3.194$ s
(e) $t = 3.317$ s
(f) $t = 3.563$ s
(g) $t = 3.685$ s
(h) $t = 3.845$ s

Figure 1: Relatively more slender jets emerge at higher values of $\epsilon = 1.7$ as seen in the panels here. Weakly nonlinear nonlinear theory describes the onset of the dimple in 1c, but cannot capture it when it becomes more pronounced in figure 1d. The thinning of the jet in panels 1g and 1h is not captured. The instant of maximum overshoot of the jet is not shown here and occurs beyond $t = 4$ in both theory and simulations. Case 8, grid:(b) in table 3 of the paper

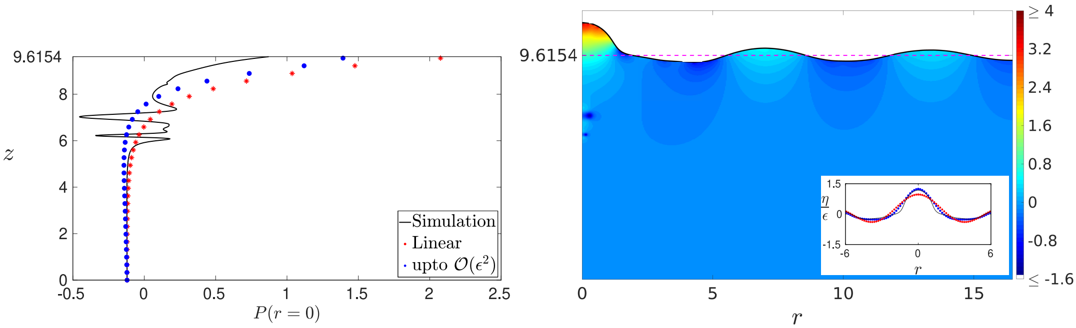
3. Comparison of pressure profiles at $\epsilon > 1$



(a) $t = 1.228$ s



(b) $t = 3.071$ s

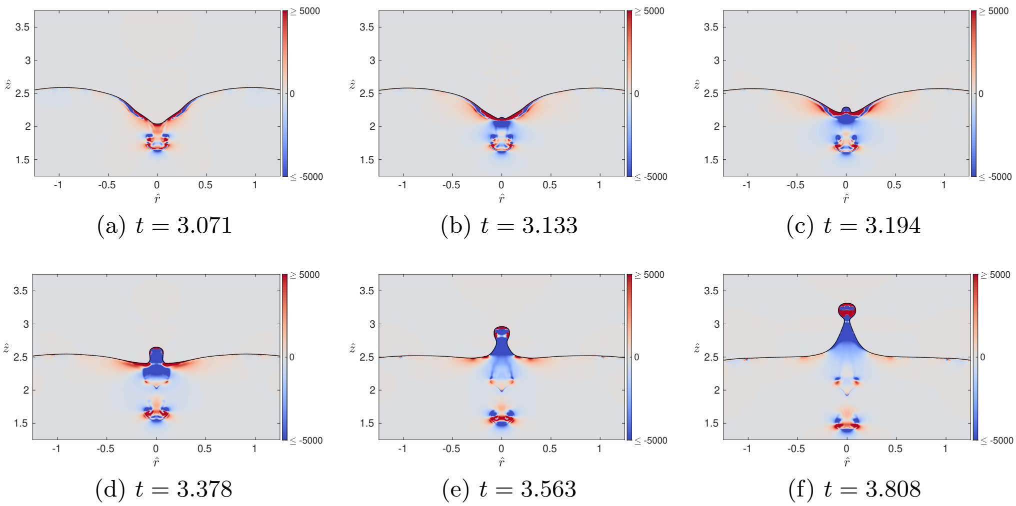


(c) $t = 4.177$ s

Figure 2: Left panels: (Nondimensional) pressure profiles $P(r, z, t)$ at the axis of symmetry $r = 0$ at various time instants for $\epsilon = 1.2$. Right panels: pressure contours from numerical simulations. The dashed line on the contour plots represents the depth of the unperturbed liquid at $t = 0$. The insets depict the instantaneous interface shape. As the weakly nonlinear theory has been derived using a Taylor series expansion about the unperturbed state for the liquid side only, we plot data at most upto the (scaled) unperturbed depth ($z = 9.6154$). For parameters, see case 5 in table 3, grid:(a)

4. Fluid particle acceleration: dimple and jet formation

In figures 3a-3f, we plot contours of the vertical component of the instantaneous fluid particle acceleration $\left(\dfrac{D\hat{u}_z}{D\hat{t}}\right)$ for $\epsilon = 1.7$. These are obtained numerically by evaluating the local acceleration and the convective acceleration between two snapshots using a first order, forward in time and second order, central difference in space approximation, respectively. It is seen in figures 3c and 3e that a region of very high acceleration develops at the base of the dimple. Note that the peak value of this acceleration is more than five times the acceleration due to gravity, consistent with the assertion in Zeff *et al.* (2000) that close to the singularity, surface tension is the dominant force and gravity effect is negligible. The contours of vertical acceleration inside the dimple in figure 3e, agree qualitatively with structure described in Gekle *et al.* (2009), where the acceleration inside the jet has been divided into three regions viz. an acceleration region, a ballistic region and the tip. It is clearly seen from later snapshots that while the tip strongly accelerates, the base of the jet develops strong deceleration triggering an ejection of a droplet eventually. Note the regions of alternate high and low accelerations in 3a, 3c and 3e, below the interface with qualitative resemblance to a quadrupole, but with a lot of additional structure underneath.



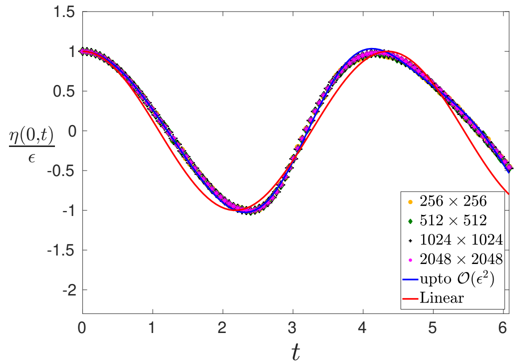
(a) $t = 3.071$ (b) $t = 3.133$ (c) $t = 3.194$

(d) $t = 3.378$ (e) $t = 3.563$ (f) $t = 3.808$

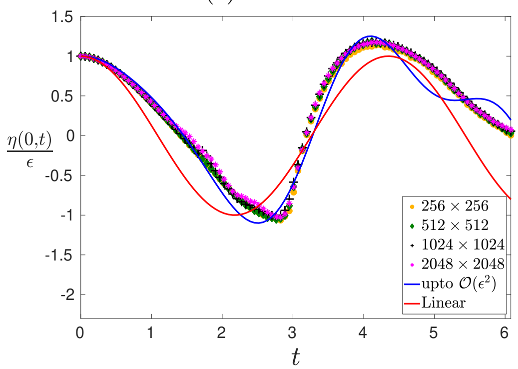
Figure 3: Contours of the fluid acceleration $\dfrac{D\hat{u}_z}{D\hat{t}}$ field for $\epsilon = 1.7$, case 8, grid:(a) in table 3 of the paper.

5. Grid convergence tests

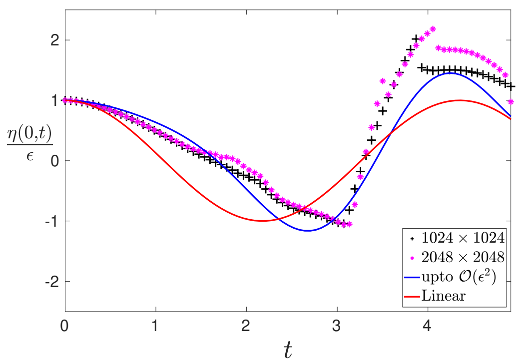
In this section, we provide the variation of the time signals in the paper with grid size as seen in figures 4a, 4b and 4c.



(a) $\epsilon = 0.5$



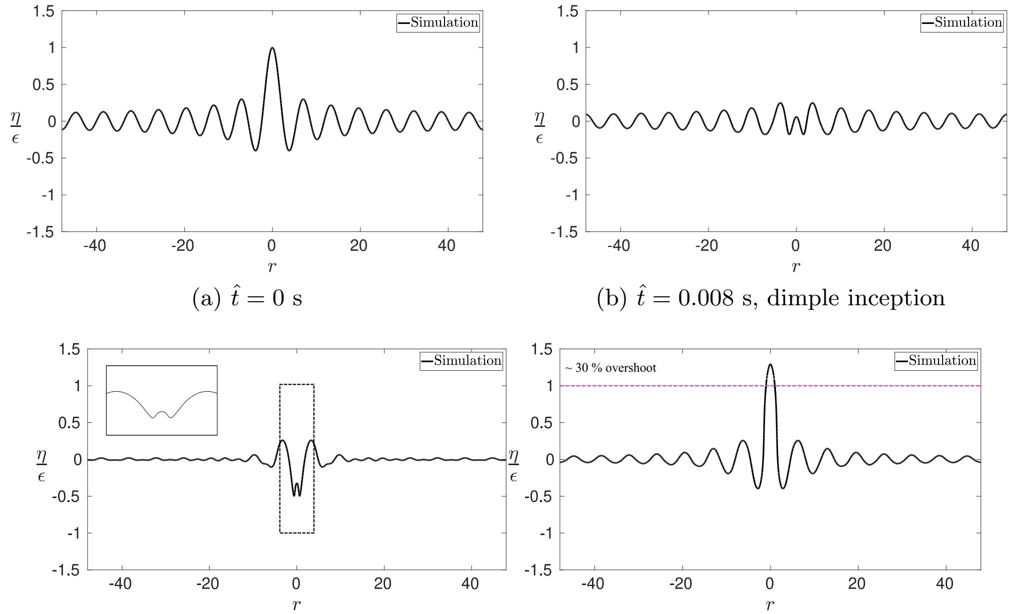
(b) $\epsilon = 1.2$



(c) $\epsilon = 1.7$

Figure 4: Grid convergence of time signals provided in manuscript. Note that $\epsilon = 1.7$ corresponds to a slender jet which produces a droplet. This reflects in the discontinuity in the signal at $t \approx 4$. The weakly nonlinear model captures the overshoot (within 10% accuracy) but is not expected to resolve droplet ejection. For $\epsilon = 1.2$ and 1.7, the data reported in the paper is for the highest grid resolution 2048×2048.

6. Jetting from pure capillary waves



(a) $\hat{t} = 0$ s (b) $\hat{t} = 0.008$ s, dimple inception

(c) $\hat{t} = 0.0155$ s, pronounced dimple (d) $\hat{t} = 0.024$ s, jetting and overshoot

Figure 5: Snapshots of the interface as function of time. Parameters: $\epsilon = 2$, $\sigma = \infty$. All features presented for capillary-gravity waves viz. dimple inception, pronounced dimple formation and a slender jet are also seen in these purely capillary wave simulations, see panel (b), (c) and (d). In dimensional CGS units, $\hat{R}_0 = 4.282$, $l_q = l_{15} = 47.901$, $\rho = 1$, $T = 72$, $g = 0$, Grid $= 512 \times 512$ (uniform). Note that the wavelength of the primary perturbation (≈ 5 mm) is chosen here to be below the capillary-length scale for air-water. This is consistent with the assumption of zero gravity.

In figure 5, we show that similar to the capillary-gravity case demonstrated in the manuscript, it is possible to obtain dimple and jet formation at large ϵ for pure capillary waves also. Each panel in fig. 5 is obtained from numerical simulations and shows a snapshot of the interface at the instant of time, provided in the caption below. One can see dimple inception (b), a pronounced dimple (c) and finally a jet in (d) which shows approximately 30% overshoot. This simulation is for $\epsilon = 2.0$ and $\sigma = \infty$.

7. Simulation Parameters

The set of parameters for Case 9 in table 3 in the manuscript are

$$a_{0i} \equiv \Big\{0.0026, 0.013, 0.026, 0.039, 0.052, 0.065, 0.078, 0.091, 0.104, 0.117, 0.13, 0.143, 0.156, 0.169, 0.182, 0.195, 0.208, 0.221, 0.234, 0.247, 0.26, 0.273, 0.286, 0.299, 0.312, 0.325, 0.338, 0.351, 0.364, 0.377, 0.39\Big\},$$

$$T_{0i} \equiv \Big\{6.6316, 13.2631, 19.8947, 26.5262, 33.1578, 39.7894, 46.4209, 53.0525, 59.6840, 66.3156, 72.9472, 79.5787, 86.2103, 92.8418, 99.4734, 106.1050, 112.7365, 119.3681, 125.9996, 132.6312\Big\},$$

$$\sigma_{0i} = \Big\{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2.0\Big\},$$

$$\epsilon_{0i} = \Big\{0.01, 0.05, 0.10, 0.15, 0.20, 0.25, 0.30, 0.35, 0.40, 0.45, 0.50, 0.55, 0.60, 0.65, 0.70, 0.75, 0.80, 0.85, 0.90, 0.95, 1.0, 1.05, 1.1, 1.15, 1.2, 1.25, 1.3, 1.35, 1.4, 1.45, 1.5\Big\}$$

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