Other supplementary materials

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1. Derivation of linearised equations using strongly conservative LES equations

From the strongly conservative LES equations in the mapped computational curvilinear coordinate, i.e. (4.1) and (4.2) of the main paper, the momentum and continuity equations for the wave-induced airflow can be derived, i.e. (4.7) and (4.8) of the main paper, respectively, in which the curvilinear coordinate velocity U_j and the products of physical quantities and grid transformation terms in the wave-induced pressure stress $\widetilde{\tau}^p_{jk}$ and the wave-induced viscous stress $\widetilde{\tau}^v_{jk}$ make it challenging to understand the underlying physics. In this section, the linearised equations for the wave-induced airflow are derived by using the Cartesian quantities to express the leading-order terms of U_j and $\widetilde{\tau}^p_{jk}$ in § 1.1, and of $\widetilde{\tau}^v_{jk}$ in §§ 1.2 and 1.3.

1.1. *Linearisation of curvilinear coordinate velocity and wave-induced pressure stress*

In this subsection, we apply the properties of the wave-induced quantities and use the Cartesian variables to express the curvilinear coordinate velocity U_j and the wave-induced pressure stress $\widetilde{\tau}^p_{jk}$ in the wave-induced momentum equation (4.7) in the main paper. In the wave following frame $\xi - ct$, the grid transformation terms, which appear in the equations of the wave-induced airflow, do not vary with time. We have

$$\eta = \overline{\eta} = \widetilde{\eta}, \tag{1.1}$$

$$\zeta_x = \overline{\zeta_x} = \frac{g\widetilde{\eta}_\xi}{1 - g_\zeta \widetilde{\eta}}, \qquad \zeta_y = \overline{\zeta_y} = \frac{g\widetilde{\eta}_\psi}{1 - g_\zeta \widetilde{\eta}}, \qquad \zeta_z = \overline{\zeta_z} = \frac{1}{1 - g_\zeta \widetilde{\eta}}, \tag{1.2}$$

where η is the wave surface, and ζ_x, ζ_y, and ζ_z are the geometrical transformation terms defined in (2.8) of the main paper. Here, $\overline{(\cdot)}$ and $\widetilde{(\cdot)}$ are the phase-averaged and wave-induced quantities, respectively, defined in § 2.2 of the main paper.

To simplify the products of the velocity components and grid transformation terms, we utilize the following properties of the phase average (Finnigan 1988),

$$\overline{f_1 \widetilde{f_2}} = \overline{f_1}\widetilde{f_2}, \qquad \overline{\langle f_1 \rangle f_2} = \langle f_1 \rangle \overline{f_2}, \tag{1.3}$$

where f_1 and f_2 are two arbitrary physical quantities, and $\langle \cdot \rangle$ denotes average in time and over the $(\xi,\ \psi)$ plane. It is also worth noting that the correlation between a wave-induced quantity and the surface elevation or slope does not have direct contributions to a wave-induced quantity, because

$$\widetilde{f}\widetilde{\eta} = (\widehat{f}\mathrm{e}^{\mathrm{i}k\xi} + \widehat{f}^*\mathrm{e}^{-\mathrm{i}k\xi})(\widehat{\eta}\mathrm{e}^{\mathrm{i}k\xi} + \widehat{\eta}^*\mathrm{e}^{-\mathrm{i}k\xi}) = 2\,\mathrm{Re}[\widehat{f}\widehat{\eta}^*] + 2\,\mathrm{Re}[\widehat{f}\widehat{\eta}\mathrm{e}^{\mathrm{i}2k\xi}], \tag{1.4}$$

$$\widetilde{f}\widetilde{\eta}_\xi = (\widehat{f}\mathrm{e}^{\mathrm{i}k\xi} + \widehat{f}^*\mathrm{e}^{-\mathrm{i}k\xi})(\mathrm{i}k\widehat{\eta}\mathrm{e}^{\mathrm{i}k\xi} - \mathrm{i}k\widehat{\eta}^*\mathrm{e}^{-\mathrm{i}k\xi}) = 2\,\mathrm{Re}[\mathrm{i}k\widehat{f}^*\widehat{\eta}] + 2\,\mathrm{Re}[\mathrm{i}k\widehat{f}\widehat{\eta}\mathrm{e}^{\mathrm{i}2k\xi}], \tag{1.5}$$

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where $\widetilde{f}$ is an arbitrary wave-induced quantity. As illustrated in (1.4) and (1.5), both $\widetilde{f\eta}$ and $\widetilde{f\eta_\xi}$ have two components, one with a zero wavenumber corresponding to the mean and the other corresponding to the harmonic with a wavenumber of $2k$, and thus they have no direct contributions to the wave-coherent fluctuation with a wavenumber k.

Using (1.1)–(1.3), the phase-averaged curvilinear coordinate velocity, which is defined in (4.1) and (4.2) of the main paper, can be represented as

$$\overline{U} = \overline{J^{-1}u_j\frac{\partial\xi}{\partial\xi_j}} = \overline{u-c} - \overline{(u-c)g_\zeta\widetilde{\eta}} = (\overline{u}-c)(1-g_\zeta\widetilde{\eta}), \tag{1.6}$$

$$\overline{W} = \overline{J^{-1}u_j\frac{\partial\zeta}{\partial\xi_j}} = \overline{(u-c)g\widetilde{\eta}_\xi} + \overline{w} = (\overline{u}-c)g\widetilde{\eta}_\xi + \overline{w}, \tag{1.7}$$

where $J=\zeta_z$ is the determinant of the transformation matrix $\boldsymbol{J}$, defined in (2.8) of the main paper. The last step in deriving $\overline{U}$ and $\overline{W}$ has utilized the fact that $g=\langle g\rangle$ and $g_\zeta=\langle g_\zeta\rangle$. We can obtain the mean curvilinear coordinate velocity by averaging (1.6) and (1.7) over the (ξ,ψ) plane

$$\langle U\rangle = \langle(\overline{u}-c)(1-g_\zeta\widetilde{\eta})\rangle = \langle u\rangle - c + 2\,\mathrm{Re}[g_\zeta\widehat{u}\widehat{\eta}^*], \tag{1.8}$$

$$\langle W\rangle = \langle(\overline{u}-c)g\widetilde{\eta}_\xi + \overline{w}\rangle = \langle w\rangle + 2\,\mathrm{Re}[g\mathrm{i}k\widehat{u}^*\widehat{\eta}], \tag{1.9}$$

where (1.4) and (1.5) are applied. Considering that $\widetilde{u}_i = O(ak)$, $\widetilde{\eta} = O(ak)$, and that the mean quantities are of $O(1)$, one can deduce that $2\,\mathrm{Re}[g_\zeta\widehat{u}\widehat{\eta}^*]$ and $2\,\mathrm{Re}[g\mathrm{i}k\widehat{u}^*\widehat{\eta}]$ are of $O((ak)^2)$, which are high-order corrections to the mean velocity. The wave-correlated curvilinear coordinate velocity is obtained by subtracting its mean value from its phase-averaged value,

$$\widetilde{U} = \overline{U} - \langle U\rangle = \widetilde{u} - (\langle u\rangle - c)g_\zeta\widetilde{\eta}, \tag{1.10}$$

$$\widetilde{W} = \overline{W} - \langle W\rangle = \widetilde{w} + (\langle u\rangle - c)g\widetilde{\eta}_\xi, \tag{1.11}$$

where the terms related to $\widetilde{u\eta}$ and $\widetilde{u\eta_\xi}$ do not present because they do not contribute to the wave-induced fluctuation as illustrated in (1.4) and (1.5).

Using (1.1)–(1.5), we can express the wave-induced pressure stress $\widetilde{\tau}^p_{ij}$ by eliminating the grid transformation terms,

$$\widetilde{\tau}^p_{11} = \overline{J^{-1}p\xi_x} - \langle J^{-1}p\xi_x\rangle = \overline{p(1-g_\zeta\widetilde{\eta})} - \langle p(1-g_\zeta\widetilde{\eta})\rangle = \widetilde{p}, \tag{1.12}$$

$$\widetilde{\tau}^p_{13} = \overline{J^{-1}p\zeta_x} - \langle J^{-1}p\zeta_x\rangle = \overline{pg\widetilde{\eta}_\xi} - \langle pg\widetilde{\eta}_\xi\rangle = \langle p\rangle g\widetilde{\eta}_\xi = 0, \tag{1.13}$$

$$\widetilde{\tau}^p_{31} = \overline{J^{-1}p\xi_z} - \langle J^{-1}p\xi_z\rangle = 0, \tag{1.14}$$

$$\widetilde{\tau}^p_{33} = \overline{J^{-1}p\zeta_z} - \langle J^{-1}p\zeta_z\rangle = \overline{p} - \langle p\rangle = \widetilde{p}, \tag{1.15}$$

where $\langle p\rangle = 0$ and $\xi_z = 0$ are applied.

To simplify the wave-induced momentum equation (4.7) in the main paper, (1.8)–(1.9) and (1.10)–(1.11) are employed to replace the mean curvilinear coordinate velocity $\langle U_j\rangle$ and the wave-induced curvilinear coordinate velocity $\widetilde{U}_j$, respectively, and (1.12)–(1.15) are used to express the wave-induced pressure stress $\widetilde{\tau}^p_{jk}$. With (1.10) and (1.11), the wave-induced continuity equation (4.8) in the main paper can be represented as

$$\frac{\partial\widetilde{u}}{\partial\xi} + \frac{\partial\widetilde{w}}{\partial\zeta} + \frac{\mathrm{d}\langle u\rangle}{\mathrm{d}\zeta}g\widetilde{\eta}_\xi = 0 + O((ak)^2), \tag{1.16}$$

where $O((ak)^2)$ denotes the neglected high-order contributions to the mean curvilinear coordinate velocity.

1.2. *Linearisation of products of velocity derivatives and grid transformation terms*

In this subsection, we show how to extract the leading-order terms in the phase-averaged products of the velocity derivatives and the grid transformation terms by employing (1.1)–(1.5). The products frequently appear in the wave-induced viscous stress $\widetilde{\tau}^v_{jk}$ in the momentum equation of the wave-induced airflow, i.e. (4.7) of the main paper. The simplification of $\widetilde{\tau}^v_{jk}$ is further discussed in the subsequent section.

First, we simplify the phase-averaged products of the determinant of the transformation Jacobian and the velocity derivatives, i.e. $1/J \cdot \partial u_j/\partial x$ and $1/J \cdot \partial u_j/\partial z$,

$$\overline{\frac{1}{J}\frac{\partial u_j}{\partial x}} = \overline{\left(\frac{\partial u_j}{\partial \xi} + \frac{\partial u_j}{\partial \zeta}\zeta_x\right)/\zeta_z} = \frac{\partial \overline{u}_j}{\partial \xi} - \frac{\partial \overline{u}_j}{\partial \xi} g_\zeta \widetilde{\eta} + \frac{\partial \overline{u}_j}{\partial \zeta} g \widetilde{\eta}_\xi, \tag{1.17}$$

$$\overline{\frac{1}{J}\frac{\partial u_j}{\partial z}} = \overline{\frac{\partial u_j}{\partial \zeta}\zeta_z/\zeta_z} = \frac{\partial \overline{u}_j}{\partial \zeta}. \tag{1.18}$$

We then simplify the phase-averaged products of the determinant of the transformation Jacobian, the streamwise derivatives of the velocity, and the transformation terms, i.e. $1/J \cdot \partial u_j/\partial x \cdot \zeta_x$ and $1/J \cdot \partial u_j/\partial x \cdot \zeta_z$, as the following

$$\overline{\frac{1}{J}\frac{\partial u_j}{\partial x}\zeta_x} = \frac{\partial \overline{u}_j}{\partial \xi} g \widetilde{\eta}_\xi + \overline{\frac{\partial u_j}{\partial \zeta}\frac{g^2 \widetilde{\eta}_\xi^2}{1 - g_\zeta \widetilde{\eta}}} = \frac{\partial \overline{u}_j}{\partial \xi} g \widetilde{\eta}_\xi + O\left((ak)^2 \frac{\partial \overline{u}_j}{\partial \zeta}\right), \tag{1.19}$$

$$\overline{\frac{1}{J}\frac{\partial u_j}{\partial x}\zeta_z} = \frac{\partial \overline{u}_j}{\partial \xi} + \overline{\frac{\partial u_j}{\partial \zeta} g \widetilde{\eta}_\xi + \frac{\partial u_j}{\partial \zeta}\frac{g g_\zeta \widetilde{\eta}_\xi \widetilde{\eta}}{1 - g_\zeta \widetilde{\eta}}} = \frac{\partial \overline{u}_j}{\partial \xi} + \frac{\partial \overline{u}_j}{\partial \zeta} g \widetilde{\eta}_\xi + O\left((ak)^2 \frac{\partial \overline{u}_j}{\partial \zeta}\right), \tag{1.20}$$

where we replace all the grid transformation terms related to $\widetilde{\eta}_\xi \widetilde{\eta}$ and $\widetilde{\eta}_\xi^2$ with $O((ak)^2)$ by considering that η is of $O(ak)$ and g is of $O(1)$. Similarly, the phase-averaged products of the determinant of the transformation Jacobian, the vertical derivatives of the velocity, and the transformation terms, i.e. $1/J \cdot \partial u_j/\partial z \cdot \zeta_x$ and $1/J \cdot \partial u_j/\partial z \cdot \zeta_z$, can be simplified as

$$\overline{\frac{1}{J}\frac{\partial u_j}{\partial z}\zeta_x} = \overline{\frac{\partial u_j}{\partial \zeta}\frac{g \widetilde{\eta}_\xi}{1 - g_\zeta \widetilde{\eta}}} = \overline{\frac{\partial u_j}{\partial \zeta} g \widetilde{\eta}_\xi + \frac{\partial u_j}{\partial \zeta}\frac{g g_\zeta \widetilde{\eta}_\xi \widetilde{\eta}}{1 - g_\zeta \widetilde{\eta}}} = \frac{\partial \overline{u}_j}{\partial \zeta} g \widetilde{\eta}_\xi + O\left((ak)^2 \frac{\partial \overline{u}_j}{\partial \zeta}\right), \tag{1.21}$$

$$\begin{aligned}\overline{\frac{1}{J}\frac{\partial u_j}{\partial z}\zeta_z} &= \overline{\frac{\partial u_j}{\partial \zeta}\frac{1}{1 - g_\zeta \widetilde{\eta}}} = \overline{\frac{\partial u_j}{\partial \zeta} + \frac{\partial u_j}{\partial \zeta} g_\zeta \widetilde{\eta} + \frac{\partial u_j}{\partial \zeta}\frac{g_\zeta^2 \widetilde{\eta}^2}{1 - g_\zeta \widetilde{\eta}}} \\ &= \frac{\partial \overline{u}_j}{\partial \zeta} + \frac{\partial \overline{u}_j}{\partial \zeta} g_\zeta \widetilde{\eta} + O\left((ak)^2 \frac{\partial \overline{u}_j}{\partial \zeta}\right),\end{aligned} \tag{1.22}$$

where the grid transformation terms with $\widetilde{\eta}^2$, $\widetilde{\eta}_\xi \widetilde{\eta}$, and $\widetilde{\eta}_\xi^2$ are of $O((ak)^2)$, similar to (1.19) and (1.20). In (1.17)–(1.22), the equations are expanded up to $O((ak)^2)$ to capture the dominant physical processes. In the subsequent section, (1.17)–(1.22) are employed to simplify the wave-induced viscous stress $\widetilde{\tau}^v_{jk}$ in (4.7) of the main paper.

1.3. *Linearisation of wave-induced viscous stress*

In this subsection, we show how to obtain the leading-order terms of the wave-induced viscous stress $\widetilde{\tau}^v_{jk}$ in (4.7) of the main paper, by employing the relationships (1.17)–(1.22) derived in § 1.2. With (1.17), the phase-averaged τ^v_{11} is represented as

$$\overline{\tau}^v_{11} = \nu \overline{\frac{1}{J} 2 \frac{\partial u}{\partial x}} = 2\nu \frac{\partial \overline{u}}{\partial \xi} - 2\nu \frac{\partial \overline{u}}{\partial \xi} g_\zeta \widetilde{\eta} + 2\nu \frac{\partial \overline{u}}{\partial \zeta} g \widetilde{\eta}_\xi, \tag{1.23}$$

and the wave-induced τ^v_{11} is obtained as

$$\widetilde{\tau}^v_{11} = \overline{\tau}^v_{11} - \langle \tau^v_{11} \rangle = 2\nu \frac{\partial \widetilde{u}}{\partial \xi} - 2\nu \frac{\partial \langle u \rangle}{\partial \xi} \widetilde{\eta} + 2\nu \frac{\partial \langle u \rangle}{\partial \zeta} g \widetilde{\eta}_\xi = -2\nu \frac{\partial \widetilde{w}}{\partial \zeta}, \tag{1.24}$$

where $\partial \langle u \rangle / \partial \xi = 0$ and the continuity equation (1.16) are applied. With (1.19), (1.20), and (1.22), the phase-averaged τ^v_{13} is simplified as

$$\begin{aligned}\overline{\tau}^v_{13} = 2\nu \Big(\frac{\partial \overline{u}}{\partial \xi} g \widetilde{\eta}_\xi + O\Big(\nu (ak)^2 \frac{\partial \overline{u}}{\partial \zeta} \Big) \Big) + \nu \Big(\frac{\partial \overline{u}}{\partial \zeta} + \frac{\partial \overline{u}}{\partial \zeta} g_\zeta \widetilde{\eta} + O\Big((ak)^2 \nu \frac{\partial \overline{u}}{\partial \zeta} \Big) \Big) \\ + \nu \Big(\frac{\partial \overline{w}}{\partial \xi} + \frac{\partial \overline{w}}{\partial \zeta} g \widetilde{\eta}_\xi + O\Big((ak)^2 \nu \frac{\partial \overline{w}}{\partial \zeta} \Big) \Big) \\ = \nu \Big(\frac{\partial \overline{u}}{\partial \zeta} + \frac{\partial \overline{w}}{\partial \xi} \Big) + \nu \frac{\partial \overline{u}}{\partial \zeta} g_\zeta \widetilde{\eta} + \nu \frac{\partial \overline{w}}{\partial \zeta} g \widetilde{\eta}_\xi + 2\nu \frac{\partial \overline{u}}{\partial \xi} g \widetilde{\eta}_\xi + O\Big((ak)^2 \nu \Big(\frac{\partial \overline{u}}{\partial \zeta} + \frac{\partial \overline{w}}{\partial \zeta} \Big) \Big),\end{aligned} \tag{1.25}$$

and the wave-induced τ^v_{13} is expressed as

$$\widetilde{\tau}^v_{13} = \overline{\tau}^v_{13} - \langle \tau^v_{13} \rangle = \nu \Big(\frac{\partial \widetilde{u}}{\partial \zeta} + \frac{\partial \widetilde{w}}{\partial \xi} \Big) + \nu \frac{\mathrm{d} \langle u \rangle}{\mathrm{d} \zeta} g_\zeta \widetilde{\eta} + O\Big((ak)^2 \nu \Big(\frac{\partial \widetilde{u}}{\partial \zeta} + \frac{\partial \widetilde{w}}{\partial \zeta} \Big) \Big), \tag{1.26}$$

where $\partial \langle u \rangle / \partial \xi = 0$ and $\langle w \rangle = 0$ are applied. Using (1.17) and (1.18), the phase-averaged τ^v_{31} and wave-induced τ^v_{31} can be expressed as

$$\overline{\tau}^v_{31} = \nu \overline{\frac{1}{J} \frac{\partial u}{\partial z}} + \nu \overline{\frac{1}{J} \frac{\partial w}{\partial x}} = \nu \frac{\partial \overline{u}}{\partial \zeta} + \nu \Big(\frac{\partial \overline{w}}{\partial \xi} - \frac{\partial \overline{w}}{\partial \xi} g_\zeta \widetilde{\eta} + \frac{\partial \overline{w}}{\partial \zeta} g \widetilde{\eta}_\xi \Big), \tag{1.27}$$

$$\widetilde{\tau}^v_{31} = \overline{\tau}^v_{31} - \langle \tau^v_{31} \rangle = \nu \Big(\frac{\partial \widetilde{u}}{\partial \zeta} + \frac{\partial \widetilde{w}}{\partial \xi} \Big), \tag{1.28}$$

where $\langle w \rangle = 0$ and $\xi_z = 0$ are applied. With (1.19), (1.21), and (1.22), the phase-averaged τ^v_{33} is represented as

$$\begin{aligned}\overline{\tau}^v_{33} &= \nu \Big(\frac{\partial \overline{u}}{\partial \zeta} g \widetilde{\eta}_\xi + O\Big((ak)^2 \nu \frac{\partial \overline{u}}{\partial \zeta} \Big) \Big) + \nu \Big(\frac{\partial \overline{w}}{\partial \xi} g \widetilde{\eta}_\xi + O\Big((ak)^2 \nu \frac{\partial \overline{w}}{\partial \zeta} \Big) \Big) \\ &+ 2\nu \Big(\frac{\partial \overline{w}}{\partial \zeta} + \frac{\partial \overline{w}}{\partial \zeta} g_\zeta \widetilde{\eta} + O\Big((ak)^2 \nu \frac{\partial \overline{w}}{\partial \zeta} \Big) \Big) \\ &= \nu \frac{\partial \overline{w}}{\partial \zeta} - \nu \frac{\partial \overline{u}}{\partial \xi} + O\Big((ak)^2 \nu \Big(\frac{\partial \overline{w}}{\partial \zeta} + \frac{\partial \overline{u}}{\partial \zeta} \Big) \Big),\end{aligned} \tag{1.29}$$

where the continuity equation (1.16) is utilized in the last step and the wave-induced τ^v_{33} is obtained as

$$\widetilde{\tau}^v_{33} = -\nu \frac{\partial \widetilde{u}}{\partial \xi} + \nu \frac{\partial \widetilde{w}}{\partial \zeta} + O\Big((ak)^2 \nu \Big(\frac{\partial \widetilde{w}}{\partial \zeta} + \frac{\partial \widetilde{u}}{\partial \zeta} \Big) \Big). \tag{1.30}$$

Equations (1.24), (1.26), (1.28), and (1.30) are employed to represent the wave-induced viscous stress $\widetilde{\tau}^v_{jk}$ in (4.7) of the main paper.

To conclude § 1, we have linearised the mean curvilinear coordinate velocity $\langle U_j \rangle$, the wave-induced curvilinear coordinate velocity $\widetilde{U}_j$, and the wave-induced pressure stress $\widetilde{\tau}^p_{jk}$ in § 1.1, and the wave-induced viscous stress $\widetilde{\tau}^v_{jk}$ has been linearised in §§ 1.2 and 1.3. These results are used to simplify the equations for the wave-induced airflow, i.e. (4.7) and (4.8) of the main paper, and the resultant linearised equations for the wave-induced airflow are (4.9)–(4.11) of the main paper.

2. Derivation of linearised equations using weakly conservative LES equations

In this section, we derive the viscous linearised equations for the wave-induced airflow from the weakly conservative LES equations in the mapped computational curvilinear coordinates, i.e. (2.10) and (2.11) of the main paper. Because some derivation details are similar to those in § 1, we only show the key steps of the derivation in this section. The phase-averaged weakly conservative LES equations in the wave following frame read

$$(\overline{u}-c)\Big(\frac{\partial\overline{u}}{\partial\xi}+\overline{\zeta}_x\frac{\partial\overline{u}}{\partial\zeta}\Big)+\overline{w}\overline{\zeta}_z\frac{\partial\overline{u}}{\partial\zeta}=-\Big(\frac{\partial\overline{p}}{\partial\xi}+\overline{\zeta}_x\frac{\partial\overline{p}}{\partial\zeta}\Big) \\ -\Big(\frac{\partial\overline{\sigma}_{11}}{\partial\xi}+\overline{\zeta}_x\frac{\partial\overline{\sigma}_{11}}{\partial\zeta}\Big)-\overline{\zeta}_z\frac{\partial\overline{\sigma}_{13}}{\partial\zeta}+Tub., \tag{2.1}$$

$$(\overline{u}-c)\Big(\frac{\partial\overline{w}}{\partial\xi}+\overline{\zeta}_x\frac{\partial\overline{w}}{\partial\zeta}\Big)+\overline{w}\zeta_z\frac{\partial\overline{w}}{\partial\zeta}=-\overline{\zeta}_z\frac{\partial\overline{p}}{\partial\zeta} \\ -\Big(\frac{\partial\overline{\sigma}_{31}}{\partial\xi}+\overline{\zeta}_x\frac{\partial\overline{\sigma}_{31}}{\partial\zeta}\Big)-\overline{\zeta}_z\frac{\partial\overline{\sigma}_{33}}{\partial\zeta}+Tub., \tag{2.2}$$

$$\frac{\partial\overline{u}}{\partial\xi}+\overline{\zeta}_x\frac{\partial\overline{u}}{\partial\zeta}+\overline{\zeta}_z\frac{\partial\overline{w}}{\partial\zeta}=0, \tag{2.3}$$

where $\sigma_{jk}=-\nu(\partial u_j/\partial x_k+\partial u_k/\partial x_j)$ is the viscous stress and 'Tub' represents the neglected terms associated with the correlation between turbulent velocity fluctuations.

Dividing (2.1) by $\overline{\zeta}_z$ and applying the triple decomposition to all of the terms, we can extract the streamwise momentum equation for the wave-coherent velocity

$$(\langle u\rangle-c)\frac{\partial\widetilde{u}}{\partial\xi}+(\widetilde{w}+(\langle u\rangle-c)g\widetilde{\eta}_\xi)\frac{\mathrm{d}\langle u\rangle}{\mathrm{d}\zeta}+\frac{\partial\widetilde{p}}{\partial\xi}=-\frac{\partial\widetilde{\sigma}_{11}}{\partial\xi} \\ -g\widetilde{\eta}_\xi\frac{\mathrm{d}\langle\sigma_{11}\rangle}{\mathrm{d}\zeta}-\frac{\partial\widetilde{\sigma}_{13}}{\partial\zeta}+O((ak)^2)\ \ +n.l.f. \tag{2.4}$$

where $O((ak)^2)$ denotes the neglected second and higher -order terms, and the '$n.l.f.$' represents the neglected nonlinear forcing, i.e. the correlation between turbulent velocity fluctuations and the correlation between wave-induced velocity components. By linearising the viscous stresses $\overline{\sigma}_{11}$ and $\overline{\sigma}_{13}$ as

$$\widetilde{\sigma}_{11}=-2\nu(\frac{\partial\widetilde{u}}{\partial\xi}+g\widetilde{\eta}_\xi\frac{\mathrm{d}\langle u\rangle}{\mathrm{d}\zeta})+O((ak)^2)=2\nu\frac{\partial\widetilde{w}}{\partial\zeta}+O((ak)^2), \tag{2.5}$$

$$\langle\sigma_{11}\rangle=0+O((ak)^2), \tag{2.6}$$

$$\widetilde{\sigma}_{13}=-\nu(\frac{\partial\widetilde{u}}{\partial\zeta}+\frac{\partial\widetilde{w}}{\partial\xi}+g_\zeta\widetilde{\eta}\frac{\mathrm{d}\langle u\rangle}{\mathrm{d}\zeta})+O((ak)^2), \tag{2.7}$$

where $\overline{\zeta}_z=1+g_\zeta\widetilde{\eta}+O((ak)^2)$ has been applied, we can express (2.4) as

$$(\langle u\rangle-c)\frac{\partial\widetilde{u}}{\partial\xi}+(\widetilde{w}+(\langle u\rangle-c)g\widetilde{\eta}_\xi)\frac{\mathrm{d}\langle u\rangle}{\mathrm{d}\zeta}+\frac{\partial\widetilde{p}}{\partial\xi}=\nu\Big(\frac{\partial^2\widetilde{u}}{\partial\zeta^2}-\frac{\partial^2\widetilde{w}}{\partial\xi\partial\zeta}\Big) \\ +\frac{\mathrm{d}}{\mathrm{d}\zeta}\Big(g_\zeta\frac{\mathrm{d}\langle u\rangle}{\mathrm{d}\zeta}\Big)\widetilde{\eta}+O((ak)^2)+n.l.f. \tag{2.8}$$

Dividing (2.2) by $\overline{\zeta}_z$ and applying the triple decomposition to all of the quantities, we can extract the vertical momentum equation for the wave-coherent velocity

$$(\langle u\rangle-c)\frac{\partial\widetilde{w}}{\partial\xi}+\frac{\partial\widetilde{p}}{\partial\zeta}=-\frac{\partial\widetilde{\sigma}_{31}}{\partial\xi}-g\widetilde{\eta}_\xi\frac{\mathrm{d}\langle\sigma_{31}\rangle}{\mathrm{d}\zeta}-\frac{\partial\widetilde{\sigma}_{33}}{\partial\zeta}+O((ak)^2)+n.l.f. \tag{2.9}$$

By linearising the viscous stresses $\overline{\sigma}_{31}$ and $\overline{\sigma}_{33}$ as

$$\widetilde{\sigma}_{31} = -\nu\frac{\partial\widetilde{u}}{\partial\zeta} - \nu\frac{\partial\widetilde{w}}{\partial\xi} - \nu g_\zeta\widetilde{\eta}\frac{\mathrm{d}\langle u\rangle}{\mathrm{d}\zeta}) + O((ak)^2), \tag{2.10}$$

$$\langle\sigma_{31}\rangle = -\nu\frac{\mathrm{d}\langle u\rangle}{\mathrm{d}\zeta} + O((ak)^2), \tag{2.11}$$

$$\widetilde{\sigma}_{33} = -2\nu\frac{\partial\widetilde{w}}{\partial\zeta} + O((ak)^2), \tag{2.12}$$

we can express (2.9) as

$$(\langle u\rangle - c)\frac{\partial\widetilde{w}}{\partial\xi} + \frac{\partial\widetilde{p}}{\partial\zeta} = \nu\Big(\frac{\partial^2\widetilde{w}}{\partial\xi^2} + \frac{\partial^2\widetilde{w}}{\partial\zeta^2}\Big) + O((ak)^2) + n.l.f. \tag{2.13}$$

where $g_\zeta\langle u\rangle_\zeta + g\mathrm{d}\langle u\rangle_\zeta/\mathrm{d}\zeta = \mathrm{d}(g\langle u\rangle_\zeta)/d\zeta$ has been applied.

Similarly, dividing (2.3) by $\overline{\zeta}_z$ and applying the triple decomposition to all of the terms, we can extract the continuity equation for the wave-coherent velocity

$$\frac{\partial\widetilde{u}}{\partial\xi} + g\widetilde{\eta}_\xi\frac{\mathrm{d}\langle u\rangle}{\mathrm{d}\zeta} + \frac{\partial\widetilde{w}}{\partial\zeta} = 0 + O((ak)^2). \tag{2.14}$$

Equations (2.8), (2.13), and (2.14) are the same as the linearised equations derived from the strongly conservative LES equations, i.e. (4.9)–(4.11) of the main paper.

3. Equations for in-phase and out-of-phase wave-induced pressure

In this section, we obtain the leading-order contributions to the in-phase and out-of-phase wave-induced pressure $\widetilde{p}$ using the linearised vertical momentum equation for wave-induced airflow, i.e. (4.10) of the main paper. The integration of (4.10) in the main paper gives the expression for $\widetilde{p}$ at height ζ

$$\widehat{p}|_{\zeta=\lambda} - \widehat{p}|_\zeta = \underbrace{\int_\zeta^\lambda -(\langle u\rangle - c)\frac{\partial\widetilde{w}}{\partial\xi}\mathrm{d}\zeta}_{\mathcal{A}} + \underbrace{\int_\zeta^\lambda \nu(\frac{\partial^2\widetilde{w}}{\partial\xi^2} + \frac{\partial^2\widetilde{w}}{\partial\zeta^2})\mathrm{d}\zeta}_{\mathcal{V}} + \underbrace{\int_\zeta^\lambda (-\frac{\partial\widetilde{\tau}_{3j}}{\partial\xi_j} - \frac{\partial\widetilde{\tau}^w_{3j}}{\partial\xi_j})\mathrm{d}\zeta}_{\mathcal{N}} + O((ak)^2), \tag{3.1}$$

where $\mathcal{A}$, $\mathcal{V}$, and $\mathcal{N}$ represent the contributions from the advection by $\widetilde{w}$, viscous stress, and nonlinear forcing, respectively. Equation (3.1) indicates that $\widetilde{p}$ results from the integration of all of the stress terms in the wave boundary layer. Because the nonlinear forcing and viscous stress are important only in the vicinity of the wave surface and thus have a very thin integration thickness, $\mathcal{V}$ and $\mathcal{N}$ have the secondary effects compared with $\mathcal{A}$. Then we obtain the leading order contribution to the generation of $\widetilde{p}$ as

$$\widehat{p}|_\zeta \approx \int_\zeta^\lambda -(\langle u\rangle - c)\frac{\partial\widetilde{w}}{\partial\xi}\mathrm{d}\zeta, \tag{3.2}$$

where $\widetilde{p}|_{\zeta=\lambda} = 0$ has been applied. Taking the imaginary and real parts of (3.2) with respect to $\widetilde{p}$, we obtain the leading order contribution to the in-phase pressure $\mathrm{Im}[\widehat{p}]$,

$$\mathrm{Im}[\widehat{p}]\big|_\zeta \approx \int_\zeta^\lambda k(\langle u\rangle - c)\,\mathrm{Re}[\widehat{w}]\mathrm{d}\zeta, \tag{3.3}$$

and the out-of-phase pressure $\mathrm{Re}[\widehat{p}]$,

$$\mathrm{Re}[\widehat{p}]\big|_{\zeta} \approx \int_{\zeta}^{\lambda} -k(\langle u \rangle - c)\,\mathrm{Im}[\widehat{w}]\mathrm{d}\zeta. \tag{3.4}$$

In § 5.3 of the main paper, (3.3) and (3.4) are used to investigate the physical processes associated with $\mathrm{Im}[\widehat{p}]$ and $\mathrm{Re}[\widehat{p}]$, respectively.

REFERENCES

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