CONTENTS OF SUPPLEMENTAL MATERIALS

This document lists and describes the contents of the Supplemental Materials.

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DESCRIPTION OF CONTENTS

Each of the figures in the SM is discussed below in detail, with information provided on the exact content and the algorithms, parameter values, etc. used in their preparation.

I. CONVERGENCE STUDY IN Δs: We checked convergence in the time-step Δs of our Monte Carlo Lagrangian method, where this step-size appears in the Euler-Maruyama scheme (3.1) for the stochastic particle trajectories and also in the corresponding Euler scheme (3.4) for the deformation gradient tensor. All of the results presented in the main text over the time-interval $-150< \delta s^+<0$ used time-step $\Delta s=10^{-3}$ with number of particles $N=10^7$. We tested for convergence by comparing results for the time-step $\Delta s=10^{-3}$ with those for half smaller time-step $\Delta s=5\times10^{-4}$. In order to identify the effects of Δs alone on convergence, we compared the results for the two step sizes using the same number of particles $N=5\times10^6$ over shorter interval $-100< \delta s^+<0$. As the simplest test for convergence, we compared the components $i=x,y,z,\|,\perp$ of the expected values of the stochastic Cauchy invariant, corresponding to the results in Fig.2a of the main text for the ejection and in Fig.4a for the sweep. The set of plots are:

par-converge-eject.tif: Plot of expected value of $\tilde{\omega}_{s\|}(\boldsymbol{x},t)$ vs. δs^+ for two Δs, in the ejection

perp-converge-eject.tif: Plot of the norm of the expected value of $\tilde{\boldsymbol{\omega}}_{s\perp}(\boldsymbol{x},t)$ vs. δs^+ for two Δs, in the ejection

x-converge-eject.tif: Plot of expected value of $\tilde{\omega}_{sx}(\boldsymbol{x},t)$ vs. δs^+ for two Δs, in the ejection

y-converge-eject.tif: Plot of expected value of $\tilde{\omega}_{sy}(\boldsymbol{x},t)$ vs. δs^+ for two Δs, in the ejection

z-converge-eject.tif: Plot of expected value of $\tilde{\omega}_{sz}(\boldsymbol{x},t)$ vs. δs^+ for two Δs, in the ejection

par-converge-sweep.tif: Plot of expected value of $\tilde{\omega}_{s\|}(\boldsymbol{x},t)$ vs. δs^+ for two Δs, in the sweep

perp-converge-sweep.tif: Plot of the norm of the expected value of $\tilde{\boldsymbol{\omega}}_{s\perp}(\boldsymbol{x},t)$ vs. δs^+ for two Δs, in the sweep

x-converge-sweep.tif: Plot of expected value of $\tilde{\omega}_{sx}(\boldsymbol{x},t)$ vs. δs^+ for two Δs, in the sweep

y-converge-sweep.tif: Plot of expected value of $\tilde{\omega}_{sy}(\boldsymbol{x},t)$ vs. δs^+ for two Δs, in the sweep

z-converge-sweep.tif: Plot of expected value of $\tilde{\omega}_{sz}(\boldsymbol{x},t)$ vs. δs^+ for two Δs, in the sweep

II. NAÏVE CAUCHY INVARIANTS: The standard Cauchy invariants $\boldsymbol{\omega}_s(\boldsymbol{x},t)$ that are conserved for smooth solutions of the ideal incompressible Euler equations are not, of course, *a priori* expected to be conserved for viscous Navier-Stokes solutions. We have computed these naïve Cauchy invariants, with the stochastic noise set to zero in the equations for particle trajectories, as a simple control experiment to demonstrate that the stochasticity is necessary to obtain conservation for viscous flows. The naïve invariants were calculated using a 4th-order Runge-Kutta method to solve the ODE's for Lagrangian particle trajectories $\boldsymbol{A}^s_t(\boldsymbol{x})$. With $\Delta s<0$, the following well-known RK4 method yields $\boldsymbol{A}=\boldsymbol{A}^s_t(\boldsymbol{x})$ for $s=s_n=t+n(\Delta s)$, n=0,1,2,…:

$$\boldsymbol{k}_1=\boldsymbol{u}(\boldsymbol{A}_n,s_n),\quad \boldsymbol{k}_2=\boldsymbol{u}(\boldsymbol{A}_n+\Delta s\,\boldsymbol{k}_1/2,s_n+\Delta s/2),\quad \boldsymbol{k}_3=\boldsymbol{u}(\boldsymbol{A}_n+\Delta s\,\boldsymbol{k}_2/2,s_n+\Delta s/2),\quad \boldsymbol{k}_4=\boldsymbol{u}(\boldsymbol{A}_n+\Delta s\,\boldsymbol{k}_3,s_n+\Delta s)$$
$$\boldsymbol{A}_{n+1} = \boldsymbol{A}_n+(\boldsymbol{k}_1+2\boldsymbol{k}_2+2\boldsymbol{k}_3+\boldsymbol{k}_4)\,\Delta s/6$$

with velocity $\boldsymbol{u}$ retrieved as needed from the channel-flow database. To calculate the naïve Cauchy invariant we use the standard formula $\boldsymbol{\omega}_s(\boldsymbol{x},t)=\boldsymbol{G}^{\top -1}\,\boldsymbol{\omega}(\boldsymbol{A}^s_t(\boldsymbol{x}),s)$, which needs also the gradient matrix $\boldsymbol{G}=\nabla_{\boldsymbol{x}}\boldsymbol{A}^s_t(\boldsymbol{x})$. The latter can be approximated at the times $s=s_n=t+n(\Delta s)$, n=0,1,2,… by taking the gradient with respect to $\boldsymbol{x}$ of the Runge-Kutta solution $\boldsymbol{A}=\boldsymbol{A}^s_t(\boldsymbol{x})$. Applying the chain rule yields the corresponding integration formula:

$$\boldsymbol{G}_{n+1}=\boldsymbol{G}_n+(\boldsymbol{K}_1+2\boldsymbol{K}_2+2\boldsymbol{K}_3+\boldsymbol{K}_4)\,\Delta s/6,$$

with the matrix products

$$\boldsymbol{K}_1=\boldsymbol{G}_n\boldsymbol{J}_1,\quad \boldsymbol{K}_2=(\boldsymbol{G}_n+\boldsymbol{K}_1\Delta s/2)\boldsymbol{J}_2,\quad \boldsymbol{K}_3=(\boldsymbol{G}_n+\boldsymbol{K}_2\Delta s/2)\boldsymbol{J}_3,\quad \boldsymbol{K}_4=(\boldsymbol{G}_n+\boldsymbol{K}_3\Delta s)\boldsymbol{J}_4,$$

and with

$$\boldsymbol{J}_1=\boldsymbol{J}(\boldsymbol{A}_n,s_n),\quad \boldsymbol{J}_2=\boldsymbol{J}(\boldsymbol{A}_n+\Delta s\,\boldsymbol{k}_1/2,s_n+\Delta s/2),\quad \boldsymbol{J}_3=\boldsymbol{J}(\boldsymbol{A}_n+\Delta s\,\boldsymbol{k}_2/2,s_n+\Delta s/2),\quad \boldsymbol{J}_4=\boldsymbol{J}(\boldsymbol{A}_n+\Delta s\,\boldsymbol{k}_3,s_n+\Delta s),$$

where $\boldsymbol{J}=\nabla_{\boldsymbol{a}}\boldsymbol{u}(\boldsymbol{a},s)$ is retrieved as needed from the channel-flow database. We have verified that this RK4 algorithm yields conserved values of the Cauchy invariants when applied to nontrivial exact solutions of the incompressible Euler equations, e.g. ABC flows. We then applied this same algorithm to the channel-flow database for the space-time points $(\boldsymbol{x},t)$ in Table 2 of the main text, for the ejection and the sweep. Using this 4th-order algorithm both with $\Delta s=10^{-2}$ and with $\Delta s=10^{-3}$ over the range $-150<\delta s^+<0$, we obtained identical results to single-precision accuracy, verifying convergence. We then plotted over this range of δs^+ the three Cartesian components $i=x,y,z$ of the naive Cauchy invariant $\boldsymbol{\omega}_s(\boldsymbol{x},t)$, for both the ejection and the sweep. The set of figures are:

<u>naivecauchy-eject.tif</u>: log-linear plots of $|\omega_{si}(\boldsymbol{x},t)|$, $i=x,y,z$ vs. δs^+, for the ejection

<u>naivecauchy-sweep.tif</u>: log-linear plots of $|\omega_{si}(\boldsymbol{x},t)|$, $i=x,y,z$ vs. δs^+, for the sweep

III. TEST OF DIVERGENCE-FREE VELOCITY-GRADIENT: The velocity-gradient matrices for the channel-flow database returned by *getVelocityGradient* are not exactly traceless, as required by incompressibility of the flow. Although the original channel-flow simulation had spectral accuracy in streamwise and spanwise directions and 7^{th}-order accuracy in the wall-normal direction, only the velocity fields from the original simulation were stored in the database and not the velocity-gradients. The matrices returned by *getVelocityGradient* are calculated at simulation grid points by Lagrange-interpolation differentiation formulas from the archived velocity data and, between grid points, are further approximated by Lagrange interpolation. The deviation of the velocity-gradient matrices that are returned by *getVelocityGradient* from being traceless is thus a reasonable measure of their deviation from true velocity-gradients of a Navier-Stokes solution, both because of finite-resolution effects in the original simulation and because of the additional Lagrange interpolations performed in the database. We have therefore quantified this deviation from the divergence-free condition by calculating the dimensionless ratios

$$\langle |\nabla\cdot\boldsymbol{u}|\rangle/\langle\|\nabla\boldsymbol{u}\|\rangle,\quad \langle |\nabla\cdot\boldsymbol{u}|/\|\nabla\boldsymbol{u}\|\rangle,$$

where $\langle\cdot\rangle$ represents an average in streamwise position x over the full range from 0 to 8π and where $\|\nabla\boldsymbol{u}\|$ represents the Frobenius matrix norm. We have calculated these ratios at the final time in the database, for space-points at the spanwise midpoint z=3π/2 and at various y-positions in the wall-normal direction. The resulting figure is:

DivFreeTest.tif : Plot of the ratios $\langle |\nabla\cdot\boldsymbol{u}|\rangle/\langle\|\nabla\boldsymbol{u}\|\rangle$, $\langle |\nabla\cdot\boldsymbol{u}|/\|\nabla\boldsymbol{u}\|\rangle$ vs. y^+ distances from the upper wall of the channel