Supplementary material

First the integrals I_m defined by (B 1)-(B 4) are related to the auxiliary integrals M_q^v and N_q^v, introduced by (B 11)-(B 14). While the integrals I_3 and I_4 are given in Appendix B it is also found that, for $m = 1, 2$

$$
\begin{aligned}
I_v(n,\alpha) &= (\cosh\alpha)^2 N_2^v(n,\alpha) - 2\cosh\alpha N_1^v(n,\alpha) + N_0^v(n,\alpha), && \text{(C 1)}\\
I_{v+4}(n,\alpha) &= (\cosh\alpha)^2 M_2^{v-1}(n,\alpha) - 2\cosh\alpha M_1^{v-1}(n,\alpha) + M_0^{v-1}(n,\alpha), && \text{(C 2)}\\
I_{v+6}(n,\alpha) &= -(\cosh\alpha)^2 M_4^{v+1}(n,\alpha) + 2\cosh\alpha M_3^{v+1}(n,\alpha) \\
&\quad +(\sinh\alpha)^2 M_2^{v+1}(n,\alpha) - 2\cosh\alpha M_1^{v+1}(n,\alpha) + M_0^{v+1}(n,\alpha), && \text{(C 3)}\\
I_{v+8}(n,\alpha) &= (\cosh\alpha)^2\{a_n N_2^{v+1}(n-1,\alpha)\mathrm{e}^{\alpha} + b_n N_2^{v+1}(n+1,\alpha)\mathrm{e}^{-\alpha}\} \\
&\quad -2\cosh\alpha\{a_n N_1^{v+1}(n-1,\alpha)\mathrm{e}^{\alpha} + b_n N_1^{v+1}(n+1,\alpha)\mathrm{e}^{-\alpha}\} \\
&\quad +a_n N_0^{v+1}(n-1,\alpha)\mathrm{e}^{\alpha} + b_n N_0^{v+1}(n+1,\alpha)\mathrm{e}^{-\alpha} && \text{(C 4)}\\
a_n &= \frac{(n+1)(n+2)}{(2n+1)},\ b_n = -\frac{n(n-1)}{(2n+1)}. && \text{(C 5)}
\end{aligned}
$$

Each integral N_q^v occurring in (C 1) and (C 4) has been given in Appendix B except the integrals N_1^1 and N_2^1. Using one time or twice the second identity (B 16) easily gives

$$
N_1^1(n,\alpha) = \frac{n+1}{2n+1}N_0^1(n-1,\alpha)\mathrm{e}^{\alpha} + \frac{n}{2n+1}N_0^1(n+1,\alpha)\mathrm{e}^{-\alpha}\ \text{for } n \ge 1, \quad \text{(C 6)}
$$

$$
\begin{aligned}
N_2^1(1,\alpha) = 8\mathrm{e}^{1.5\alpha}\{&\frac{32(\cosh\alpha)^4 + 80(\cosh\alpha)^3 + 58\cosh^2\alpha + 5\cosh\alpha - 5}{15(\cosh\alpha+1)^{5/2}} \\
&-\frac{32(\cosh\alpha)^4 - 80(\cosh\alpha)^3 + 58(\cosh\alpha)^2 - 5\cosh\alpha - 5}{15(\cosh\alpha-1)^{5/2}}\}, \quad \text{(C 7)}
\end{aligned}
$$

$$
N_2^1(n,\alpha) = o_n N_0^1(n-2,\alpha)\mathrm{e}^{2\alpha} + p_n N_0^1(n,\alpha) + q_n N_0^1(n+2,\alpha)\mathrm{e}^{-2\alpha}\ \text{for } n \ge 2 \quad \text{(C 8)}
$$

with coefficient o_n, p_n and q_n already defined in Appendix B by (B 29).

The determination of each integral M_q^v uses differentiations with respect to α of the analytical result (B 15) for M_0^0 and the first relation (B 16) expressing $tP_n(t)$ in terms of $P_{n+1}(t)$ and $P_{n-1}(t)$. After elementary but lengthy algebra one then analytically gets each desired integral M_q^v. The obtained results are listed below.
(i) Results for the integrals M_q^0 when $q = 1, 2$ are

$$
M_1^0(0,\alpha) = \mathrm{e}^{\alpha/2}\{\frac{2\cosh\alpha+3}{(\cosh\alpha+1)^{3/2}} - \frac{2\cosh\alpha-3}{(\cosh\alpha-1)^{3/2}}\}, \quad \text{(C 9)}
$$

$$
M_1^0(n,\alpha) = \frac{n+1}{2n+1}M_0^0(n+1,\alpha)\mathrm{e}^{-\alpha} + \frac{n}{2n+1}M_0^0(n-1,\alpha)\mathrm{e}^{\alpha}\ \text{for } n \ge 1, \quad \text{(C 10)}
$$

$$
M_2^0(0,\alpha) = 2\mathrm{e}^{\alpha/2}\{\frac{8(\cosh\alpha)^2 + 12\cosh\alpha + 3}{3(\cosh\alpha+1)^{3/2}} - \frac{8(\cosh\alpha)^2 - 12\cosh\alpha + 3}{3(\cosh\alpha-1)^{3/2}}\}, \quad \text{(C 11)}
$$

$$
\begin{aligned}
M_2^0(1,\alpha) = 2\mathrm{e}^{3\alpha/2}\{&\frac{16(\cosh\alpha)^3 + 24(\cosh\alpha)^2 + 6\cosh\alpha - 1}{3(\cosh\alpha+1)^{3/2}} \\
&-\frac{16(\cosh\alpha)^3 - 24(\cosh\alpha)^2 + 6\cosh\alpha + 1}{3(\cosh\alpha-1)^{3/2}}\}, \quad \text{(C 12)}
\end{aligned}
$$

$$
M_2^0(n,\alpha) = c_n M_0^0(n+2,\alpha)\mathrm{e}^{-2\alpha} + d_n M_0^0(n,\alpha) + e_n M_0^0(n-2,\alpha)\mathrm{e}^{2\alpha}\ \text{for } n \ge 2, \quad \text{(C 13)}
$$

with the following coefficients

$$c_n = \frac{(n+1)(n+2)}{(2n+1)(2n+3)}, e_n = \frac{n(n-1)}{4n^2-1}, \; d_n = \frac{(n+1)^2(2n-1)+n^2(2n+3)}{(2n-1)(2n+1)(2n+3)}. \tag{C 14}$$

(ii) Results for the integrals M_q^1 when $q = 0, 1, 2$ are

$$M_0^1(n,\alpha) = \frac{2\sqrt{2}}{15(\sinh\alpha)^5}[(\cosh\alpha)^2(4n^2+4n+9) + \cosh\alpha\sinh\alpha(12n+6) - 4n^2 - 4n] \text{ for } n \geq 0, \tag{C 15}$$

$$M_1^1(0,\alpha) = 2\mathrm{e}^{\alpha/2}\{\frac{2\cosh\alpha+5}{15(\cosh\alpha+1)^{5/2}} - \frac{2\cosh\alpha-5}{15(\cosh\alpha-1)^{5/2}}\}, \tag{C 16}$$

$$M_1^1(n,\alpha) = \frac{n+1}{2n+1}M_0^1(n+1,\alpha)\mathrm{e}^{-\alpha} + \frac{n}{2n+1}M_0^1(n-1,\alpha)\mathrm{e}^{\alpha} \text{ for } n \geq 1, \tag{C 17}$$

$$M_2^1(0,\alpha) = -2\mathrm{e}^{\alpha/2}\{\frac{8(\cosh\alpha)^2+2\cosh\alpha+15}{15(\cosh\alpha+1)^{5/2}} - \frac{8(\cosh\alpha)^2-2\cosh\alpha+15}{15(\cosh\alpha-1)^{5/2}}\}, \tag{C 18}$$

$$M_2^1(1,\alpha) = -2\mathrm{e}^{3\alpha/2}\{\frac{16(\cosh\alpha)^3+40(\cosh\alpha)^2+30\cosh(\alpha)+5}{5(\cosh\alpha+1)^{5/2}} - \frac{16(\cosh\alpha)^3-40(\cosh\alpha)^2+30\cosh(\alpha)-5}{5(\cosh\alpha-1)^{5/2}}\}, \tag{C 19}$$

$$M_2^1(n,\alpha) = c_n M_0^2(n+2,\alpha)\mathrm{e}^{-2\alpha} + d_n M_0^2(n,\alpha) + e_n M_0^2(n-2,\alpha)\mathrm{e}^{2\alpha} \text{ for } n \geq 2. \tag{C 20}$$

(iii) Results for the integrals M_q^2 (taking $q = 0, 1, 2, 3, 4$) are given by the relations $M_3^2(0,\alpha) = M_2^2(1,\alpha)\mathrm{e}^{-\alpha}, M_4^2(0,\alpha) = M_3^2(1,\alpha)\mathrm{e}^{-\alpha}$ and the formulae

$$M_0^2(n,\alpha) = \frac{2\sqrt{2}}{105(\sinh\alpha)^7}[(\cosh\alpha)^3(48n^2+48n+60) + (\cosh\alpha)^2\sinh\alpha(8n^3+12n^2+94n+45) - \cosh\alpha(48n^2+48n-60) - \sinh\alpha(8n^3+12n^2-26n-15)\} \text{ for } n \geq 0, \tag{C 21}$$

$$M_1^2(0,\alpha) = \frac{2}{35}(\mathrm{e}^{\alpha/2})\{\frac{2\cosh\alpha+7}{(\cosh\alpha+1)^{7/2}} - \frac{2\cosh\alpha-7}{(\cosh\alpha-1)^{7/2}}\}, \tag{C 22}$$

$$M_1^2(n,\alpha) = \frac{n+1}{2n+1}M_0^2(n+1,\alpha)\mathrm{e}^{-\alpha} + \frac{n}{2n+1}M_0^2(n-1,\alpha)\mathrm{e}^{\alpha} \text{ for } n \geq 1, \tag{C 23}$$

$$M_2^2(0,\alpha) = -2\mathrm{e}^{\alpha/2}\{\frac{8(\cosh\alpha)^2+28\cosh\alpha+35}{105(\cosh\alpha+1)^{7/2}} - \frac{8(\cosh\alpha)^2-28\cosh\alpha+35}{105(\cosh\alpha-1)^{7/2}}\}, \tag{C 24}$$

$$M_2^2(1,\alpha) = 2\mathrm{e}^{3\alpha/2}\{\frac{16(\cosh\alpha)^3+56(\cosh\alpha)^2+70\cosh\alpha+35}{35(\cosh\alpha+1)^{7/2}} - \frac{16(\cosh\alpha)^3-56(\cosh\alpha)^2+70\cosh\alpha-35}{35(\cosh\alpha-1)^{7/2}}\}, \tag{C 25}$$

$$M_2^2(n,\alpha) = c_n M_0^2(n+2,\alpha)\mathrm{e}^{-2\alpha} + d_n M_0^2(n,\alpha) + e_n M_0^2(n-2,\alpha)\mathrm{e}^{2\alpha} \text{ for } n \geq 2, \tag{C 26}$$

$$M_3^2(1,\alpha) = 2\mathrm{e}^{3\alpha/2}\{\frac{128(\cosh\alpha)^4+448(\cosh\alpha)^3+560(\cosh\alpha)^2+280\cosh\alpha+35}{35(\cosh\alpha+1)^{7/2}} - \frac{128(\cosh\alpha)^4-448(\cosh\alpha)^3+560(\cosh\alpha)^2-280\cosh\alpha+35}{35(\cosh\alpha-1)^{7/2}}\}, \tag{C 27}$$

$$M_3^2(2,\alpha) = \frac{2}{35}(\mathrm{e}^{5\alpha/2})\{\frac{640(\cosh\alpha)^5 + 2240(\cosh\alpha)^4 + 2792(\cosh\alpha)^3}{(\cosh\alpha+1)^{7/2}}$$
$$-\frac{640(\cosh\alpha)^5 - 2240(\cosh\alpha)^4 + 2792(\cosh\alpha)^3}{(\cosh\alpha-1)^{7/2}}$$
$$+\frac{1372(\cosh\alpha)^2 + 140\cosh\alpha - 35}{(\cosh\alpha+1)^{7/2}} + \frac{1372(\cosh\alpha)^2 - 140\cosh\alpha - 35}{(\cosh\alpha-1)^{7/2}}\}, \quad \text{(C 28)}$$
$$M_3^2(n,\alpha) = f_n M_0^2(n+3,\alpha)\mathrm{e}^{-3\alpha} + g_n M_0^2(n+1,\alpha)\mathrm{e}^{-\alpha}$$
$$+h_n M_0^2(n-1,\alpha)\mathrm{e}^{\alpha} + i_n M_0^2(n-3,\alpha)\mathrm{e}^{3\alpha} \text{ for } n \geq 3, \quad \text{(C 29)}$$
$$f_n = \frac{c_n(n+3)}{(2n+5)}, g_n = \frac{c_n(n+2)}{(2n+5)} + \frac{d_n(n+1)}{(2n+1)}, \quad \text{(C 30)}$$
$$h_n = \frac{d_n n}{(2n+1)} + \frac{e_n(n-1)}{(2n-3)}, i_n = \frac{e_n(n-2)}{(2n-3)}. \quad \text{(C 31)}$$

It is recalled that the coefficients c_n, d_n and e_n appearing in (C 30)-(C 31) are defined in (C 14). Finally, the integral $M_4^2(n,\alpha)$ is computed from the relations

$$M_4^2(1,\alpha) = 2\mathrm{e}^{3\alpha/2}\{\frac{256(\cosh\alpha)^5 + 896(\cosh\alpha)^4 + 1120(\cosh\alpha)^3}{21(\cosh\alpha+1)^{7/2}}$$
$$+\frac{560(\cosh\alpha)^2 + 70\cosh\alpha - 7}{21(\cosh\alpha+1)^{7/2}} + \frac{560(\cosh\alpha)^2 - 70\cosh\alpha - 7}{21(\cosh\alpha-1)^{7/2}}$$
$$-\frac{256(\cosh\alpha)^5 - 896(\cosh\alpha)^4 + 1120(\cosh\alpha)^3}{21(\cosh\alpha-1)^{7/2}}\}, \quad \text{(C 32)}$$
$$M_4^2(2,\alpha) = \frac{2}{35}(\mathrm{e}^{5\alpha/2})\{\frac{1536(\cosh\alpha)^6 + 5376(\cosh\alpha)^5 + 6656(\cosh\alpha)^4 + 3136(\cosh\alpha)^3}{(\cosh\alpha+1)^{7/2}}$$
$$+\frac{140(\cosh\alpha)^2 - 182\cosh\alpha - 7}{(\cosh\alpha+1)^{7/2}} - \frac{140(\cosh\alpha)^2 + 182\cosh\alpha - 7}{(\cosh\alpha-1)^{7/2}}$$
$$-\frac{1536(\cosh\alpha)^6 - 5376(\cosh\alpha)^5 + 6656(\cosh\alpha)^4 - 3136(\cosh\alpha)^3}{(\cosh\alpha-1)^{7/2}}\}, \quad \text{(C 33)}$$
$$M_4^2(3,\alpha) = \frac{2}{7}(\mathrm{e}^{7\alpha/2})\{\frac{1024(\cosh\alpha)^7 + 3584(\cosh\alpha)^6 + 4352(\cosh\alpha)^5}{(\cosh\alpha+1)^{7/2}}$$
$$+\frac{1792(\cosh\alpha)^4 - 280(\cosh\alpha)^3 - 308(\cosh\alpha)^2 - 28\cosh\alpha + 1}{(\cosh\alpha+1)^{7/2}}$$
$$-\frac{1024(\cosh\alpha)^7 - 3584(\cosh\alpha)^6 + 4352(\cosh\alpha)^5}{(\cosh\alpha-1)^{7/2}}$$
$$+\frac{1792(\cosh\alpha)^4 + 280(\cosh\alpha)^3 - 308(\cosh\alpha)^2 + 28\cosh\alpha + 1}{(\cosh\alpha-1)^{7/2}}\}, \quad \text{(C 34)}$$
$$M_4^2(n,\alpha) = j_n M_0^2(n+4,\alpha)e^{-4\alpha} + k_n M_0^2(n+2,\alpha)e^{-2\alpha} + l_n M_0^2(n,\alpha)$$
$$+m_n M_0^2(n-2,\alpha)e^{2\alpha} + n_n M_0^2(n-4,\alpha)e^{4\alpha} \text{ for } n \geq 4, \quad \text{(C 35)}$$
$$j_n = \frac{f_n(n+4)}{(2n+7)}, k_n = \frac{f_n(n+3)}{(2n+7)} + \frac{g_n(n+2)}{(2n+3)}, l_n = \frac{g_n(n+1)}{(2n+3)} + \frac{nh_n}{(2n-1)}, \quad \text{(C 36)}$$
$$m_n = \frac{h_n(n-1)}{(2n-1)} + \frac{i_n(n-2)}{(2n-5)}, n_n = \frac{i_n(n-3)}{(2n-5)}. \quad \text{(C 37)}$$

The coefficients f_n, g_n, h_n and i_n involved in (C 36)-(C 37) have been defined by (C 30)-(C 31). In summary, the integrals M_q^2 for $q = 1, ..., 4$ are deduced from M_0^2.

(iv) Results for the integrals M_q^3 (taking $q = 0,1,2,3,4$) are given by $M_3^3(0,\alpha) = M_2^3(1,\alpha)\mathrm{e}^{-\alpha}$ and the following formulae

$$M_0^3(n,\alpha) = \frac{2\sqrt{2}}{945(\sinh\alpha)^9}\{(\cosh\alpha)^4(16n^4+32n^3+584n^2+568n+525)$$
$$+(\cosh\alpha)^3\sinh\alpha(160n^3+240n^2+920n+420)$$
$$-(\cosh\alpha)^2(32n^4+64n^3+448n^2+416n-1050)$$
$$-\cosh\alpha\sinh\alpha(16n^3+240n^2-760n-420)$$
$$+16n^4+32n^3-136n^2-152n+105\}\text{ for } n\geq 0, \tag{C 38}$$

$$M_1^3(0,\alpha) = \frac{2}{63}(\mathrm{e}^{\alpha/2})\{\frac{2\cosh\alpha+9}{(\cosh\alpha+1)^{9/2}} - \frac{2\cosh\alpha-9}{(\cosh\alpha-1)^{9/2}}\}, \tag{C 39}$$

$$M_1^3(n,\alpha) = \frac{n+1}{2n+1}M_0^3(n+1,\alpha)\mathrm{e}^{-\alpha} + \frac{n}{2n+1}M_0^3(n-1,\alpha)\mathrm{e}^{\alpha}\text{ for } n\geq 1, \tag{C 40}$$

$$M_2^3(0,\alpha) = -2\mathrm{e}^{\alpha/2}\{\frac{8(\cosh\alpha)^2+36\cosh\alpha+63}{315(\cosh\alpha+1)^{9/2}} - \frac{8(\cosh\alpha)^2-36\cosh\alpha+63}{315(\cosh\alpha-1)^{9/2}}\}, \tag{C 41}$$

$$M_2^3(1,\alpha) = 2\mathrm{e}^{3\alpha/2}\{\frac{16(\cosh\alpha)^3+72(\cosh\alpha)^2+126\cosh\alpha+105}{315(\cosh\alpha+1)^{9/2}}$$
$$-\frac{16(\cosh\alpha)^3-72(\cosh\alpha)^2+126\cosh\alpha-105}{315(\cosh\alpha-1)^{9/2}}\}, \tag{C 42}$$

$$M_2^3(n,\alpha) = c_nM_0^3(n+2,\alpha)\mathrm{e}^{-2\alpha} + d_nM_0^3(n,\alpha) + e_nM_0^3(n-2,\alpha)\mathrm{e}^{2\alpha}\text{ for } n\geq 2, \tag{C 43}$$

$$M_3^3(1,\alpha) = -2\mathrm{e}^{3\alpha/2}\{\frac{128(\cosh\alpha)^4+576(\cosh\alpha)^3+1008(\cosh\alpha)^2+840\cosh\alpha+315}{315(\cosh\alpha+1)^{9/2}}$$
$$-\frac{128(\cosh\alpha)^4-576(\cosh\alpha)^3+1008(\cosh\alpha)^2-840\cosh\alpha+315}{315(\cosh\alpha-1)^{9/2}}\}, \tag{C 44}$$

$$M_3^3(2,\alpha) = -\frac{2}{315}(\mathrm{e}^{5\alpha/2})\{\frac{1920(\cosh\alpha)^5+8640(\cosh\alpha)^4+15128(\cosh\alpha)^3}{(\cosh\alpha+1)^{9/2}}$$
$$+\frac{12636(\cosh\alpha)^2+4788\cosh\alpha+525}{(\cosh\alpha+1)^{9/2}} + \frac{12636(\cosh\alpha)^2-4788\cosh\alpha+525}{(\cosh\alpha-1)^{9/2}}$$
$$-\frac{1920(\cosh\alpha)^5-8640(\cosh\alpha)^4+15128(\cosh\alpha)^3}{(\cosh\alpha-1)^{9/2}}\}, \tag{C 45}$$

$$M_3^3(n,\alpha) = f_nM_0^3(n+3,\alpha)\mathrm{e}^{-3\alpha} + g_nM_0^3(n+1,\alpha)\mathrm{e}^{-\alpha}$$
$$+h_nM_0^3(n-1,\alpha)\mathrm{e}^{\alpha} + i_nM_0^3(n-3,\alpha)\mathrm{e}^{3\alpha}\text{ for } n\geq 3. \tag{C 46}$$

In addition, $M_4^3(0,\alpha) = M_3^3(1,\alpha)\mathrm{e}^{-\alpha}$ while the integral $M_4^3(n,\alpha)$ is obtained for $n\geq 1$ using the identities

$$M_4^3(n,\alpha) = j_nM_0^3(n+4,\alpha)\mathrm{e}^{-4\alpha} + k_nM_0^3(n+2,\alpha)\mathrm{e}^{-2\alpha} + l_nM_0^3(n,\alpha)$$
$$+m_nM_0^3(n-2,\alpha)\mathrm{e}^{2\alpha} + n_nM_0^3(n-4,\alpha)\mathrm{e}^{4\alpha}\text{ for } n\geq 4, \tag{C 47}$$

$$M_4^3(1,\alpha) = -2\mathrm{e}^{3\alpha/2}\{\frac{256(\cosh\alpha)^5+1152(\cosh\alpha)^4+2016(\cosh\alpha)^3}{63(\cosh\alpha+1)^{9/2}}$$
$$+\frac{1680(\cosh\alpha)^2+630\cosh\alpha+63}{63(\cosh\alpha+1)^{9/2}} + \frac{1680(\cosh\alpha)-630\cosh\alpha+63}{63(\cosh\alpha-1)^{9/2}}$$
$$-\frac{256(\cosh\alpha)^5-1152(\cosh\alpha)^4+2016(\cosh\alpha)^3}{63(\cosh\alpha-1)^{9/2}}\}, \tag{C 48}$$

and also

$$M_4^3(2,\alpha) = -2\mathrm{e}^{5\alpha/2}\{\frac{7680(\cosh\alpha)^6 + 34560(\cosh\alpha)^5 + 60416(\cosh\alpha)^4 + 50112(\cosh\alpha)^3}{155(\cosh\alpha+1)^{9/2}}$$
$$+\frac{18396(\cosh\alpha)^2 + 1470\cosh\alpha - 315}{155(\cosh\alpha+1)^{9/2}} - \frac{18396(\cosh\alpha)^2 - 1470\cosh\alpha - 315}{155(\cosh\alpha-1)^{9/2}}$$
$$-\frac{7680(\cosh\alpha)^6 - 34560(\cosh\alpha)^5 + 60416(\cosh\alpha)^4 - 50112(\cosh\alpha)^3}{155(\cosh\alpha-1)^{9/2}}\}, \qquad (\mathrm{C}\,49)$$

$$M_4^3(3,\alpha) = -\frac{2}{63}(\mathrm{e}^{7\alpha/2})\{\frac{7168(\cosh\alpha)^7 + 32256(\cosh\alpha)^6 + 56064(\cosh\alpha)^5}{(\cosh\alpha+1)^{9/2}}$$
$$+\frac{45312(\cosh\alpha)^4 + 14616(\cosh\alpha)^3 - 756(\cosh\alpha)^2 - 1092\cosh\alpha - 63}{(\cosh\alpha+1)^{9/2}}$$
$$-\frac{7168(\cosh\alpha)^7 - 32256(\cosh\alpha)^6 + 56064(\cosh\alpha)^5}{(\cosh\alpha-1)^{9/2}}$$
$$+\frac{45312(\cosh\alpha)^4 - 14616(\cosh\alpha)^3 - 756(\cosh\alpha)^2 + 1092\cosh\alpha - 63}{(\cosh\alpha-1)^{9/2}}\}. \qquad (\mathrm{C}\,50)$$