Non-Newtonian effects on the slip and mobility of a self-propelling active particle: Supplementary Material

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In this Supplementary Material we provide details of the derivation of slip velocity in a second-order fluid.

S.1 Evaluation of $O(De)$ solution in the inner region

The flow field at $\mathcal{O}(De)$ is governed by

$$\frac{\partial \hat{u}_1}{\partial \hat{x}} + \frac{\partial \hat{v}_1}{\partial \hat{y}} = 0, \tag{1a}$$

$$-\frac{\partial \hat{p}_1}{\partial \hat{x}} + \frac{\partial^2 \hat{u}_1}{\partial \hat{y}^2} + \frac{\partial \hat{S}_{xx\,0}}{\partial \hat{x}} + \frac{\partial \hat{S}_{xy\,0}}{\partial \hat{y}} = 0, \tag{1b}$$

$$-\frac{\partial \hat{p}_1}{\partial \hat{y}} + \frac{\partial \hat{S}_{yy\,0}}{\partial \hat{y}} = 0. \tag{1c}$$

Here the subscripts on S denote the components of polymeric stress tensor. The components of the polymeric stress tensor ($\mathbf{S}$) are:

$$\hat{S}_{xx} = \left(\frac{\partial \hat{u}}{\partial \hat{y}}\right)^2, \quad \hat{S}_{yy} = (1+2\delta)\left(\frac{\partial \hat{u}}{\partial \hat{y}}\right)^2, \tag{2a}$$

$$\hat{S}_{xy} = \hat{S}_{yx} = 2\delta\left(\frac{\partial \hat{u}}{\partial \hat{y}}\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\hat{v}\frac{\partial^2 \hat{u}}{\partial \hat{y}^2} + \hat{u}\frac{\partial^2 \hat{u}}{\partial \hat{y}\partial \hat{x}}}{2}\right). \tag{2b}$$

Using the pressure decay condition: $\hat{p} \to 0$ as $\hat{y} \to \infty$ [1, pg.579], the solution to (1c) yields $\hat{p}_1 = \hat{S}_{yy\,0} - \mathcal{J}_0(\hat{x})^2(1+2\delta)$, where $\mathcal{J}_0$ is later shown to be zero through matching. Substituting $\hat{p}_1$ in (1b), we obtain

$$\frac{\partial^2 \hat{u}_1}{\partial \hat{y}^2} + 2\mathcal{J}_0(\hat{x})\mathcal{J}_0'(\hat{x})(1+2\delta) = \frac{\partial \hat{S}_{yy0}}{\partial \hat{x}} - \left[\frac{\partial \hat{S}_{xx0}}{\partial \hat{x}} + \frac{\partial \hat{S}_{xy0}}{\partial \hat{y}}\right], \tag{3a}$$

$$= 2(1+2\delta)u_y u_{yx} - \left[2u_y u_{yx} + 2\delta\left(u_{yy}u_x + u_y u_{yx} + \frac{v_y u_{yy} + v u_{yyy} + u_y u_{yx} + u u_{yyx}}{2}\right)\right]. \tag{3b}$$

Here we have substituted (2a) in (3a) to obtain (3b). The subscripts in above equation (and the next) denote the derivative of u and v with respect to x or y. We also have temporarily dropped the hat symbol and subscript 0 in the right hand side. The coloured terms (blue) get canceled out and using the continuity equation ($v_y = -u_x$), we simplify the equation as:

$$\frac{\partial^2 \hat{u}_1}{\partial \hat{y}^2} + 2\mathcal{J}_0(\hat{x})\mathcal{J}_0'(\hat{x})(1+2\delta) = 2\delta\left\{2u_y u_{yx} - \left[u_{yy}u_x + u_y u_{yx} + \frac{-u_x u_{yy} + v u_{yyy} + u_y u_{yx} + u u_{yyx}}{2}\right]\right\}, \tag{4a}$$

$$= 2\delta\left\{2u_y u_{yx} - \left[\frac{u_{yy}u_x}{2} + \frac{3\,u_y u_{yx}}{2} + \frac{v u_{yyy} + u u_{yyx}}{2}\right]\right\}, \tag{4b}$$

$$= 2\delta\left\{\frac{u_y u_{yx}}{2} - \left[\frac{u_{yy}u_x}{2} + \frac{v u_{yyy} + u u_{yyx}}{2}\right]\right\}, \tag{4c}$$

$$= -\delta\left\{-u_y u_{yx} + u_{yy}u_x + v u_{yyy} + u u_{yyx}\right\}. \tag{4d}$$

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Dropping the subscript notation (for derivative), we obtain the differential equation for $\hat{u}_1$ as:

$$\frac{\partial^2 \hat{u}_1}{\partial \hat{y}^2} = -\delta \left\{ -\frac{\partial \hat{u}_0}{\partial \hat{y}}\frac{\partial^2 \hat{u}_0}{\partial \hat{y}\partial \hat{x}} + \frac{\partial \hat{u}_0}{\partial \hat{x}}\frac{\partial^2 \hat{u}_0}{\partial \hat{y}^2} + \hat{v}_0 \frac{\partial^3 \hat{u}_0}{\partial \hat{y}^3} + \hat{u}_0 \frac{\partial^3 \hat{u}_0}{\partial \hat{x}\partial \hat{y}^2} \right\} - 2\mathcal{J}_0(\hat{x})\mathcal{J}_0'(\hat{x})(1+2\delta). \tag{5}$$

We will now substitute the $O(De^0)$ solution in the above equation. $\hat{u}_0$ and its derivatives, and $\hat{v}_0$ (obtained from the continuity equation) are:

$$\hat{u}_0 = -\mathcal{I}'(\hat{x}) \int_0^{\hat{y}} \int_t^{\infty} \mathcal{F}(s) ds dt \; + \mathcal{J}_0(x)\hat{y}, \tag{6a}$$

$$\frac{\partial \hat{u}_0}{\partial \hat{y}} = -\mathcal{I}'(\hat{x}) \int_{\hat{y}}^{\infty} \mathcal{F}(s) ds \; + \mathcal{J}_0(x), \tag{6b}$$

$$\frac{\partial^2 \hat{u}_0}{\partial \hat{y}^2} = \mathcal{I}'(\hat{x})\mathcal{F}(\hat{y}), \tag{6c}$$

$$\hat{v}_0 = -\int_0^{\hat{y}} \frac{\partial \hat{u}_0}{\partial \hat{x}} dy + \mathcal{C}(\hat{x}) = +\mathcal{I}''(\hat{x}) \int_0^{\hat{y}} \int_0^{r} \int_t^{\infty} \mathcal{F}(s)\, ds dt dr - \frac{\mathcal{J}_0'(\hat{x})\hat{y}^2}{2} + \mathcal{C}(\hat{x}). \tag{6d}$$

Using the no-penetration condition at the surface, we find that $\mathcal{C}(\hat{x}) = 0$.

Substituting (6) in the bracket terms of (5), we obtain the four terms $\{A+B+C+D\}$ as:

$$A = -\left[\left(-\mathcal{I}'(\hat{x}) \int_{\hat{y}}^{\infty} \mathcal{F}(s) ds \; + \mathcal{J}_0(x) \right) \left(-\mathcal{I}''(\hat{x}) \int_{\hat{y}}^{\infty} \mathcal{F}(s) ds \; + \mathcal{J}_0'(\hat{x}) \right) \right], \tag{7a}$$

$$B = \left(-\mathcal{I}''(\hat{x}) \int_0^{\hat{y}} \int_t^{\infty} \mathcal{F}(s) ds dt \; + \mathcal{J}_0'(x)\hat{y} \right) \left(\mathcal{I}'(\hat{x})\mathcal{F}(\hat{y}) \right), \tag{7b}$$

$$C = \left(\mathcal{I}''(\hat{x}) \int_0^{\hat{y}} \int_0^{r} \int_t^{\infty} \mathcal{F}(s)\, ds dt dr - \frac{\mathcal{J}_0'(\hat{x})\hat{y}^2}{2} \right) \left(\mathcal{I}'(\hat{x})\mathcal{F}'(\hat{y}) \right), \tag{7c}$$

$$D = \left(-\mathcal{I}'(\hat{x}) \int_0^{\hat{y}} \int_t^{\infty} \mathcal{F}(s) ds dt \; + \mathcal{J}_0(\hat{x})\hat{y} \right) \left(\mathcal{I}''(\hat{x})\mathcal{F}(\hat{y}) \right). \tag{7d}$$

These terms can be further simplified as

$$A = -\left[+\mathcal{I}'(\hat{x})\mathcal{I}''(\hat{x}) \left(\int_{\hat{y}}^{\infty} \mathcal{F}(s) ds \right)^2 - \mathcal{J}_0(\hat{x})\mathcal{I}''(\hat{x}) \left(\int_{\hat{y}}^{\infty} \mathcal{F}(s) ds \right) - \mathcal{I}'(\hat{x})\mathcal{J}_0'(\hat{x}) \left(\int_{\hat{y}}^{\infty} \mathcal{F}(s) ds \right) + \mathcal{J}_0(\hat{x})\mathcal{J}_0'(\hat{x}) \right], \tag{8a}$$

$$B = -\mathcal{I}'(\hat{x})\mathcal{I}''(\hat{x})\mathcal{F}(\hat{y}) \left(\int_0^{\hat{y}} \int_t^{\infty} \mathcal{F}(s) ds dt \right) + \mathcal{I}'(\hat{x})\mathcal{J}_0'(\hat{x})\hat{y}\mathcal{F}(\hat{y}), \tag{8b}$$

$$C = \mathcal{I}'(\hat{x})\mathcal{I}''(\hat{x})\mathcal{F}'(\hat{y}) \left(\int_0^{\hat{y}} \int_0^{r} \int_t^{\infty} \mathcal{F}(s) ds dt dr \right) - \mathcal{J}_0'(\hat{x})\mathcal{I}'(\hat{x})\mathcal{F}'(\hat{y})\frac{\hat{y}^2}{2}, \tag{8c}$$

$$D = -\mathcal{I}'(\hat{x})\mathcal{I}''(\hat{x})\mathcal{F}(\hat{y}) \left(\int_0^{\hat{y}} \int_t^{\infty} \mathcal{F}(s) ds dt \right) + \mathcal{J}_0(\hat{x})\mathcal{I}''(\hat{x})\hat{y}\mathcal{F}(\hat{y}). \tag{8d}$$

Substituting the above four terms in (5) and simplifying we get:

$$\begin{aligned} \frac{\partial^2 \hat{u}_1}{\partial y^2} = &-\delta \mathcal{I}'(\hat{x})\mathcal{I}''(\hat{x}) \left\{ -\left(\int_{\hat{y}}^{\infty} \mathcal{F}(s)\,\mathrm{d}s \right)^2 - 2\,\mathcal{F}(\hat{y}) \left(\int_0^{\hat{y}} \int_t^{\infty} \mathcal{F}(s) ds dt \right) + \mathcal{F}'(\hat{y}) \left(\int_0^{r} \int_0^{\omega} \int_t^{\infty} \mathcal{F}(s) ds dt d\omega \right) \right. \\ &+ \left(\frac{\mathcal{J}_0(\hat{x})}{\mathcal{I}'(\hat{x})} + \frac{\mathcal{J}_0'(\hat{x})}{\mathcal{I}''(\hat{x})} \right) \left(\hat{y}\mathcal{F}(\hat{y}) + \int_{\hat{y}}^{\infty} \mathcal{F}(s) ds \right) - \frac{\mathcal{J}_0(\hat{x})}{\mathcal{I}''(\hat{x})}\mathcal{F}'(\hat{y})\frac{\hat{y}^2}{2} - \frac{\mathcal{J}_0(\hat{x})\mathcal{J}_{(0)}'(\hat{x})}{\mathcal{I}'(\hat{x})\mathcal{I}''(\hat{x})} \\ &\left. + \frac{2\mathcal{J}_0(\hat{x})\mathcal{J}_0'(\hat{x})(1+2\delta)}{\delta \mathcal{I}'(\hat{x})\mathcal{I}''(\hat{x})} \right\}. \end{aligned} \tag{9}$$

Integrating the above equation once, we obtain

$$\frac{\partial \hat{u}_1}{\partial y} = C_1 - \delta \mathcal{I}'(\hat{x})\mathcal{I}''(\hat{x}) \int_0^{\hat{y}} \Bigg\{ -\left(\int_r^\infty \mathcal{F}(s)\,\mathrm{d}s\right)^2 - 2\,\mathcal{F}(r)\left(\int_0^r \int_t^\infty \mathcal{F}(s) ds dt\right) + \mathcal{F}'(r)\left(\int_0^r \int_0^\omega \int_t^\infty \mathcal{F}(s) ds dt d\omega\right) + \left(\frac{\mathcal{J}_0(\hat{x})}{\mathcal{I}'(\hat{x})} + \frac{\mathcal{J}_0'(\hat{x})}{\mathcal{I}''(\hat{x})}\right)\left(r\mathcal{F}(r) + \int_r^\infty \mathcal{F}(s) ds\right) - \frac{\mathcal{J}_0(\hat{x})}{\mathcal{I}''(\hat{x})}\mathcal{F}'(r)\frac{r^2}{2} - \frac{\mathcal{J}_0(\hat{x})\mathcal{J}_{(0)}'(\hat{x})}{\mathcal{I}'(\hat{x})\mathcal{I}''(\hat{x})} + \frac{2\mathcal{J}_0(\hat{x})\mathcal{J}_0'(\hat{x})(1+2\delta)}{\delta \mathcal{I}'(\hat{x})\mathcal{I}''(\hat{x})} \Bigg\} dr. \tag{10}$$

As $\hat{y} \to \infty$, $\frac{\partial \hat{u}_1}{\partial y} = \mathcal{J}_1(\hat{x})$ which is to be determined through matching. Simplifying the above equation (replacing $C_1 = \mathcal{J}_1 + \delta \mathcal{I}'\mathcal{I}'' \int_0^\infty \cdots dr$), we get:

$$\frac{\partial \hat{u}_1}{\partial y} = \mathcal{J}_1(\hat{x}) + \delta \mathcal{I}'(\hat{x})\mathcal{I}''(\hat{x}) \int_{\hat{y}}^{\infty} \Bigg\{ -\left(\int_r^\infty \mathcal{F}(s)\mathrm{d}s\right)^2 - 2\mathcal{F}(r)\left(\int_0^r \int_t^\infty \mathcal{F}(s) ds dt\right) + \mathcal{F}'(r)\left(\int_0^r \int_0^\omega \int_t^\infty \mathcal{F}(s) ds dt d\omega\right) + \left(\frac{\mathcal{J}_0(\hat{x})}{\mathcal{I}'(\hat{x})} + \frac{\mathcal{J}_0'(\hat{x})}{\mathcal{I}''(\hat{x})}\right)\left(r\mathcal{F}(r) + \int_r^\infty \mathcal{F}(s) ds\right) - \frac{\mathcal{J}_0(\hat{x})}{\mathcal{I}''(\hat{x})}\mathcal{F}'(r)\frac{r^2}{2} - \frac{\mathcal{J}_0(\hat{x})\mathcal{J}_{(0)}'(\hat{x})}{\mathcal{I}'(\hat{x})\mathcal{I}''(\hat{x})} + \frac{2\mathcal{J}_0(\hat{x})\mathcal{J}_0'(\hat{x})(1+2\delta)}{\delta \mathcal{I}'(\hat{x})\mathcal{I}''(\hat{x})} \Bigg\} dr. \tag{11}$$

Integrating once more and using the no-slip condition (renders the constant C_2 zero), we get:

$$\hat{u}_1 = \mathcal{J}_1(\hat{x})\hat{y} - \delta \mathcal{I}'(\hat{x})\mathcal{I}''(\hat{x}) \int_0^{\hat{y}} dp \int_p^\infty \Bigg\{ \left(\int_r^\infty \mathcal{F}(s)\,\mathrm{d}s\right)^2 + 2\,\mathcal{F}(r)\left(\int_0^r \int_t^\infty \mathcal{F}(s) ds dt\right) - \mathcal{F}'(r)\left(\int_0^r \int_0^\omega \int_t^\infty \mathcal{F}(s) ds dt d\omega\right) - \left(\frac{\mathcal{J}_0(\hat{x})}{\mathcal{I}'(\hat{x})} + \frac{\mathcal{J}_0'(\hat{x})}{\mathcal{I}''(\hat{x})}\right)\left(r\mathcal{F}(r) + \int_r^\infty \mathcal{F}(s) ds\right) + \frac{\mathcal{J}_0(\hat{x})}{\mathcal{I}''(\hat{x})}\mathcal{F}'(r)\frac{r^2}{2} + \frac{\mathcal{J}_0(\hat{x})\mathcal{J}_{(0)}'(\hat{x})}{\mathcal{I}'(\hat{x})\mathcal{I}''(\hat{x})} - \frac{2\mathcal{J}_0(\hat{x})\mathcal{J}_0'(\hat{x})(1+2\delta)}{\delta \mathcal{I}'(\hat{x})\mathcal{I}''(\hat{x})} \Bigg\} dr. \tag{12}$$

Here $\mathcal{F}(r) = -1 + e^{-\hat{\psi}(r)}$ and $\mathcal{F}'(r) = -\hat{\psi}'(r)e^{-\hat{\psi}(r)}$.

We now reduce the integrals (inside the bracket) by changing the order of integration:

$$2\mathcal{F}(r)\left(\int_0^r \int_t^\infty \mathcal{F}(s)\mathrm{d}s\,\mathrm{d}t\right)\mathrm{d}r = 2\mathcal{F}(r)\left(\int_0^r s\mathcal{F}(s)\mathrm{d}s + r\int_r^\infty \mathcal{F}(s)\mathrm{d}s\right)\mathrm{d}r \text{ and}$$
$$\int_0^r \int_0^\omega \int_t^\infty \mathcal{F}(s)\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\omega = \int_0^r \left(r - \frac{s}{2}\right) s\mathcal{F}(s)\mathrm{d}s + \frac{r^2}{2}\int_r^\infty \mathcal{F}(s)\mathrm{d}s. \tag{13}$$

Substituting the above simplification in (12), we obtain

$$\hat{u}_1 = \mathcal{J}_1(\hat{x})\hat{y} - \delta \mathcal{I}'(\hat{x})\mathcal{I}''(\hat{x}) \int_0^{\hat{y}} dp \int_p^\infty \Bigg\{ \left(\int_r^\infty \mathcal{F}(s)\,\mathrm{d}s\right)^2 + 2\mathcal{F}(r)\left(\int_0^r s\mathcal{F}(s)\mathrm{d}s + r\int_r^\infty \mathcal{F}(s)\mathrm{d}s\right)\mathrm{d}r + \hat{\psi}'(r)e^{-\hat{\psi}(r)}\left(\int_0^r \left(r - \frac{s}{2}\right) s\mathcal{F}(s)\mathrm{d}s + \frac{r^2}{2}\int_r^\infty \mathcal{F}(s)\mathrm{d}s\right) - \left(\frac{\mathcal{J}_0(\hat{x})}{\mathcal{I}'(\hat{x})} + \frac{\mathcal{J}_0'(\hat{x})}{\mathcal{I}''(\hat{x})}\right)\left(r\mathcal{F}(r) + \int_r^\infty \mathcal{F}(s) ds\right) + \frac{\mathcal{J}_0(\hat{x})}{\mathcal{I}''(\hat{x})}\mathcal{F}'(r)\frac{r^2}{2} + \frac{\mathcal{J}_0(\hat{x})\mathcal{J}_{(0)}'(\hat{x})}{\mathcal{I}'(\hat{x})\mathcal{I}''(\hat{x})} - \frac{2\mathcal{J}_0(\hat{x})\mathcal{J}_0'(\hat{x})(1+2\delta)}{\delta \mathcal{I}'(\hat{x})\mathcal{I}''(\hat{x})} \Bigg\} dr. \tag{14}$$

S.2 Matching inner and outer solutions

For any field variable f (representing concentration and velocity), the matching condition at $\mathcal{O}(\epsilon^0)$ is

$$\lim_{y\to 0}(f_0^{(0)} + Def_1^{(0)} + \cdots) = \lim_{\hat{y}\to\infty}(\hat{f}_0^{(0)} + De\hat{f}_1^{(0)} + \cdots). \tag{15}$$

The matching condition for the concentration field ($\hat{c}$) yields: $\mathcal{I} = \lim_{y\to 0} c^{(0)}(x,y) + C_\infty$. At $O(1)$, the matching condition for velocity yields:

$$u_0^{(0)}\Big|_{y=0} = \lim_{\hat{y}\to\infty} \hat{u}_0^{(0)}.$$

We substitute (6a) in the above equation, which yields

$$u_0^{(0)}\Big|_{y=0} = -\left(\frac{\partial c}{\partial x}\Big|_{y=0}\right)\int_0^\infty\int_t^\infty \left(e^{-\hat{\psi}(s)} - 1\right)\mathrm{d}s\,\mathrm{d}t + \lim_{\hat{y}\to\infty}\mathcal{J}_0\,\hat{y}. \tag{16}$$

For a bounded solution, $\mathcal{J}_0 = 0$. We thus obtain the solution reported previously in the literature [1–3]. At $O(De)$, we use (14) and obtain

$$u_1^{(0)}\Big|_{y=0} = -\delta\left(\frac{\partial c}{\partial x}\Big|_{y=0}\frac{\partial^2 c}{\partial x^2}\Big|_{y=0}\right)\int_0^\infty \mathrm{d}p\int_p^\infty \mathcal{G}(r)\mathrm{d}r + \lim_{\hat{y}\to\infty}\mathcal{J}_1\,\hat{y}. \tag{17}$$

$$\text{Here, } \mathcal{G}(r) = \Bigg\{\left(\int_r^\infty \mathcal{F}(s)\,\mathrm{d}s\right)^2 + 2\,\mathcal{F}(r)\left(\int_0^r s\mathcal{F}(s)\mathrm{d}s + r\int_r^\infty \mathcal{F}(s)\mathrm{d}s\right)$$
$$+\,\hat{\psi}'(r)\,e^{-\hat{\psi}(r)}\left(\int_0^r\left(r-\frac{s}{2}\right)s\,\mathcal{F}(s)\mathrm{d}s + \frac{r^2}{2}\int_r^\infty \mathcal{F}(s)\mathrm{d}s\right)\Bigg\}. \tag{18}$$

Similar to the $O(1)$ solution, we obtain $\mathcal{J}_1 = 0$.

2.1 Intermediate matching

The above results for velocity can also be obtained using intermediate matching [4]. In an arbitrary intermediate region ($y \sim \epsilon^\alpha$, where $0<\alpha<1$), the matching condition for both outer and inner region is

$$\lim_{\epsilon\to 0}(u_0^{(0)} + De\,u_1^{(0)} + \cdots) = \lim_{\epsilon\to 0}(\hat{u}_0^{(0)} + De\,\hat{u}_1^{(0)} + \cdots). \tag{19}$$

Using the Taylor series in the LHS of (19), at $O(1)$ the matching condition yields

$$\lim_{\epsilon\to 0} u_0^{(0)}\Big|_{y=0} + y\left(\frac{\partial u_0^{(0)}}{\partial y}\Big|_{y=0}\right) + \cdots = \lim_{\epsilon\to 0}\hat{u}_0^{(0)}. \tag{20}$$

We substitute (6a) in the RHS of the above equation and obtain

$$\lim_{\epsilon\to 0}\left[u_0^{(0)}\Big|_{y=0} + y\left(\frac{\partial u_0^{(0)}}{\partial y}\Big|_{y=0}\right) + \cdots\right] = \lim_{\epsilon\to 0}\left[-\left(\frac{\partial c}{\partial x}\Big|_{y=0}\right)\int_0^{\hat{y}}\int_t^\infty \mathcal{F}(s)dsdt\; + \mathcal{J}_0\,\hat{y}\right]. \tag{21}$$

Rescaling y and $\hat{y}$ in terms of the intermediate coordinate ($\bar{y}$): $y = \bar{y}\,\epsilon^\alpha$ and $\hat{y} = \bar{y}\,\epsilon^{-\alpha}$.

$$\lim_{\epsilon\to 0}\left[u_0^{(0)}\Big|_{y=0} + \epsilon^\alpha\bar{y}\left(\frac{\partial u_0^{(0)}}{\partial y}\Big|_{y=0}\right) + \cdots\right] = \lim_{\epsilon\to 0}\left[-\left(\frac{\partial c}{\partial x}\Big|_{y=0}\right)\int_0^{\bar{y}\epsilon^{-\alpha}}\int_t^\infty \mathcal{F}(s)dsdt\; + \mathcal{J}_0\,\bar{y}\epsilon^{-\alpha}\right]. \tag{22}$$

Comparing the coefficients of $\bar{y}$, we obtain

$$\mathcal{J}_0 = \epsilon^{2\alpha}\left(\frac{\partial u_0^{(0)}}{\partial y}\Big|_{y=0}\right). \tag{23}$$

Since the velocity gradient at the surface ($y=0$) is less than or equal to $O(1)$, $\mathcal{J}_0$ can be neglected at the leading order (as it is $O(\epsilon^{2\alpha})$, where $0<\alpha<1$). In the limit $\epsilon\to 0$, equation (22) yields

$$\lim_{\epsilon\to 0} u_0^{(0)}\Big|_{y=0} = -\left(\frac{\partial c}{\partial x}\Big|_{y=0}\right)\int_0^\infty\int_t^\infty \mathcal{F}(s)dsdt. \tag{24}$$

S.3 Validation with literature

3.1 Concentration field and Newtonian slip velocity

The slip velocity and concentration field for different surface coverage is shown in fig. 1. For step change in activity, our results agree well with that of [1] (i.e. Newtonian fluid). To obtain these results, the Newtonian mobility coefficient M_0 is fixed to be -1 (as performed by [1]). It should be noted that M_0 in the main text is -1.1465 (corresponding to $\Phi_0 = -1$).

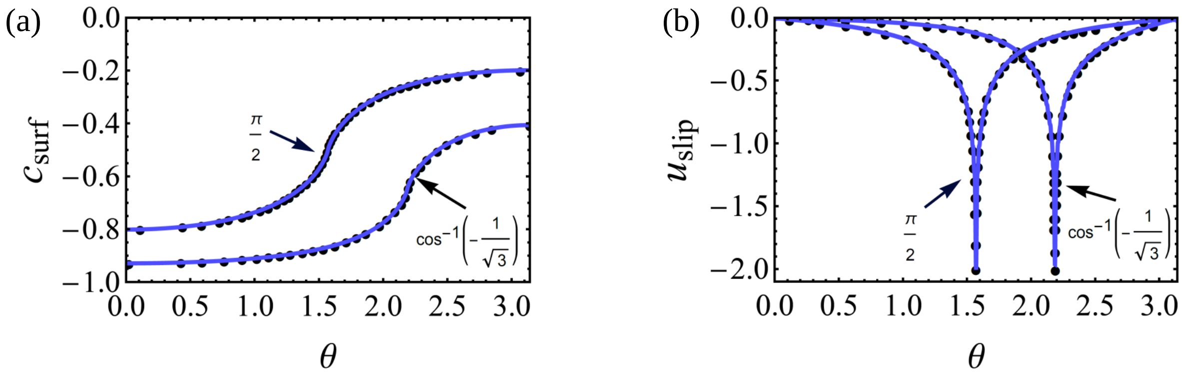

FIG. 1: (a) Surface concentration profile and (b) slip velocity profile for $\cos\theta_c = 0$ and $-1/\sqrt{3}$. The filled circles represent the results of [1].

3.2 U_B for shear-thinning fluid

The velocity field around an axisymmetric squirmer was provided by [6], which was later used by [5] to find the bulk non-Newtonian effects on the swimming of an axisymmetric Janus sphere. The radial component of the outer region disturbance field is

$$u_r = \alpha_1 \frac{P_1}{r^3} + \sum_{m=2}^{\infty} \left(\frac{1}{r^{m+2}} - \frac{1}{r^m} \right) \left(m + \frac{1}{2} \right) \alpha_n P_m, \tag{25}$$

and the tangential component is

$$u_\theta = \alpha_1 \frac{V_1}{2r^3} + \sum_{m=2}^{\infty} \left[\frac{m}{2r^{m+2}} - \left(\frac{n}{2} - 1 \right) \frac{1}{r^n} \right] \left(m + \frac{1}{2} \right) \alpha_m V_m. \tag{26}$$

Here, $V_m = [-2\sin\theta/(m(m+1))]\, P_m^1(\cos\theta)$, P_m^1 is an associated Legendre polynomial of the first kind, $\alpha_m = m\mathcal{K}_m/(2m+1)$, and $\mathcal{K}_m$ is the m$^{\text{th}}$ spectral mode for the step activity which is given by (2.26) in the main text. We convert the above field in Cartesian coordinates and substitute it in the expression for U_B:

$$U_B = -\frac{1}{6\pi}\chi \int_{V_f} \mu_1(\gamma_0)\, \mathbf{A}_0 : \nabla \boldsymbol{u}^t \mathrm{d}V. \tag{27}$$

Here $\mu_1 = \left(1 + Cu_B^2 |\gamma_0|^2\right)^{\frac{n-1}{2}} - 1$ and $|\gamma_0| = (\mathbf{A}_0 : \mathbf{A}_0/2)^{1/2}$. We use inbuilt Gauss-Kronrod rule in Mathematica 12 to numerically evaluate U_B. Fig.2 shows an agreement with the results reported by [5] for $\theta_c = \pi/2$.

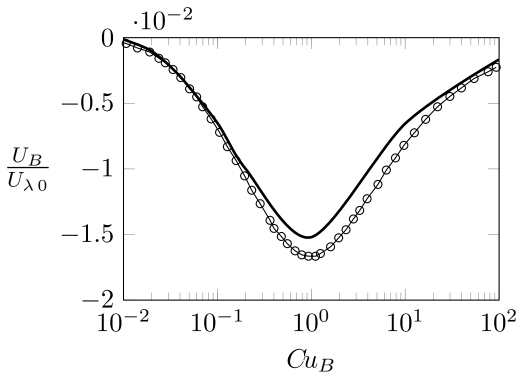

FIG. 2: Comparison of numerical calculation of U_B (——) with [5] (—o—) for $\chi = 0.1$, $\theta_c = \pi/2$, $n = 0.25, m = 15$.

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