Electrokinetically Enhanced Cross-stream Particle Migration in Viscoelastic Flows: Supplementary Material

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I. ADDITIONAL EXPERIMENTAL DETAILS

Here we provide additional details of the experiments carried out. We dissolved polyethylene oxide (PEO, molecular weight = 2×10^6 g/mol, Sigma-Aldrich) powder into a buffer solution (0.01 mM phosphate buffer mixed with 0.5% Tween 20 (Fisher Scientific)) at a concentration of 250 ppm. The particle suspension was prepared by suspending polystyrene spheres (Thermo Scientific) in the viscoelastic solution. The particle concentration was kept low (<0.1% in volume fraction) in the fluid, and hence particle-particle interaction and its effect on fluid viscosity can be neglected. The microchannel was primed with the particle-free suspending fluid prior to the introduction of the particle suspension. The microchannel used in the experiment was 2 cm long with a rectangular cross-section of 66 μm $\times$ 54 μm. Particle motion was visualized at the outlet of the microchannel through an inverted microscope (Nikon Eclipse TE2000U) equipped with a CCD camera (Nikon DS-Qi1MC). Digital videos were recorded at a rate of 15 frames per second, from which the superimposed images were obtained. We further processed these images in the Nikon imaging software (NIS-Elements AR 3.22).

II. RECIPROCAL THEOREM FOR CROSS-STREAM MIGRATION

We follow Ho and Leal [1] to find the migration velocity in this section. The momentum of the unknown $O(De)$ field is governed by

$$\nabla\cdot\boldsymbol{\sigma}_{H(1)}=\mathbf{0},\;\;\text{where }\boldsymbol{\sigma}_{H(1)}=-p_{(1)}\mathbf{I}+(\nabla\boldsymbol{v}_{(1)}+\nabla\boldsymbol{v}_{(1)}^T)+\mathbf{s}_{(0)},\tag{1}$$

where $\mathbf{I}$ is the identity matrix and T denotes transpose. The boundary conditions at the particle surface are

$$\boldsymbol{v}_{(1)}=\boldsymbol{U}_{s(1)}+\boldsymbol{\Omega}_{s(1)}\times\boldsymbol{r}\quad\text{at }r=1\tag{2}$$

The test field is chosen to be that generated by a sphere moving in the positive z-direction with a unit velocity in a quiescent Newtonian medium.

$$\nabla\cdot\boldsymbol{\sigma}_t=\mathbf{0},\;\;\text{where }\boldsymbol{\sigma}_t=-p^t\mathbf{I}+(\nabla\boldsymbol{u}^t+\nabla\boldsymbol{u}^{tT}).\tag{3}$$

The boundary condition for test field is

$$\boldsymbol{u}^t=\boldsymbol{e}_z\quad\text{at }r=1.\tag{4}$$

Taking the inner product of (1) with $\boldsymbol{u}^t$ and (3) with $\boldsymbol{v}_{(1)}$ over the entire fluid domain (from particle surface to infinity) and equating:

$$\int_{V_f} \boldsymbol{u}^t \cdot (\nabla \cdot \boldsymbol{\sigma}_{H(1)}) dV = \int_{V_f} \boldsymbol{v}_{(1)} \cdot (\nabla \cdot \boldsymbol{\sigma}_t) dV. \tag{5}$$

A rearrangement provides

$$\int_{V_f} \nabla \cdot (\boldsymbol{u}^t \cdot \boldsymbol{\sigma}_{H(1)}) dV - \int_{V_f} \boldsymbol{\sigma}_{H(1)} : \nabla \boldsymbol{u}^t \, dV = \int_{V_f} \nabla \cdot (\boldsymbol{v}_{(1)} \cdot \boldsymbol{\sigma}_t) dV - \int_{V_f} \boldsymbol{\sigma}_t : \nabla \boldsymbol{v}_{(1)} \, dV. \tag{6}$$

Using Gauss-divergence theorem and rearranging the integrals, we obtain

$$-\int_{S_p} (\boldsymbol{u}^t \cdot \boldsymbol{\sigma}_{H(1)}) \cdot \boldsymbol{n} \, dS + \int_{S_p} (\boldsymbol{v}_{(1)} \cdot \boldsymbol{\sigma}_t) \cdot \boldsymbol{n} \, dS = \int_{V_f} \boldsymbol{\sigma}_{H(1)} : \nabla \boldsymbol{u}^t \, dV - \int_{V_f} \boldsymbol{\sigma}_t : \nabla \boldsymbol{v}_{(1)} \, dV, \tag{7}$$

where $\boldsymbol{n}$ is the outward unit vector, normal to the surface. Using the boundary conditions (2) and (4), we write

$$-\boldsymbol{e}_z \cdot \int_{S_p} \boldsymbol{\sigma}_{H(1)} \cdot \boldsymbol{n} \, dS + \boldsymbol{U}_{s\,(1)} \cdot \int_{S_p} \boldsymbol{\sigma}_t \cdot \boldsymbol{n} \, dS = \int_{V_f} \boldsymbol{\sigma}_{H(1)} : \nabla \boldsymbol{u}^t \, dV - \int_{V_f} \boldsymbol{\sigma}_t : \nabla \boldsymbol{v}_{(1)} \, dV. \tag{8}$$

The first term on the left hand side in the above equation is zero for a freely suspended neutrally buoyant sphere. Accounting for the leading order wall correction[1], the second term on the left hand side is the hydrodynamic drag which is $-6\pi(1+O(\kappa))U_{s\,(1)z}$. The $O(\kappa)$ corrections are due to wall effects. Expanding the right hand side, we obtain

$$-6\pi(1+O(\kappa))U_{s\,z(1)} = \int_{V_f} \left(-p_{(1)}\mathbf{I} + \nabla \boldsymbol{v}_{(1)} + \nabla \boldsymbol{v}_{(1)}^T + \mathbf{s}_{(0)}\right) : \nabla \boldsymbol{u}^t \, dV - \int_{V_f} \left(-p^t\mathbf{I} + \nabla \boldsymbol{u}^t + \nabla \boldsymbol{u}^{t\,T}\right) : \nabla \boldsymbol{v}_{(1)} \, dV. \tag{9}$$

The incompressibility condition results in: $-p(\mathbf{I} : \nabla \boldsymbol{u}^t) = -p(\mathbf{I} : \nabla \boldsymbol{u}_{(1)}) = 0$. Upon further simplifications, we obtain (9) as

$$6\pi(1+O(\kappa))U_{s\,z(1)} = -\int_{V_f} \mathbf{s}_{(0)} : \nabla \boldsymbol{u}^t \, dV. \tag{10}$$

Thus, we obtain the migration velocity as

$$De\, U_{s\,z(1)} = U^H_{mig} = -\frac{1}{6\pi(1+O(\kappa))} De \int_{V_f} \mathbf{s}_{(0)} : \nabla \boldsymbol{u}^t \, dV. \tag{11}$$

Superscript H denotes the hydrodynamic contribution.

III. EVALUATION OF TRANSLATIONAL AND ROTATIONAL VELOCITY

Here we provide the details of evaluation of $U_{s\,x\,(0)}$ and $\Omega_{s\,y\,(0)}$. To estimate the wall correction, we require the third reflection of velocity field because the particle is absent in the second reflection[3].

[1] For $\kappa \ll 1$, the wall correction can be calculated using both method of reflections and the expression provided by Brenner [2, p.246] (originally derived by Hendrik Lorentz *Theoret. Phys.* 1907 1 23). In Section IV, we compare the wall correction, obtained from method of reflections, with [2].

A. Second reflection of the disturbance velocity

In this subsection, we provide the details of the evaluation of ${}_{(2)}\boldsymbol{v}_{(0)}$. Equations governing the second reflection are

$$\left.\begin{aligned} &\nabla\cdot{}_{(2)}\boldsymbol{v}_{(0)}=0,\quad \nabla^2{}_{(2)}\boldsymbol{v}_{(0)}-\nabla{}_{(2)}p_{(0)}=\boldsymbol{0},\\ &{}_{(2)}\boldsymbol{v}_{(0)}=Ha\zeta_w\left(\nabla_{(1)}\psi+\nabla_{(2)}\psi\right)-{}_{(1)}\boldsymbol{v}_{(0)}\quad\text{at the walls},\\ &{}_{(2)}\boldsymbol{v}_{(0)}\to\boldsymbol{0}\quad\text{at } r\to\infty. \end{aligned}\right\} \tag{12}$$

The solution is determined by the form of non-homogeneity in the boundary condition at the walls. Following the procedure described in Appendix A.1 (main text), the non-homogeneities ($Ha\zeta_w\nabla_{(1)}\psi$, $Ha\zeta_w\nabla_{(2)}\psi$ and ${}_{(1)}\boldsymbol{v}_{(0)}$) are represented in the outer scale coordinates before applying Faxén's integral transformation. ${}_{(2)}\psi$ is already defined in the integral form (see A10 of main text), whereas ${}_{(1)}\psi$ and ${}_{(1)}\boldsymbol{v}_{(0)}$ have been defined in the particle scale (A4 and 3.34, respectively). Upon performing Faxén transformation of the non-homogeneities, we find that $\tilde{\nabla}_{(2)}\tilde{\psi}$ has a different integral form in comparison to ${}_{(1)}\tilde{\boldsymbol{v}}_{(0)}$ and $\tilde{\nabla}_{(1)}\tilde{\psi}$. Therefore, we use superposition and seek ${}_{(2)}\tilde{\boldsymbol{v}}_{(0)}$ as ${}_{(2\,i)}\tilde{\boldsymbol{v}}_{(0)}+{}_{(2\,ii)}\tilde{\boldsymbol{v}}_{(0)}$. These components satisfy the following boundary conditions:

$${}_{(2\,i)}\tilde{\boldsymbol{v}}_{(0)}=Ha\zeta_w\kappa\tilde{\nabla}({}_{(1)}\tilde{\psi})-{}_{(1)}\tilde{\boldsymbol{v}}_{(0)}\quad\text{at the walls}, \tag{13}$$

$${}_{(2\,ii)}\tilde{\boldsymbol{v}}_{(0)}=Ha\zeta_w\kappa\tilde{\nabla}({}_{(2)}\tilde{\psi})\quad\text{at the walls}. \tag{14}$$

Solution to ${}_{(2\,i)}\tilde{\boldsymbol{v}}_{(0)}$: In view of the wall boundary condition in (13), ${}_{(1)}\psi$ (A4) and ${}_{(1)}\boldsymbol{v}_{(0)}$ (3.34) are represented in the outer coordinates (${}_{(1)}\tilde{\psi}$ and ${}_{(1)}\tilde{\boldsymbol{v}}_{(0)}$) and then Faxén transformation is applied. $\tilde{\psi}_1$ has already been represented in outer coordinates and transformed into integral form (see (A9)); ${}_{(1)}\boldsymbol{v}_{(0)}$ is represented into outer coordinates and then Faxén's transformation is applied. Various terms present in the expression for ${}_{(1)}\boldsymbol{v}_{(0)}$ (such as: $1/r$, $x^2/r^3,\cdots$ in (3.34)) are transformed into Faxén's integral form. Upon deriving each term, we obtain the RHS of wall boundary condition (13) as:

$$Ha\zeta_w\kappa\tilde{\nabla}_{(1)}\tilde{\psi}-{}_{(1)}\tilde{\boldsymbol{v}}_{(0)}=-\frac{1}{2\pi}\begin{bmatrix}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\mathrm{e}^{\left(\mathrm{i}\Theta-\frac{\lambda|\tilde{z}|}{2}\right)}\left(\ell_1+(\xi^2/\lambda^2)\left(\ell_2+\frac{\lambda|\tilde{z}|}{2}\ell_3\right)\right)d\xi d\eta\\ \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\mathrm{e}^{\left(\mathrm{i}\Theta-\frac{\lambda|\tilde{z}|}{2}\right)}\left(\ell_2+\frac{\lambda|\tilde{z}|}{2}\ell_3\right)((\eta\xi)/\lambda^2)\,d\xi d\eta\\ \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\mathrm{e}^{\left(\mathrm{i}\Theta-\frac{\lambda|\tilde{z}|}{2}\right)}\left(\ell_1+\ell_2+\left(1+\frac{\lambda|\tilde{z}|}{2}\right)\ell_3\right)((\mathrm{i}\xi)/\lambda)\frac{\tilde{z}}{|\tilde{z}|}\,d\xi d\eta\end{bmatrix} \tag{15}$$

Here, $\Theta=(\tilde{x}\xi+\tilde{y}\eta)/2$ and $\lambda=(\xi^2+\eta^2)^{1/2}$. The terms ℓ_1, ℓ_2, and ℓ_3 are given by:

$$\begin{aligned}\ell_1&=\frac{A_1\kappa}{\lambda}+\frac{\kappa^2}{4}\left(C_1+\frac{D_1}{3}\right)\frac{\tilde{z}}{|\tilde{z}|}-\frac{F_1\kappa^3}{12}\lambda+\frac{5G_1\kappa^3}{12}\lambda,\\ \ell_2&=\frac{-A_1\kappa}{2\lambda}-\frac{B_1\lambda\kappa^3}{8}+\frac{Ha\zeta_w b_1\lambda\kappa^3}{2}+\frac{F_1\kappa^3}{24}\lambda+\frac{13G_1\kappa^3}{24}\lambda+\frac{E_1\kappa^4\lambda^2}{48}\frac{\tilde{z}}{|\tilde{z}|}-\frac{H_1\kappa^5}{96}\lambda^3,\\ \ell_3&=\frac{-A_1\kappa}{2\lambda}-\frac{D_1\kappa^2}{12}\frac{\tilde{z}}{|\tilde{z}|}-\frac{5G_1\kappa^3}{8}\lambda.\end{aligned} \tag{16}$$

Following the procedure carried out in Appendix A.1, we assume the form of ${}_{(2\,i)}\tilde{\boldsymbol{v}}_{(0)}=\{{}_{(2\,i)}\tilde{u}_{(0)},\,{}_{(2\,i)}\tilde{v}_{(0)},\,{}_{(2\,i)}\tilde{w}_{(0)}\}$ as:

$$
{}_{(2\,i)}\tilde{u}_{(0)} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\Theta} \begin{pmatrix} \mathrm{e}^{\left(\frac{-\lambda\tilde{z}}{2}\right)}\left(\ell_4+\frac{\xi^2}{\lambda^2}\left(\ell_5+\frac{\lambda\tilde{z}}{2}\ell_6\right)\right) \\ +\,\mathrm{e}^{\left(\frac{+\lambda\tilde{z}}{2}\right)}\left(\ell_7+\frac{\xi^2}{\lambda^2}\left(\ell_8-\frac{\lambda\tilde{z}}{2}\ell_9\right)\right) \end{pmatrix} \mathrm{d}\xi\mathrm{d}\eta \tag{17}
$$

$$
{}_{(2\,i)}\tilde{v}_{(0)} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\Theta} \begin{pmatrix} \mathrm{e}^{\left(\frac{-\lambda\tilde{z}}{2}\right)}\left(\ell_5+\frac{\lambda\tilde{z}}{2}\ell_6\right) \\ +\,\mathrm{e}^{\left(\frac{+\lambda\tilde{z}}{2}\right)}\left(\ell_8-\frac{\lambda\tilde{z}}{2}\ell_9\right) \end{pmatrix} \left(\frac{\xi\eta}{\lambda^2}\right) \mathrm{d}\xi\mathrm{d}\eta \tag{18}
$$

$$
{}_{(2\,i)}\tilde{w}_{(0)} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\Theta} \begin{pmatrix} \mathrm{e}^{\left(\frac{-\lambda\tilde{z}}{2}\right)}\left(\ell_4+\ell_5+\left(1+\frac{\lambda\tilde{z}}{2}\right)\ell_6\right) \\ -\,\mathrm{e}^{\left(\frac{+\lambda\tilde{z}}{2}\right)}\left(\ell_7+\ell_8+\left(1-\frac{\lambda\tilde{z}}{2}\right)\ell_9\right) \end{pmatrix} \left(\frac{\mathrm{i}\xi}{\lambda}\right) \mathrm{d}\xi\mathrm{d}\eta \tag{19}
$$

Here, the terms $\ell_4,\ \ell_5, \cdots \ell_9$ are functions of the Fourier variable λ and coefficients ($A_1,\ B_1,\ C_1$ and D_1) defined in eq.3.36 (of main text). The terms $\ell_4,\ \ell_5, \cdots \ell_9$ in the above equations can be expressed in terms of the known ℓ_1, ℓ_2, and ℓ_3. Towards this, we form a system of six equations by substituting (15) into RHS of (13). The LHS of (13) is represented by (17)-(19). Since the integrals on both the sides are identical, we obtain the following linear system of equations:

$$
\begin{bmatrix}
e^{\frac{s\lambda}{2}} & e^{\frac{s\lambda}{2}}\xi^2/\lambda^2 & -e^{\frac{s\lambda}{2}}s\xi^2/2\lambda & e^{-\frac{s\lambda}{2}} & e^{-\frac{s\lambda}{2}}\xi^2/\lambda^2 & e^{-\frac{s\lambda}{2}}s\xi^2/2\lambda \\
0 & e^{\frac{s\lambda}{2}} & -e^{\frac{s\lambda}{2}}s\lambda/2 & 0 & e^{-\frac{s\lambda}{2}} & e^{-\frac{s\lambda}{2}}s\lambda/2 \\
e^{\frac{s\lambda}{2}} & e^{\frac{s\lambda}{2}} & e^{\frac{s\lambda}{2}}\left(1-s\lambda/2\right) & -e^{-\frac{s\lambda}{2}} & -e^{-\frac{s\lambda}{2}} & -e^{-\frac{s\lambda}{2}}\left(1+s\lambda/2\right) \\
e^{-\frac{1}{2}(1-s)\lambda} & e^{-\frac{1}{2}(1-s)\lambda}\xi^2/\lambda^2 & e^{-\frac{1}{2}(1-s)\lambda}\left(1-s\right)\xi^2/2\lambda & e^{\frac{1}{2}(1-s)\lambda} & e^{\frac{1}{2}(1-s)\lambda}\xi^2/\lambda^2 & -e^{\frac{1}{2}(1-s)\lambda}\left(1-s\right)\xi^2/2\lambda \\
0 & e^{-\frac{1}{2}(1-s)\lambda} & e^{-\frac{1}{2}(1-s)\lambda}\left(1-s\right)\lambda/2 & 0 & e^{\frac{1}{2}(1-s)\lambda} & -e^{\frac{1}{2}(1-s)\lambda}\left(1-s\right)\lambda/2 \\
e^{-\frac{1}{2}(1-s)\lambda} & e^{-\frac{1}{2}(1-s)\lambda} & e^{-\frac{1}{2}(1-s)\lambda}\left(1+\left(1-s\right)\lambda/2\right) & -e^{\frac{1}{2}(1-s)\lambda} & -e^{\frac{1}{2}(1-s)\lambda} & -e^{\frac{1}{2}(1-s)\lambda}\left(1-\left(1-s\right)\lambda/2\right)
\end{bmatrix}
\begin{bmatrix} \ell_4 \\ \ell_5 \\ \ell_6 \\ \ell_7 \\ \ell_8 \\ \ell_9 \end{bmatrix}
$$

$$
= \begin{bmatrix}
-e^{-\frac{s\lambda}{2}}\left(\ell_{1b}+\left(\ell_2+\frac{\ell_{3b}s\lambda}{2}\right)\xi^2/\lambda^2\right) \\
-e^{-\frac{s\lambda}{2}}\left(\ell_2+\ell_{3b}s\lambda/2\right) \\
e^{-\frac{s\lambda}{2}}\left(\ell_{1b}+\ell_2+\ell_{3b}\left(1+s\lambda/2\right)\right) \\
-e^{-\frac{1}{2}(1-s)\lambda}\left(\ell_{1t}+\left(\ell_2+\frac{1}{2}\ell_{3t}\left(1-s\right)\lambda\right)\xi^2/\lambda^2\right) \\
-e^{-\frac{1}{2}(1-s)\lambda}\left(\ell_2+\frac{1}{2}\ell_{3t}\left(1-s\right)\lambda\right) \\
-e^{-\frac{1}{2}(1-s)\lambda}\left(\ell_{1t}+\ell_2+\ell_{3t}\left(1+\left(1-s\right)\lambda/2\right)\right)
\end{bmatrix} \tag{20}
$$

Here, ℓ_{1b}, ℓ_{3b} and ℓ_{1t}, ℓ_{3t} correspond to the boundary condition at the bottom wall $\tilde{z}/|\tilde{z}| < 0$ and the top wall $\tilde{z}/|\tilde{z}| > 0$, respectively.

Solution to ${}_{(2\,ii)}\tilde{\boldsymbol{v}}_{(0)}$: Upon substituting ${}_{(2)}\tilde{\psi}$ in (14), we obtain ${}_{(2\,ii)}\tilde{\boldsymbol{v}}_{(0)} = \{{}_{2\,(ii)}\tilde{u}_{(0)},\ {}_{2\,(ii)}\tilde{v}_{(0)},\ {}_{2\,(ii)}\tilde{w}_{(0)}$:

$$
{}_{2\,(ii)}\tilde{u}_{(0)} = \frac{Ha\zeta_w\kappa^3}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\Theta}\left(\mathrm{e}^{\left(-\frac{\lambda\tilde{z}}{2}\right)}b_2+\mathrm{e}^{\left(+\frac{\lambda\tilde{z}}{2}\right)}b_3\right)\left(\frac{i^2\xi^2}{2\lambda}\right)\mathrm{d}\xi\mathrm{d}\eta \tag{21}
$$

$$
{}_{2\,(ii)}\tilde{v}_{(0)} = \frac{Ha\zeta_w\kappa^3}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\Theta}\left(\mathrm{e}^{\left(-\frac{\lambda\tilde{z}}{2}\right)}b_2+\mathrm{e}^{\left(+\frac{\lambda\tilde{z}}{2}\right)}b_3\right)\left(\frac{i^2\xi\eta}{2\lambda}\right)\mathrm{d}\xi\mathrm{d}\eta \tag{22}
$$

$$ {}_{2\,(ii)}\tilde{w}_{(0)} = \frac{Ha\zeta_w\kappa^3}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\Theta}\left(-\mathrm{e}^{\left(-\frac{\lambda\tilde{z}}{2}\right)}b_2 + \mathrm{e}^{\left(+\frac{\lambda\tilde{z}}{2}\right)}b_3\right)\left(\frac{\lambda\tilde{z}}{2}\right)\left(\frac{\mathrm{i}\xi}{\lambda}\right)\mathrm{d}\xi\mathrm{d}\eta \tag{23} $$

Combining (17)-(19) and (21)-(23), we obtain the second reflection of velocity field:

$$ {}_2\tilde{u}_{(0)} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\Theta}\left(\begin{array}{l}\mathrm{e}^{\left(\frac{-\lambda\tilde{z}}{2}\right)}\left(\ell_4 + \frac{\xi^2}{\lambda^2}\left(\ell_5 + \frac{\lambda\tilde{z}}{2}\ell_6 - \frac{Ha\zeta_w\kappa^3\lambda}{2}b_2\right)\right)\\ +\,\mathrm{e}^{\left(\frac{+\lambda\tilde{z}}{2}\right)}\left(\ell_7 + \frac{\xi^2}{\lambda^2}\left(\ell_8 - \frac{\lambda\tilde{z}}{2}\ell_9 - \frac{Ha\zeta_w\kappa^3\lambda}{2}b_3\right)\right)\end{array}\right)\mathrm{d}\xi\mathrm{d}\eta, \tag{24} $$

$$ {}_2\tilde{v}_{(0)} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\Theta}\left(\begin{array}{l}\mathrm{e}^{\left(\frac{-\lambda\tilde{z}}{2}\right)}\left(\ell_5 + \frac{\lambda\tilde{z}}{2}\ell_6 - \frac{Ha\zeta_w\kappa^3\lambda}{2}b_2\right)\\ +\,\mathrm{e}^{\left(\frac{+\lambda\tilde{z}}{2}\right)}\left(\ell_8 - \frac{\lambda\tilde{z}}{2}\ell_9 - \frac{Ha\zeta_w\kappa^3\lambda}{2}b_3\right)\end{array}\right)\left(\frac{\xi\eta}{\lambda^2}\right)\mathrm{d}\xi\mathrm{d}\eta, \tag{25} $$

$$ {}_2\tilde{w}_{(0)} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\Theta}\left(\begin{array}{l}\mathrm{e}^{\left(\frac{-\lambda\tilde{z}}{2}\right)}\left(\ell_4 + \ell_5 + \left(1 + \frac{\lambda\tilde{z}}{2}\right)\ell_6 - \frac{Ha\zeta_w\kappa^3\lambda\tilde{z}}{2}b_2\right)\\ -\,\mathrm{e}^{\left(\frac{+\lambda\tilde{z}}{2}\right)}\left(\ell_7 + \ell_8 + \left(1 - \frac{\lambda\tilde{z}}{2}\right)\ell_9 - \frac{Ha\zeta_w\kappa^3\lambda\tilde{z}}{2}b_3\right)\end{array}\right)\left(\frac{\mathrm{i}\xi}{\lambda}\right)\mathrm{d}\xi\mathrm{d}\eta. \tag{26} $$

Since the particle is absent in the formulation of even-numbered reflections, the hydrodynamic drag and torque due to even fields vanishes [3]. Thus, the correction to particle translational and rotational velocity arises from the odd-numbered reflections. Hence, we evaluate the next reflection.

B. Third reflection of the disturbance velocity

The equations governing the third reflection are

$$ \left.\begin{array}{r}\nabla\cdot\boldsymbol{v}^{(0)} = 0, \nabla^2\boldsymbol{v}_3^{(0)} - \nabla p_3^{(0)} = \mathbf{0},\\ \boldsymbol{v}_3^{(0)} = -\boldsymbol{v}_2^{(0)} \quad \text{at } r = 1,\\ \boldsymbol{v}_3^{(0)} \to \mathbf{0} \quad \text{at } r \to \infty.\end{array}\right\} \tag{27} $$

The particle boundary condition in (27) requires us to calculate ${}_{(2)}\tilde{\boldsymbol{v}}_{(0)}$ in the vicinity of the particle. Since, ${}_{(2)}\tilde{\boldsymbol{v}}_{(0)}$ is represented in outer scaled coordinates, the particle surface is equivalent to $\tilde{r} \to 0$. Upon expanding ${}_{(2)}\tilde{\boldsymbol{v}}_{(0)}$ about the origin, we obtain:

$$ {}_{(2)}\boldsymbol{v}_{(0)}\big|_{r=1} = \int_0^{\infty}\begin{bmatrix}(2\ell_4 + \ell_5 + 2\ell_7 + \ell_8)\,\lambda/2 \; - \; (2\ell_4 + \ell_5 - \ell_6 - 2\ell_7 - \ell_8 + \ell_9)\,\lambda^2\kappa z/4 + \cdots\\ 0\\ (-\ell_4 - \ell_5 - \ell_6 + \ell_7 + \ell_8 + \ell_9)\,\lambda^2\kappa x/4 + \cdots\end{bmatrix}\mathrm{d}\lambda \tag{28} $$

Having obtained the boundary condition for $\boldsymbol{v}_3^{(0)}$ at the particle surface, we use Lamb's method to obtain the third reflection. The resulting solution has a form similar to eq.3.34 (of main text) with coefficients A_3, B_3, C_3 and D_3. These coefficients are in the integral form, owing to the integral form of (28). Since the force-free and torque free arguments require only the coefficients of stokeslet and rotlet field [4, p. 88], here we only report the coefficients A_3 and C_3 for brevity:

$$ \begin{aligned} A_3 &= \tfrac{-3}{4}\int_0^{\infty}\tfrac{1}{2}\left(2\ell_4 + \ell_5 + 2\ell_7 + \ell_8\right)\lambda\mathrm{d}\lambda\\ C_3 &= \tfrac{1}{4}\int_0^{\infty}\left(\tfrac{1}{2}\left(\ell_4 - 2\ell_6 - \ell_7 + 2\ell_9\right)\lambda^2\kappa\right)\mathrm{d}\lambda. \end{aligned} \tag{29} $$

Substitution of $\ell_4, \ell_5, \cdots \ell_9$ (obtained from eq.20) into the above equations results in representation of A_3, C_3 in terms of A_1, B_1, C_1 and D_1:

$$A_3 \equiv A_1 \left(\kappa W_A\right) + B_1 \left(\kappa^3 W_B\right) + C_1 \left(\kappa^2 W_C\right) + D_1 \left(\kappa^2 W_D\right) + \cdots , \tag{30}$$

$$C_3 \equiv A_1 \left(\kappa^2 \mathcal{X}_A\right) + B_1 \left(\kappa^3 \mathcal{X}_B\right) + C_1 \left(\kappa^4 \mathcal{X}_C\right) + \cdots . \tag{31}$$

Here, κW_A, $\kappa^3 W_B$, $\kappa^2 W_C$ and $\kappa^2 W_D$ represent the wall corrections to hydrodynamic drag due to the reflection of stokeslet, source-dipole, rotlet and stresslet disturbances, respectively. Similarly, $\kappa^2 \mathcal{X}_\mathcal{A}$, $\kappa^3 \mathcal{X}_\mathcal{B}$, and $\kappa^4 \mathcal{X}_\mathcal{C}$ represent wall correction to the hydrodynamic torque. It should be noted that the form of equations (30)-(31) is valid for a general problem of a particle suspended in wall bounded flow [5]. These equations show that the leading order correction to viscous drag is $O(\kappa)$ (through κW_A) and that to torque is $O(\kappa^2)$ (through $\kappa^2 \mathcal{X}_A$).

C. Evaluation of Translational and Rotational velocity

Translational and rotational velocity of the particle can be found by imposing force-free and torque-free conditions on the particle at $O(De^0)$: $\boldsymbol{F}_{H\,(0)} + \boldsymbol{F}_M = \boldsymbol{0}$ and $\boldsymbol{L}_{H\,(0)} + \boldsymbol{L}_M = \boldsymbol{0}$. Since the Maxwell force $\boldsymbol{F}_M$ acts only along the z-axis ($\sim O(\kappa^4)$) and the torque $\boldsymbol{L}_M$ is zero, $U_{s\,x\,(0)}$ and $\Omega_{s\,y\,(0)}$ are found by hydrodynamic force and torque balance in x and y directions, respectively.

The force and torque on a spherical particle can be expressed through the coefficients of stokeslet and rotlet disturbances, respectively. Following Kim and Karrila [4, p. 88], we write the force-free and torque free condition as:

$$F_{H\,x\,(0)} = -4\pi \left(A_1 + A_3 + \cdots\right) \text{ and } L_{Hy\,(0)} = -8\pi \left(C_1 + C_3 + \cdots\right). \tag{32}$$

The coefficients A_1, C_1 and A_3, C_3 are associated with the Lamb's solution of the first reflection of the velocity disturbance (eq.3.36 of main text) and third reflection (30-31), respectively. Substitution of (30) and (31) into (32) results in a system of two equations for: $U_{s\,x}^{(0)}$ and $\Omega_{s\,y}^{(0)}$. Imposing the hydrodynamic force and torque to be zero and upon expanding the coefficients A_1, $B_1, \cdots H_1$, we obtain:

$$U_{s\,x\,(0)} \approx \alpha + \frac{\gamma}{3} - \frac{10\kappa^2 \beta W_D}{9(1+\kappa W_A)}, \tag{33}$$

$$\Omega_{s\,y\,(0)} \approx \frac{\beta}{2} - \frac{5\mathcal{X}_D \kappa^3 \beta/3}{(1+\kappa W_A)}. \tag{34}$$

Since the correction to particle velocity is $O(\kappa^2)$ and higher, we neglect the wall contribution to $U_{s\,x\,(0)}$ and $\Omega_{s\,y\,(0)}$.

D. Order of magnitude of the disturbance velocity field and test field

We substitute the leading order estimates derived in (33-34): $U_{s\,x(0)} = \alpha + \frac{\gamma}{3}$ and $\Omega_{s\,y(0)} = \beta/2$ in the coefficients (eq.3.36 main text). We then substitute the coefficients in ${}_{(1)}\boldsymbol{v}_{(0)}$ (eq.3.34main text) and ${}_{(2)}\boldsymbol{v}_{(0)}$

(eq.24-26). The order of magnitude of reflections of the velocity fields is obtained as:

$$ {}_{(1)}\boldsymbol{v}_{(0)} \sim Ha\zeta_p O(1/r^3) + \beta\, O(1/r^2) + \beta\, O(1/r^4) + \gamma\, O(1/r^3) + \gamma\, O(1/r^5), \tag{35a} $$

$$ {}_{(2)}\tilde{\boldsymbol{v}}_{(0)} \sim \beta\, O(\kappa^2) + Ha\zeta_p O(\kappa^3) + \cdots . \tag{35b} $$

In the second reflection, the outer scale coordinate $\tilde{r} \sim O(1)$. Similarly, following the framework of Ho and Leal [5], the order of magnitude of test velocity field can be obtained

$$ {}_{(1)}\boldsymbol{u}^t \sim O(1/r) + O(1/r^2), \tag{36a} $$

$$ {}_{(2)}\tilde{\boldsymbol{u}}^t \sim O(\kappa) + O(\kappa^3) + \cdots . \tag{36b} $$

IV. EVALUATION OF $O(\kappa)$ WALL CORRECTION TO MIGRATION VELOCITY

Here we provide the expression for first order correction to viscous drag (it appears in the denominator of eq-3.26 and eq-3.30 in the main text). The cross-stream migration of the particle generates stokeslet and source dipole disturbances. Eq. (30) showed that the leading order correction to drag arrives as κW_A. Thus, the relationship between non-dimensional cross-stream force and velocity is $F_{mig} = 6\pi(1+\kappa W_A)U_{mig}$. Here,

$$ \begin{aligned} W_A = \int_0^\infty \frac{-3e^{\lambda(-s)}}{16\left(-e^{\lambda}\left(\lambda^2+2\right)+e^{2\lambda}+1\right)} \Big(& -4e^{\lambda s} - e^{\lambda+2\lambda s}\left(\lambda^2(s-1)^2 - 2\lambda(s-1)+2\right) \\ & + e^{2\lambda s}\left(\lambda^2 s^2 - 2\lambda s + 2\right) + e^{\lambda}\left(\lambda^2(s-1)^2 + 2\lambda(s-1)+2\right) \\ & - 2e^{\lambda+\lambda s}\left(\lambda^2 + 2\lambda + \lambda^3(-(s-1))s + 2\right) - e^{2\lambda}\left(\lambda^2 s^2 + 2\lambda s + 2\right)\Big)\, \mathrm{d}\lambda. \end{aligned} \tag{37} $$

Fig.1(a) shows that W_A increases near the walls i.e. the migration velocity becomes slower as the particle approaches walls. Using a bispherical coordinate system for a particle approaching a wall, Brenner [2, p.246] reported a similar increase in viscous resistance. He provided an approximate expression for a single wall configuration: $\kappa W_A \approx \frac{9}{8s}\kappa$ (originally derived by Hendrik Lorentz *Theoret. Phys.* 1907 1 23). Fig.1(b) shows a good agreement between our predictions and Brenner [2]. The slight mismatch near $s = 0.5$ is due to the effect of second wall.

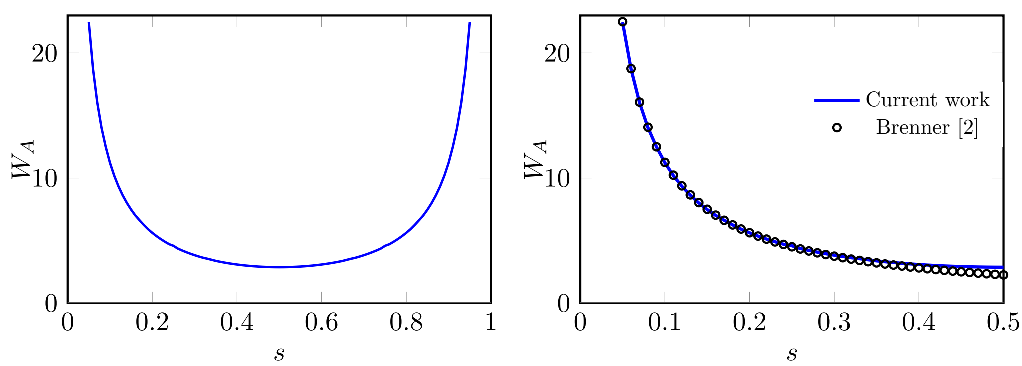

FIG. 1: (a) Variation of leading order viscous resistance to hydrodynamic drag. (b) Comparison with the single wall expression provided by Brenner [2].

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