Other supplementary materials

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1. Governing equations of two-layer fluids

The governing equations are as follows (Lamb 1932; Sutherland 2010)

$$\frac{\partial\phi_u^2}{\partial x^2}+\frac{\partial\phi_u^2}{\partial y^2}+\frac{\partial\phi_u^2}{\partial z^2}=0, \tag{1.1}$$

$$\frac{\partial\phi_l^2}{\partial x^2}+\frac{\partial\phi_l^2}{\partial y^2}+\frac{\partial\phi_l^2}{\partial z^2}=0. \tag{1.2}$$

The kinematics and dynamics boundary conditions at the surface $z=\eta_u$ are

$$\frac{\partial\eta_u}{\partial t}+\nabla\eta_u\cdot\nabla\phi_u-\phi_{u,z}=0, \tag{1.3}$$

$$\frac{\partial\phi_u}{\partial t}+g\eta_u+\frac{1}{2}\nabla\phi_u\cdot\nabla\phi_u-\frac{\sigma_s}{\rho_u}\nabla\cdot\left[\frac{\nabla\eta_u}{\sqrt{1+|\nabla\eta_u|^2}}\right]=0. \tag{1.4}$$

At the interface $z=-h_u+\eta_l$, the boundary conditions yield

$$\frac{\partial\eta_l}{\partial t}+\nabla\eta_l\cdot\nabla\phi_u-\phi_{u,z}=0, \tag{1.5}$$

$$\frac{\partial\eta_l}{\partial t}+\nabla\eta_l\cdot\nabla\phi_l-\phi_{l,z}=0, \tag{1.6}$$

$$\frac{\partial\phi_u}{\partial t}+g\eta_l+\frac{1}{2}\nabla\phi_u\cdot\nabla\phi_u+\frac{p_u}{\rho_u}=0, \tag{1.7}$$

$$\frac{\partial\phi_l}{\partial t}+g\eta_l+\frac{1}{2}\nabla\phi_l\cdot\nabla\phi_l+\frac{p_l}{\rho_l}=0, \tag{1.8}$$

$$p_l-p_u=-\sigma_i\nabla\cdot\left[\frac{\nabla\eta_l}{\sqrt{1+|\nabla\eta_l|^2}}\right], \tag{1.9}$$

where ϕ_u (respectively, ϕ_l), ρ_u (respectively, ρ_l), and h_u (respectively, h_l) are the velocity potential, density, and mean depth of the upper (respectively, lower) layer fluid, σ_s (respectively, σ_i) is the surface tension of the air-upper fluid surface (respectively, upper-lower fluid interface), p_u (respectively, p_l) is the interface pressure on the upper (respectively, lower) fluid side, and $\nabla=(\partial/\partial x,\partial/\partial y)$ denotes the gradient operator in the horizontal directions.

At the bottom $z=-h_u-h_l$, the boundary condition is

$$\phi_{l,z}=0. \tag{1.10}$$

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2. Internal wave of permanent form

The following equations are compiled in Kodaira *et al.* (2016) and summarized here.

2.1. *KdV theory*

The interface elevation is

$$\eta_l(x) = a_{iw}\mathrm{sech}^2\left(\frac{x-x_0}{D}\right), \tag{2.1}$$

where

$$D = \sqrt{\frac{12c_2}{a_{iw}c_1}}, \tag{2.2}$$

$$c_0 = \sqrt{\frac{gh_uh_l\,(1-R)}{Rh_l+h_u}}, \tag{2.3}$$

$$c_1 = \frac{3c_0}{2h_l}\frac{1+Rh_uh_l^2/\left(c_0^2/g-h_u\right)^3}{1+Rh_uh_l/\left(c_0^2/g-h_u\right)^2}, \tag{2.4}$$

$$c_2 = \frac{c_0h_l^2}{6}\frac{1+R\,(h_u/h_l)\left[3+3h_u/\left(c_0^2/g-h_u\right)+h_u^2/\left(c_0^2/g-h_u\right)^2\right]}{1+Rh_uh_l/\left(c_0^2/g-h_u\right)^2}. \tag{2.5}$$

Before used as the initial condition of our simulation, η_l is subtracted by its mean to match our definition of interface elevation.

The internal wave speed is

$$c_{KdV} = c_0 + \frac{c_1a_{iw}}{3}. \tag{2.6}$$

2.2. *MCC theory*

In the MCC theory (Miyata 1985; Choi & Camassa 1996), the surface elevation can be calculated by solving the equation

$$\frac{\mathrm{d}\eta_l}{\mathrm{d}x} = \sqrt{\frac{3\eta_l^2\left[c_{MCC}^2\,(R\eta_l+\eta_u)-g\,(1-R)\,\eta_u\eta_l\right]}{c_{MCC}^2\,(Rh_u^2\eta_l+h_l^2\eta_u)}}, \tag{2.7}$$

where the internal wave speed is

$$c_{MCC} = c_0\sqrt{\frac{g\,(h_u-a_{iw})\,(h_l+a_{iw})}{gh_uh_l-c_0^2a_{iw}}}. \tag{2.8}$$

3. Current gradient calculation

A functional is defined to calculate the spatial derivative of the current, following Chartrand (2011):

$$\mathcal{G}(f) = \alpha\int_0^{L_x}|f'|^2\,\mathrm{d}x + \frac{1}{2}\int_0^{L_x}\left|\int_0^x f(\xi)d\xi - U(x)\right|\mathrm{d}x, \tag{3.1}$$

where α is a parameter to control the balance between these two terms. Note that if $\alpha=0$, the optimal solution would be $f(x)=\mathrm{d}U/\mathrm{d}x$ in theory.

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