1. Asymptotic expansion near $k-$vertex

The leading term in the opening expansion for $x \to 0$ is given by the propagation condition $w = \ell_k^{1/2} x^{1/2}$ written in terms of the toughness lengthscale ℓ_k. To obtain the corresponding leading term in the net pressure expansion, we write the fluid balance equation in terms of the viscosity ℓ_m, leak-off $\ell_{\widetilde{m}}$, and new leak-in $\ell_{\widetilde{o}} = (\mu' Q' V^{1/2})^{2/3}$ lengthscales as follows (similar to the treatment of Appendix B of (Garagash et al. 2011))

$$w^2 \frac{dp/E'}{dx} = \ell_m + \ell_{\widetilde{m}}^{3/2} \frac{x^{1/2}}{w} + \frac{\ell_{\widetilde{o}}^{3/2}}{w} \int_0^x \frac{p(s)/E'}{2\sqrt{x-s}} ds.$$

Further, taking the integral by parts on assumption that $|p(0)| < +\infty$, we can write

$$w^2 \frac{dp/E'}{dx} = \ell_m + \left(\ell_{\widetilde{m}}^{3/2} + \ell_{\widetilde{o}}^{3/2} \frac{p(0)}{E'} \right) \frac{x^{1/2}}{w} + \frac{\ell_{\widetilde{o}}^{3/2}}{w} \int_0^x \frac{dp/E'}{ds} \sqrt{x-s} ds. \tag{1.1}$$

Further, assuming that the pressure gradient at the tip is also bounded, and using the leading term expression for the opening, we observe that the left hand side ($\propto$ fluid velocity) and the integral term vanish when $x \to 0$, suggesting that to the leading order

$$\frac{p(0)}{E'} = -\frac{\ell_{\widetilde{m}}^{3/2} + \ell_m \ell_k^{1/2}}{\ell_{\widetilde{o}}^{3/2}} = -\sigma_o' - \frac{K' V^{1/2}}{E' Q'}, \tag{1.2}$$

as recorded in table 2 of the main text.

The next order (non-constant) term in the net pressure expansion corresponds to the following linear equation on the net pressure gradient, as results from (1.1) with (1.2) and use of the opening leading asymptote,

$$x^{3/2} \frac{dp}{dx} = \left(\frac{\ell_{\widetilde{o}}}{\ell_k} \right)^{3/2} \int_0^x \frac{dp}{ds} \sqrt{x-s} ds,$$

where the lengthscale ratio can be conveniently expressed in term of the leak-in non-dimensional number ζ, $\ell_{\widetilde{o}}/\ell_k = \zeta^2$. This linear equation possesses a power law solution which we choose to write in the form $dp/dx = -p(0)\, \zeta^3\, (x/x_o)^{\gamma(\zeta)-1}$ where x_o is an unknown constant and the exponent $\gamma(\zeta)$ is given implicitly by

$$\frac{2}{\sqrt{\pi}} \frac{\Gamma\left(\gamma + \frac{3}{2}\right)}{\Gamma(\gamma)} = \zeta^3. \tag{1.3}$$

This prescribes monotonically increasing $\gamma(\zeta)$ from zero in the Carter's limit $\zeta \to 0$, $\gamma \sim \zeta^3$, to infinity when $\zeta \to \infty$, $\gamma \sim \pi^{1/3} \zeta^{2/3}$.

The corresponding k-vertex expansion of the net-pressure follows by integration

$$\frac{p}{E'} = \frac{p(0)}{E'} \left[1 - \frac{\zeta^3}{\gamma} \left(\frac{x}{x_o} \right)^{\gamma} \right]. \tag{1.4}$$

This form, upon substituting expression for $p(0)$ is given in Eq. (3.11) of the main text. The above net pressure expansion for $\zeta > 0$ can be formally reduced to the expression given by (Garagash et al. 2011) in the Carter's limit $\zeta \to 0$, as given by Eq. (3.12) of the main text. Indeed, the latter follows upon noticing that, first, $p(0)/E' = -\zeta^{-3}\, \ell_k^{1/2} / \ell_1^{1/2}$ where $\ell_1 = (\ell_{mk}^{-1/2} + \ell_{\widetilde{m}k}^{-1/2})^{-2}$ is the lengscale introduced by (Garagash et al. 2011) in the Carter's case, second, $\gamma \sim \zeta^3$ and $x^\gamma/\gamma \sim 1/\gamma + \ln x$ when $\zeta \to 0$.

To obtain the next order term(s) in the k-vertex expansion for the opening, we follow the approach of (Garagash et al. 2011) (Appendix B) by evaluating the crack elasticity integral (Eq. (2.3) of the main text) using the net-pressure expansion (1.4), truncating the upper limit of integration to some finite value X, and then expanding the result for small $x \to 0$

$$w - \ell_k^{1/2} x^{1/2} = \frac{4}{\pi} \int_0^X K(x,s) \frac{p(0)}{E'} \left[1 - \frac{\zeta^3}{\gamma} \left(\frac{s}{x_o} \right)^{\gamma} \right] ds \sim$$
$$\sim -\zeta^3 \frac{p(0)}{E'} \left[\frac{x^{3/2}}{x_1^{1/2}} + \frac{4 \tan \pi\gamma}{\gamma(1+\gamma)} \frac{x^{\gamma+1}}{x_o^{\gamma}} \right],$$

where contributions to the $x^{3/2}$-term from the both terms in the net pressure expansion are dependent on the truncated value of X, and thus, a priori unknown part of the full numerical solution for semi-infinite fracture, lumped in the above into a single unknown prefactor $x_1^{-1/2}$. As further discussed in the main text the order of the non-leading $x^{3/2}$ and $x^{\gamma+1}$ terms in the opening expansion depends on the value of ζ, specifically, the former is dominant among the two when $3/2 < \gamma(\zeta) + 1$, which takes place when $\zeta > 0.862$, and the opposite is true, i.e. $x^{\gamma+1}$ is dominant among the two, when $\zeta < 0.862$.

2. Asymptotic expansion near $\widetilde{m}-$vertex

The intermediate leak-off dominated asymptotic solution in the $mk-$scaling (Table 3 of the main text) has the form: $\Omega(\xi) = \widetilde{\beta}_0 \chi^{1/4} \xi^{5/8}$, $\Pi(\xi) = \widetilde{\delta}_0 \chi^{1/4} \xi^{-3/8}$. This asymptotic behaviour arises in the distance range $\max(\ell_{\widetilde{m}k}, \ell_{\widetilde{m}\widetilde{o}}) \ll x \ll \ell_{\widetilde{m}m}$ (in dimensional coordinate x). In the parametric space the essential condition for $\widetilde{m}$ asymptote existence is $\chi \gg 1$ and $\psi = \chi/\zeta \gg 1$ which is a consequence of transition lengthscales separation.

Firstly, we introduce "bar" variables that is normalised values on the leak-off asymptote.

$$\bar{\Omega}(\xi) = \frac{\Omega(\xi)}{\chi^{1/4}\xi^{5/8}}, \qquad \bar{\Pi}(\xi) = \frac{\Pi(\xi)}{\chi^{1/4}\xi^{-3/8}}.$$

In these new variables lubrication equation can be rewritten in the form:

$$\xi \frac{d\bar{\Pi}}{d\xi} - \frac{3\bar{\Pi}}{8} = \left(\frac{\xi}{\chi^6} \right)^{1/8} \frac{1}{\bar{\Omega}^2} + \frac{1}{\bar{\Omega}^3} + \frac{\zeta^3}{\bar{\Omega}^3 \xi^{1/2} \chi^{3/4}} \int_0^{\xi} \frac{\bar{\Pi}(\xi')\xi'^{-3/8}}{2\sqrt{\xi - \xi'}} d\xi'. \tag{2.1}$$

We find expansion terms using monomial solutions in the following form:

$$\bar{w}_\lambda(\xi) = B\xi^\lambda, \qquad \bar{\Pi}_\lambda = Bf\left(\frac{5}{8} + \lambda\right)\xi^\lambda,$$

where parameter λ should satisfy the following condition $0 < \frac{5}{8} + \lambda < 1$

Let us firstly consider zero-storage case ($\ell_{\widetilde{m}m} = \infty$). Here we should consider the far-field ($x \gg \max(\ell_{\widetilde{m}k}, \ell_{\widetilde{m}\widetilde{o}})$ or $\xi \gg \max(\chi^{-2}, \zeta^8\chi^{-2})$) of the $\widetilde{m}\widetilde{o}k$ pyramid face. In this limiting case the first term in the right-hand side of Eq. (2.1) is absent:

$$\xi \frac{d\bar{\Pi}}{d\xi} - \frac{3\bar{\Pi}}{8} = \frac{1}{\bar{\Omega}^3} + \frac{\zeta^3}{\bar{\Omega}^3 \xi^{1/2} \chi^{3/4}} \int_0^{\xi} \frac{\bar{\Pi}(\xi')\xi'^{-3/8}}{2\sqrt{\xi - \xi'}} d\xi'.$$

Further, we identity the presence of the small parameter in the interested limit ($\xi \to \infty$) that is in the pressure-dependent leak-off term. Since we consider $\bar{\Pi}(\xi)$ in the form of monomial solution: $\bar{\Pi}(\xi) \sim \xi^\lambda$, this term has the following form: $\sim \zeta^3 \cdot \xi^{\lambda - 3/8} \cdot \chi^{-3/4}$. According to the condition for λ parameter, we know that this power is less than zero and, therefore, pressure-dependent term includes the small parameter $\left(\zeta^8/(\xi\chi^2)\right)^{3/8}$. So, we could represent the "bar" solution in the form of the summation of Taylor and non-Taylor terms:

$$\bar{\Omega}(\xi) = \widetilde{\beta}_0 + \left(\frac{\zeta^8}{\xi\chi^2}\right)^{3/8} \mathring{\widetilde{\beta}}_{-3} + \frac{\widetilde{\beta}_{-1}(\chi,\zeta)}{(\chi^2\xi)^{\frac{5}{8}-\widetilde{h}}},$$

$$\bar{\Pi}(\xi) = \widetilde{\beta}_0 f(5/8) + \left(\frac{\zeta^8}{\xi\chi^2}\right)^{3/8} \mathring{\widetilde{\beta}}_{-3} f(1/4) + \frac{\widetilde{\beta}_{-1}(\chi,\zeta)}{(\chi^2\xi)^{\frac{5}{8}-\widetilde{h}}} f(\widetilde{h}).$$

Substituting this expansion into the lubrication equation for 'bar' functions in this particular limit, we could match coefficient in front of appropriate terms:

$$\widetilde{\beta}_0 = 2.53356, \qquad \widetilde{\beta}_{-3} = -0.52481,$$

and for non-Taylor term we obtain the following equation for $\widetilde{h}$:

$$-\widetilde{\beta}_{-1}(\chi,\zeta) f(\widetilde{h})(1-\widetilde{h}) \left(\frac{1}{\xi}\right)^{\frac{5}{8}-\widetilde{h}} = -\frac{3}{\widetilde{\beta}_0^4} \frac{\widetilde{\beta}_{-1}(\chi,\zeta)}{\xi^{\frac{5}{8}-\widetilde{h}}}.$$

Solving the obtained equation numerically, we obtain the following valued for parameter $\widetilde{h}$: $\widetilde{h} = 0.0699928$. The coefficient $\widetilde{\beta}_1(\chi,\zeta)$, which depends on the both parameters χ and ζ, could not be found without general numerical solution.

In order to find the second part of the $\widetilde{m}$-expansion it is necessary to consider the zero-toughness case, namely, the near-field ($x \ll \ell_{m\widetilde{m}}$) of the $\widetilde{m}m-$edge solution. Here we neglect the pressure-dependent term in the right-hand side of the Eq. (2.1) because of the near-field of this edge solution is represented by $\widetilde{m}-$vertex solution that could potentially occur in the general solution when the leak-off process becomes pressure-independent (described by Carter's law). This asymptotic solution is derived by Garagash et al. (2011).

By using the aforesaid to limiting pats, we obtain the $\widetilde{m}-$expansion.

Let us consider case of the large value of χ parameter and coordinate range $x \ll \ell_{\widetilde{m}m}$. The $O(1)$ solution is located on the zero-storage $\widetilde{m}\widetilde{o}k-$face. The next-order term corresponds to the small storage $\chi^{-1} \ll 1$ perturbation. This fact is the consequence of the lubrication equation written in the $\ell_{\widetilde{m}k}$ scaling ($\eta = \xi\chi^2$):

$$\eta\frac{d\bar{\Pi}}{d\eta} - \frac{3\bar{\Pi}}{8} = \left(\frac{\eta^{1/8}}{\chi}\right)\frac{1}{\bar{\Omega}^2} + \frac{1}{\bar{\Omega}^3} + \frac{\zeta^3}{\bar{\Omega}^3\eta^{1/2}}\int_0^\eta \frac{\bar{\Pi}(\eta')\eta'^{-3/8}}{2\sqrt{\eta-\eta'}}d\eta'.$$

The first term in the right-hand side of the previous equation is a small storage correction at distances $\eta \ll \chi^8$ (or in the dimensional form $x \ll \ell_{\widetilde{m}m}$). As a result, the next-order term in the solution for the large χ parameter could be found in the form of Taylor expansion in the small storage parameter $\epsilon = \chi^{-1} \ll 1$:

$$\bar{\Omega} = \overline{\Omega}^{(0)} + \epsilon\overline{\Omega}^{(1)}, \bar{\Pi} = \overline{\Pi}^{(0)} + \epsilon\overline{\Pi}^{(1)},$$

where $\overline{\Omega}^{(0)}, \overline{\Pi}^{(0)}$ is zero-storage $\widetilde{m}k$-edge solution.

Substituting the Taylor expansion into lubrication equation and keeping terms of order $O(\epsilon)$, we obtain the following equation:

$$\eta\frac{d\overline{\Pi}^{(1)}}{d\eta} - \frac{3\overline{\Pi}^{(1)}}{8} = \frac{\eta^{1/8}}{\overline{\Omega}^{(0)2}} - \frac{3\overline{\Omega}^{(1)}}{\overline{\Omega}^{(0)4}} -$$
$$- \frac{3\overline{\Omega}^{(1)}}{\overline{\Omega}^{(0)4}}\frac{\zeta^3}{\sqrt{\eta}}\int_0^\eta \frac{\overline{\Pi}^{(0)}(\eta')\eta'^{-3/8}}{2\sqrt{\eta-\eta'}}d\eta' + \frac{1}{\overline{\Omega}^{(0)3}}\frac{\zeta^3}{\sqrt{\eta}}\int_0^\eta \frac{\overline{\Pi}^{(1)}(\eta')\eta'^{-3/8}}{2\sqrt{\eta-\eta'}}d\eta'.$$

In the far-field ($x \gg \max\left(\ell_{\widetilde{m}\widetilde{o}}, \ell_{\widetilde{m}k}\right)$) the $O(1)$ term is given by the following equations:

$$\bar{\Omega}(\xi) = \widetilde{\beta}_0 + \frac{\mathring{\widetilde{\beta}}_{-3}}{\eta^{3/8}} + \frac{\widetilde{\beta}_{-1}(\chi,\zeta)}{\eta^{\frac{5}{8}-\widetilde{h}}}, \qquad \bar{\Pi}(\xi) = \widetilde{\beta}_0 f(5/8) + \frac{\mathring{\widetilde{\beta}}_{-3} f(1/4)}{\eta^{3/8}} + \frac{\widetilde{\beta}_{-1}(\chi,\zeta)}{\eta^{\frac{5}{8}-\widetilde{h}}} f(\widetilde{h}),$$

where the values of coefficients and $\widetilde{h}$ are derived earlier. The next-order term is found in the form of monomial solution. Using the condition for the considered coordinate range ($\eta \gg 1$), we could neglect both terms relating to the pressure-dependent leak-off because of the following reason: substituting the expression of $\overline{\Pi}^{(0)}$ into the first integral $\left(\sim \frac{1}{\sqrt{\eta}} \int_0^\eta \frac{\overline{\Pi}^{(0)}(\eta')\eta'^{-3/8}}{\sqrt{\eta-\eta'}} d\eta'\right)$, it is possible to derive that it is proportional to the η in the negative exponent; on the other hand, as shown earlier, that the second integral $\left(\sim \frac{1}{\sqrt{\eta}} \int_0^\eta \frac{\overline{\Pi}^{(1)}(\eta')\eta'^{-3/8}}{\sqrt{\eta-\eta'}} d\eta'\right)$ also provides with the term with the negative exponent of coordinate when the function $\overline{\Pi}^{(1)}$ has the form of monomial solution. As a result, both terms contain coordinate η in the negative exponent, and the equation could be simplified to the following:

$$\eta \frac{d\overline{\Pi}^{(1)}}{d\eta} - \frac{3\overline{\Pi}^{(1)}}{8} = \frac{\eta^{1/8}}{\overline{\Omega}^{(0)2}} - \frac{3\overline{\Omega}^{(1)}}{\overline{\Omega}^{(0)4}}.$$

Further, we balance this equation with the help of the next-order term in the form: $\overline{\Omega}^{(1)} = \widetilde{\beta}_1 \eta^{1/8}, \overline{\Pi}^{(1)} = \widetilde{\beta}_1 f(3/4)\eta^{1/8}$ where the numerical value of the coefficient $\widetilde{\beta}_1$ is derived earlier.

Using the obtained result, we could conclude that $\widetilde{m}-$expansion is the sum of the far-field of the $\widetilde{m}\widetilde{o}k$ face solution and the first term of the near-field of the $\widetilde{m}m$-edge expansion.

Repeating this analysis for the higher order terms $O(\epsilon^2), O(\epsilon^3)$, we could derive other terms from near-field $\widetilde{m}m$-edge expansion.

As a result, the $\widetilde{m}-$expansion in the $mk-$scaling (returning from “bar” variable to the original variables in the $mk-$scaling) has the following form:

$$\Omega(\xi) = \chi^{1/4}\xi^{5/8}\left(\widetilde{\beta}_0 + \sum_{j=1}^{3} \widetilde{\beta}_j \left(\frac{\xi}{\chi^6}\right)^{j/8} + \left(\frac{\zeta^8}{\xi\chi^2}\right)^{3/8} \mathring{\widetilde{\beta}}_{-3} + \frac{\widetilde{\beta}_{-1}(\chi,\zeta)}{(\chi^2\xi)^{\frac{5}{8}-\widetilde{h}}}\right), \tag{2.2}$$

$$\Pi(\xi) = \chi^{1/4}\xi^{-3/8}\left(\widetilde{\beta}_0 + \sum_{j=1}^{2} \widetilde{\delta}_j \left(\frac{\xi}{\chi^6}\right)^{j/8} + \frac{\widetilde{\beta}_3}{4\pi}\left(\frac{\xi}{\chi^6}\right)^{3/8} \ln\left(\frac{\xi\chi^6}{\widetilde{\xi}_0}\right) + \left(\frac{\zeta^8}{\xi\chi^2}\right)^{3/8} \mathring{\widetilde{\delta}}_{-3} + \frac{\widetilde{\delta}_{-1}(\chi,\zeta)}{(\chi^2\xi)^{\frac{5}{8}-\widetilde{h}}}\right), \tag{2.3}$$

where we use coefficients $\widetilde{\delta}_j = \widetilde{\beta}_j f\left(\frac{5}{8} + \frac{j}{8}\right)$ for $j = 0, 1, 2$, $\mathring{\widetilde{\delta}}_{-3} = \mathring{\widetilde{\beta}}_{-3} f(1/4)$ and $\widetilde{\delta}_{-1} = \widetilde{\beta}_{-1} f(\widetilde{h})$ are utilised.

3. Numerical scheme

The numerical method is an extension of the approach of Garagash et al. (2011). In the this section, we recount the main parts of the numerical algorithms and also highlight differences borne by the more general problem formulation in this study as compared to Garagash et al. (2011).

The coordinate range $0 < \xi < \infty$ is divided into three parts: $[0, \varXi_0]$, $[\varXi_0, \varXi_\infty]$, $[\varXi_\infty, +\infty]$ (the normalized problem formulation in the $mk-$scaling is utilised). The first and the last segments are approximated by the analytical asymptotic expressions for the near-field $\varPi_0^*(\xi)$ (subsection 3.3.1 of the main text) and the far-field $\varPi_\infty^*(\xi)$ (subsection 3.3.2 of the main text) correspondingly. For function $\varPi_0^*(\xi)$ (in the majority cases) we utilise only the leading term of the near-field asymptotic expansion. However, when parameter ζ is small ($\zeta < 1$) we also use the next order power term that is derived in Appendix 1. In turn, for function $\varPi_\infty^*(\xi)$, the $m-$vertex solution is utilised.

The intermediate segment, $\varXi_0 \leqslant \xi \leqslant \varXi_\infty$, is the computational domain that is discretised by n nodes into $n-1$ sub-intervals $(\xi_i, \xi_{i+1}), i = 1, ..., n-1$ where $\xi_1 = \varXi_0$ and $\xi_n = \varXi_\infty$. The value of pressure between nodes is approximated by linear combination of the constant function $\varPi_0(\xi) = -1$ and the far-field vertex solution $\varPi_\infty$. The whole representation of the pressure profile is the following:

$$\varPi(\xi) = \begin{cases} \varPi_0^*(\xi), & \xi \in (0, \xi_1), \\ a_i \varPi_0(\xi) + b_i \varPi_\infty(\xi), & \xi \in (\xi_i, \xi_{i+1}), i = 1, ..., n-1, \\ \varPi_\infty^*(\xi), & \xi \in (\xi_n, \infty). \end{cases} \tag{3.1}$$

Coefficients a_i and b_i for $i = 1, ..., n-1$ are found from values of pressure at nodes ($\varPi_i = \varPi(\xi_i)$) by imposing continuity of pressure distribution:

$$a_i = \frac{\varPi_\infty(\xi_i)\varPi_{i+1} - \varPi_\infty(\xi_{i+1})\varPi_i}{\varPi_\infty(\xi_i)\varPi_0(\xi_{i+1}) - \varPi_\infty(\xi_{i+1})\varPi_0(\xi_i)}, \quad b_i = -\frac{\varPi_0(\xi_i)\varPi_{i+1} - \varPi_0(\xi_{i+1})\varPi_i}{\varPi_\infty(\xi_i)\varPi_0(\xi_{i+1}) - \varPi_\infty(\xi_{i+1})\varPi_0(\xi_i)}.$$

Moreover, in the end nodes we define value of pressure by using analytical asymptotic expansions $\varPi_0^*(\xi)$ and $\varPi_\infty^*(\xi)$:

$$\varPi_1 = \varPi_0^*(\xi_1), \qquad \varPi_n = \varPi_\infty^*(\xi_n).$$

Fracture opening is found by integrating the inverted elasticity integral provided by Eq. (2.3) of the main text, using the above approximate representation for the net-pressure, in the following form

$$\varOmega(\xi) = \sqrt{\xi} + F[\varPi_0^*](\xi, \xi_1) + \sum_{i=1}^{n-1} \left(a_i F[\varPi_0](\xi, \eta) + b_i H[\varPi_\infty](\xi, \eta)\right)\Big|_{\eta=\xi_i}^{\eta=\xi_{i+1}} + H[\varPi_\infty^*](\xi, \xi_n),$$

where F and H are integral operators defined by

$$F[\varPi](\xi, \eta) = \frac{4}{\pi}\int_0^\eta K(\xi, \eta)\varPi(\eta)d\eta, \qquad H[\varPi](\xi, \eta) = \frac{4}{\pi}\int_\eta^\infty K(\xi, \eta)\varPi(\eta)d\eta.$$

The analytical expressions for the integrals in F and H for power and constant pressure functions that represent $\varPi_0^*(\xi)$, $\varPi_0(\xi)$, $\varPi_\infty^*(\xi)$ and $\varPi_\infty(\xi)$ are given in (Garagash and Detournay 2005; Garagash et al. 2011).

Further, in each node we write out the lubrication and elasticity equations, and together with boundary conditions for $\varPi_1$ and $\varPi_n$ they constitute the system of nonlinear algebraic equations. It is solved numerically by using Levenberg-Marquardt algorithm implemented in the SciPy library (Jones et al. 2001) of Python programming language.

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