Dirichlet-to-Neumann map for a 3-dimensional rectilinear channel

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Geometry

We consider a channel of width $2L_c$ passing through the centre of a doubly-periodic domain $(x, y) \in \Omega_0 = [-L_x, L_x] \times [-L_y, L_y]$. For $x \in (-L_c, L_c)$, the fluid depth is h_d and for $x \in \Omega_0 \backslash (-L_c, L_c)$ the fluid depth is h_s, where $h_d < h_s$. The velocity potential $\phi = \phi(x, y, z)$ satisfies periodic conditions at $x = \pm L_x$ and $y = \pm L_y$, and no-penetration conditions through the topography. Within the domain, ϕ satisfies Laplace's equation

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0.$$

We consider the case where $\phi|_{z=0} = a$ (where $a(x, y)$ is given) and we need to compute the Dirichlet-to-Neumann map $\partial_z \phi|_{z=0}$.

We note that when the channel width parameter $L_c < L_x$ satisfies $L_c \ll L_x$, the topography may be regarded as a deep trench about $x = 0$. However, when $L_c \lesssim L_x$, the doubly-periodic boundary conditions imply that one may instead think of the domain as a shallow ridge about $x = L_x$.

Domain decomposition

We decompose the domain, Ω, into two regions: $\Omega = \Omega_S \cup \Omega_D$, where $(x, y, z) \in \Omega_S = \Omega_0 \times [-h_s, 0]$ and $(x, y, z) \in \Omega_D = [-L_c, L_c] \times [-L_y, L_y] \times [-h_d, -h_s]$. We denote the velocity potential as ϕ^S in the shallow region, Ω_S, and ϕ^D in the deep region, Ω_D. We define boundary conditions $\phi^S|_{z=0} = a$, while we introduce unknown functions $b^S(x, y)$ and $b^D(x, y)$ satisfying $\phi^S = b^S$ and $\phi^D = b^D$ on $z = -h_s$.

To match the two domains, we introduce collocation points x_j $(j = 1, \dots, N_x)$ and y_l $(l = 1, \dots, N_y)$. As with a standard discrete Fourier transform, we define

$$x_j = -L_x + 2(j-1)\frac{L_x}{N_x}$$

for $j = 1, \dots, N_x$ (and similarly for y_l), where we ensure that each step lies at a collocation point, namely $x_{j'} = -L_c$ and $x_{j''} = L_c$ for some $1 < j' < j'' < N_x$, yielding $N_D = j'' - j' - 1$ collocation points in the deep region, which excludes the steps. As discussed in the main text, this approach improves the convergence of the numerical method. By defining the set $\mathcal{D} = \{j' + 1, \dots, j'' - 1\}$ so that if $j \in \mathcal{D}$ then $-L_c < x_j < L_c$, the matching conditions are (for all $l = 1, \dots, N_y$):

$$b^S(x_j, y_l) = b^D(x_j, y_l), \qquad \forall j \in \mathcal{D}, \tag{1a}$$

$$\partial_z \phi^S(x_j, y_l, -h_s) = \partial_z \phi^D(x_j, y_l, -h_s), \quad \forall j \in \mathcal{D}, \tag{1b}$$

$$\partial_z \phi^S(x_j, y_l, -h_s) = 0, \qquad \forall j \notin \mathcal{D}, \tag{1c}$$

corresponding to continuity of the horizontal and vertical velocity between the domains, and the no-flux boundary condition through the base of the shallow layer.

We proceed to analytically solve Laplace's equation in each of the two domains, Ω_S and Ω_D, using discrete Fourier transforms.

Shallow layer

In the shallow layer Ω_S, we perform a discrete Fourier transform to ϕ^S and b^S, yielding

$$\phi^S(x,y,z) = \sum_{j=1}^{N_x}\sum_{l=1}^{N_y} \phi^S_{jl}(z)\Phi_j(x)\Psi_l(y) \quad \text{and} \quad b^S(x,y) = \sum_{j=1}^{N_x}\sum_{l=1}^{N_y} b^S_{jl}\Phi_j(x)\Psi_l(y),$$

where we utilise two families of basis functions:

$$\begin{aligned} \Phi_j(x) &= \exp(\mathrm{i}k_j x), \qquad k_j = \pi j/L_x \quad \text{for} \quad j = -N_x/2, \ldots, (N_x/2-1), \\ \Psi_j(y) &= \exp(\mathrm{i}\xi_j y), \qquad \xi_j = \pi j/L_y \quad \text{for} \quad j = -N_y/2, \ldots, (N_y/2-1). \end{aligned}$$

Substituting into Laplace's equation yields

$$\partial_{zz}\phi^S_{jl} = k^2_{jl}\phi^S_{jl}, \qquad \forall z \in (-h_s, 0),$$

where $k_{jl} = \sqrt{k_j^2 + \xi_l^2}$. Similarly, the boundary conditions give $\phi^S_{jl} = a_{jl}$ on $z = 0$ (where the Fourier coefficients a_{jl} of a are defined in the same manner as b^S_{jl}) and $\phi^S_{jl} = b^S_{jl}$ on $z = -h_s$. Hence,

$$\phi^S_{jl}(z) = a_{jl}\frac{\sinh(k_{jl}(z+h_s))}{\sinh(k_{jl}h_s)} - b^S_{jl}\frac{\sinh(k_{jl}z)}{\sinh(k_{jl}h_s)}, \tag{2}$$

where the case $k_{jl} = 0$ can be derived using L'hôpital's rule. By differentiating, we thus obtain

$$\begin{aligned} \partial_z\phi^S_{jl}(0) &= k_{jl}\big(a_{jl}\coth(k_{jl}h_s) - b^S_{jl}\,\mathrm{cosech}(k_{jl}h_s)\big), \\ \partial_z\phi^S_{jl}(-h_s) &= k_{jl}\big(a_{jl}\,\mathrm{cosech}(k_{jl}h_s) - b^S_{jl}\coth(k_{jl}h_s)\big). \end{aligned}$$

Deep region

In the deep region Ω_D, we perform a similar spectral decomposition for ϕ^D and b^D, namely

$$\phi^D(x,y,z) = \sum_{j=1}^{N_D}\sum_{l=1}^{N_y} \phi^D_{jl}(z)\Pi_j(x)\Psi_l(y), \qquad b^D(x,y) = \sum_{j=1}^{N_D}\sum_{l=1}^{N_y} b^D_{jl}\Pi_j(x)\Psi_l(y),$$

where $\Pi_j(x) = \cos(\kappa_j(x+L_c))$ and $\kappa_j = j\pi/(2L_c)$ for $j = 1, \ldots, N_D$. Solving Laplace's equation with respect to the boundary conditions yields the well-known Dirichlet-to-Neumann map for finite depth

$$\partial_z\phi^D_{jl}(-h_s) = \xi_{jl}\tanh(\xi_{jl}(h_d - h_s))b^D_{jl}, \tag{3}$$

where $\xi_{jl} = \sqrt{\kappa_j^2 + \xi_l^2}$.

System of equations

To satisfy the matching condition on the horizontal velocity, given by equation (1a), for given Dirichlet data a_{jl}, we require b^S_{jl} and b^D_{jl} such that

$$\sum_{j=1}^{N_x}\sum_{l=1}^{N_y} b^S_{jl}\Phi_j(x_m)\Psi_l(y_n) = \sum_{j=1}^{N_D}\sum_{l=1}^{N_y} b^D_{jl}\Pi_j(x_m)\Psi_l(y_n) \tag{4}$$

for all $m \in \mathcal{D}$ and all $1 \le n \le N_y$. Furthermore, to satisfy the conditions for the vertical velocity (given by equations (1b)–(1c)) at the collocation points (x_m, y_n), we require

$$\sum_{j=1}^{N_x}\sum_{l=1}^{N_y} k_{jl}\big(a_{jl}\operatorname{cosech}(k_{jl}h_s) - b^S_{jl}\coth(k_{jl}h_s)\big)\Phi_j(x_m)\Psi_l(y_n)$$
$$= \begin{cases} \sum_{j=1}^{N_D}\sum_{l=1}^{N_y} b^D_{jl}\xi_{jl}\tanh(\xi_{jl}(h_d - h_s))\Pi_j(x_m)\Psi_l(y_n), & \forall m \in \mathcal{D}, \\ 0, & \forall m \notin \mathcal{D}. \end{cases} \tag{5}$$

Combining the $N_y(N_x + N_D)$ equations (4)–(5), we may solve for the $N_y(N_x + N_D)$ unknowns. However, the matrix inversion is prohibitively expensive for large N_x or N_y.

System reduction

To reduce system (4)–(5) to a series of smaller problems, we recall the orthogonality relation

$$\sum_{n=1}^{N_y}\Psi_l(y_n)\Psi^*_p(y_n) = N_y\delta_{lp}$$

for $-N_y/2 \le l, p \le (N_y/2 - 1)$, where δ_{lp} is the Kronecker-delta. Hence, by applying the weighted sum $\sum_{n=1}^{N}\Psi^*_p(y_n)$ to both sides of (4), we obtain that for each $l = 1, \ldots, N_y$:

$$\sum_{j=1}^{N_x} b^S_{jl}\Phi_j(x_m) = \sum_{j=1}^{N_D} b^D_{jl}\Pi_j(x_m), \quad \forall m \in \mathcal{D}. \tag{6}$$

Similarly, (5) yields

$$\sum_{j=1}^{N_x} k_{jl}\big(a_{jl}\operatorname{cosech}(k_{jl}h_s) - b^S_{jl}\coth(k_{jl}h_s)\big)\Phi_j(x_m) = \begin{cases} \sum_{j=1}^{N_D} b^D_{jl}\xi_{jl}\tanh(\xi_{jl}(h_d - h_s))\Pi_j(x_m), & \forall m \in \mathcal{D}, \\ 0, & \forall m \notin \mathcal{D}. \end{cases} \tag{7}$$

Hence, system (6)–(7) gives N_y problems each with $(N_x + N_D)$ unknowns (b^S_{jl} for $j = 1, \ldots, N_x$ and b^D_{jl} for $j \in \mathcal{D}$), representing a significant computational saving.

Following the computation of b^S_{jl} and b^D_{jl} for given a_{jl}, the Dirichlet-to-Neumann map is

$$\partial_z\phi_{jl}(0) = k_{jl}\big(a_{jl}\coth(k_{jl}h_s) - b^S_{jl}\operatorname{cosech}(k_{jl}h_s)\big).$$