The Dirichlet-to-Neumann map for an annular channel

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We compute the Dirichlet-to-Neumann map for ϕ in an annular cavity, Ω, using a Domain Decomposition method similar to that implemented for the corral geometry. We decompose $\Omega = \Omega_S \cup \Omega_D$, where

$$\Omega_S = \{(r,\theta,z) : r < r_\infty,\ \theta \in [0,2\pi),\ z \in (-h_s,0)\},$$
$$\Omega_D = \{(r,\theta,z) : r_1 < r < r_2,\ \theta \in [0,2\pi),\ z \in (-h_d,-h_s)\},$$

with $0 < r_1 < r_2 < r_\infty < \infty$ and $0 < h_s < h_d$. We consider $\Delta\phi + \phi_{zz} = 0$ in Ω, where Δ is the horizontal Laplacian operator. For a given function $a(r,\theta)$, the boundary conditions for ϕ are

$$\phi = a \quad \text{on} \quad z = 0, \quad r < r_\infty, \tag{1a}$$
$$\phi = 0 \quad \text{on} \quad r = r_\infty, \quad z \in (-h_s,0), \tag{1b}$$
$$\partial_r\phi = 0 \quad \text{on} \quad r = r_1, \quad z \in (-h_d,-h_s), \tag{1c}$$
$$\partial_r\phi = 0 \quad \text{on} \quad r = r_2, \quad z \in (-h_d,-h_s), \tag{1d}$$
$$\partial_z\phi = 0 \quad \text{on} \quad z = -h_d, \quad r \in (r_1,r_2), \tag{1e}$$
$$\partial_z\phi = 0 \quad \text{on} \quad z = -h_s, \quad r \notin (r_1,r_2), \tag{1f}$$

where we assume a bounded solution in the limit $r \to 0$. This configuration is depicted in figure 1.

We then define ϕ^S and ϕ^D such that

$$\phi(r,\theta,z) = \begin{cases} \phi^S(r,\theta,z), & (r,\theta,z) \in \Omega_S, \\ \phi^D(r,\theta,z), & (r,\theta,z) \in \Omega_D. \end{cases}$$

Both ϕ^S and ϕ^D satisfy Laplace's equation in their respective domains, equipped with boundary conditions consistent with (1). The aim is to solve for ϕ given the Dirichlet data $\phi|_{z=0} = a$, from which we determine the corresponding Neumann data, $\phi_z|_{z=0}$. To solve for ϕ^S and ϕ^D concurrently, we introduce the unknown functions $b^S(r,\theta)$ and $b^D(r,\theta)$, satisfying

$$\phi^S = b^S \quad \text{and} \quad \phi^D = b^D \quad \text{on} \quad z = -h_s.$$

The functions b^S and b^D are chosen to be consistent with the fluid flow, specifically

$$b^S = b^D \quad \text{on} \quad z = -h_s, \quad r_1 < r < r_2,$$
$$\partial_z\phi^S = \partial_z\phi^D \quad \text{on} \quad z = -h_s, \quad r_1 < r < r_2,$$
$$\partial_z\phi^S = 0 \quad \text{on} \quad z = -h_s, \quad r \notin (r_1,r_2).$$

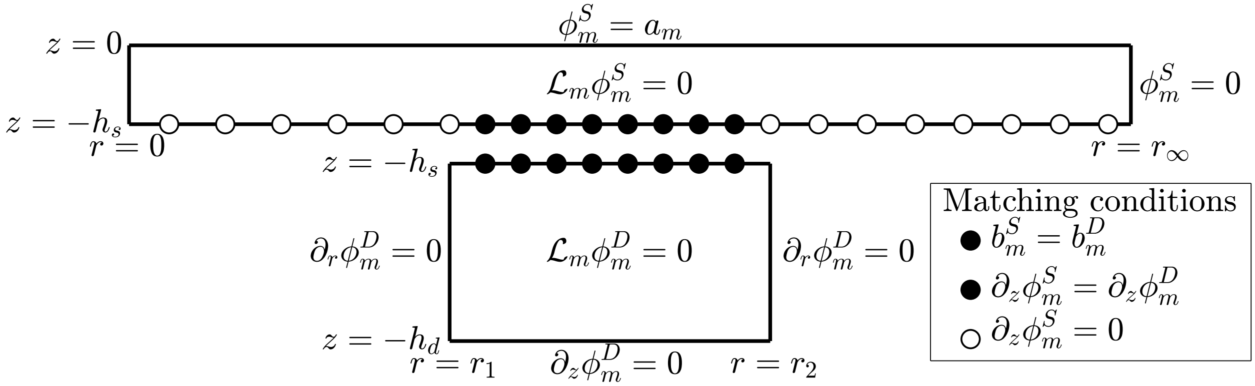

Figure 1: Cross-sectional schematic diagram for the Domain Decomposition method. Circles denote the N_S radial collocation points, of which N_D points (black circles) lie in the deep region. The differential operator is $\mathcal{L}_m = \partial_{rr} + r^{-1}\partial_r + \partial_{zz} - (m/r)^2$. At $r = 0$, we seek a bounded solution.

We proceed by expanding ϕ^S, ϕ^D, a, b^S, and b^D into Fourier angular modes, for example $\phi^S(r, \theta, z) = \sum_{m=-\infty}^{\infty} \phi_m^S(r, z)\mathrm{e}^{\mathrm{i}m\theta}$, where $\mathrm{i}^2 = -1$. As ϕ^S and ϕ^D are both harmonic functions, it is necessary to solve

$$\begin{aligned} \mathcal{L}_m \phi_m^S &= 0, \quad \text{for} \quad (r, z) \in [0, r_\infty) \times (-h_s, 0), \\ \mathcal{L}_m \phi_m^D &= 0, \quad \text{for} \quad (r, z) \in (r_1, r_2) \times (-h_d, -h_s), \end{aligned}$$

where $\mathcal{L}_m = \partial_{rr} + r^{-1}\partial_r + \partial_{zz} - (m/r)^2$ is Laplace's operator for angular mode m. As depicted in figure 1, the boundary conditions for the shallow region are $\phi_m^S = 0$ on $r = r_\infty$, $\phi_m^S = a_m$ on $z = 0$, $\phi_m^S = b_m^S$ on $z = -h_s$, with ϕ_m^S bounded as $r \to 0$. Similarly, we require $\partial_r \phi_m^D = 0$ on $r = r_1$, $\partial_r \phi_m^D = 0$ on $r = r_2$, $\phi_m^D = b_m^D$ on $z = -h_s$, $\partial_z \phi_m^D = 0$ on $z = -h_d$, and boundedness of ϕ_m^D as $r \to 0$.

We first solve for ϕ_m^S and ϕ_m^D before determining b_m^S and b_m^D according to the matching conditions. To solve $\mathcal{L}_m \phi_m^S = 0$ with respect to the radial boundary conditions, we employ the Hankel Transform $\phi_m^S(r, z) = \sum_{p=1}^{\infty} \phi_{mp}^S(z)\mathrm{J}_m(k_{mp}r)$ (similarly for b_m^S and a_m), where J_m is the Bessel function of the first kind of order m. The distinct wavenumbers $k_{mp} > 0$ satisfy the Dirichlet boundary condition (1a) at $r = r_\infty$, specifically $\mathrm{J}_m(k_{mp}r_\infty) = 0$. The analytic solution for $\phi_m^S(r, z)$ is then

$$\phi_m^S(r, z) = \sum_{p=1}^{\infty} \left[a_{mp} \frac{\sinh(k_{mp}(z + h_s))}{\sinh(k_{mp}h_s)} - b_{mp}^S \frac{\sinh(k_{mp}z)}{\sinh(k_{mp}h_s)} \right] \mathrm{J}_m(k_{mp}r). \tag{2}$$

For solving for ϕ_m^D, we now modify the method adopted for the circular corral since one must also consider the Bessel function Y_m of the second kind of order m (as $r_1 > 0$). Specifically, we define the family of cylinder functions $\mathcal{C}_m(\varrho, \vartheta) = \mathrm{J}_m(\varrho)\cos(\pi\vartheta) + \mathrm{Y}_m(\varrho)\sin(\pi\vartheta)$ for $\varrho > 0$ and $\vartheta \in (0, 1)$. We express

$$\phi_m^D(r, z) = \sum_{p=1}^{\infty} \phi_{mp}^D(z)\mathcal{C}_m(\xi_{mp}r, \vartheta_{mp}),$$

where the pair $(\xi_{mp}, \vartheta_{mp})$ is chosen to satisfy the radial boundary conditions (1c)–(1d), namely

$$\mathrm{J}_m'(\xi_{mp}r_1)\cos(\vartheta_{mp}\pi) + \mathrm{Y}_m'(\xi_{mp}r_1)\sin(\vartheta_{mp}\pi) = 0, \tag{3a}$$

$$\mathrm{J}_m'(\xi_{mp}r_2)\cos(\vartheta_{mp}\pi) + \mathrm{Y}_m'(\xi_{mp}r_2)\sin(\vartheta_{mp}\pi) = 0. \tag{3b}$$

We first note that the pair $(\xi_{01}, \vartheta_{01}) = (0, 0)$ corresponds to the unit eigenfunction. To compute the remaining pairs $(\xi_{mp}, \vartheta_{mp})$ for $\xi_{mp} > 0$, we eliminate ϑ_{mp} from (3) to obtain a single equation for ξ_{mp} given r_1 and r_2, namely

$$\mathrm{J}'_m(\xi_{mp} r_1)\mathrm{Y}'_m(\xi_{mp} r_2) - \mathrm{J}'_m(\xi_{mp} r_2)\mathrm{Y}'_m(\xi_{mp} r_1) = 0.$$

The discrete values of ξ_{mp} may be computed numerically, where we consider the order $0 \leq \xi_{m1} < \xi_{m2} < \ldots$. Given $\xi_{mp} > 0$, we then use (3a) to determine $\vartheta_{mp} \in (0, 1)$, namely

$$\vartheta_{mp} = -\frac{1}{\pi} \arctan\left(\frac{\mathrm{J}'_m(\xi_{mp} r_1)}{\mathrm{Y}'_m(\xi_{mp} r_1)}\right) \pmod 1.$$

Through consideration of the boundary conditions for ϕ^D_{mp}, we then obtain the well-known DtN map for finite depth, namely

$$\partial_z \phi^D_{mp}(-h_s) = b^D_{mp} \xi_{mp} \tanh(\xi_{mp}(h_d - h_s)).$$

Hence,

$$\partial_z \phi^D_m(r, -h_s) = \sum_{p=1}^{\infty} b^D_{mp} \xi_{mp} \tanh(\xi_{mp}(h_d - h_s)) \mathcal{C}_m(\xi_{mp} r, \vartheta_{mp}). \tag{4}$$

To match the velocity potential along $z = -h_s$ for each angular mode m, we utilise N_S radial collocation points ρ_j, where $0 \leq \rho_1 < \rho_2 < \ldots < \rho_{N_S} < r_\infty$, which are defined similarly to the main text. To satisfy the velocity continuity between the domains Ω_S and Ω_D, as well as the no-flux condition through the base of the corral (see equation (1f)), at each radial collocation point, ρ_j, we require

$$b^S_m = b^D_m, \qquad \rho_j \in (r_1, r_2), \tag{5a}$$
$$\partial_z \phi^S_m = \partial_z \phi^D_m, \quad \rho_j \in (r_1, r_2), \quad z = -h_s, \tag{5b}$$
$$\partial_z \phi^S_m = 0, \qquad \rho_j \notin (r_1, r_2), \quad z = -h_s. \tag{5c}$$

Following a similar procedure to the corral problem, $N_D < N_S$ collocation points satisfy $\rho_j \in (r_1, r_2)$. For a well-posed problem, we must similarly truncate the wavenumbers k_{mp} and ξ_{mp} to $p \leq N_S$ and $p \leq N_D$ in the shallow and deep regions, respectively.

To satisfy conditions (5b)–(5c), we evaluate (2)–(4) at the collocation points, yielding a system of N_S equations for $b^S_{m1}, \ldots, b^S_{mN_S}$ and $b^D_{m1}, \ldots, b^D_{mN_D}$:

$$\sum_{p=1}^{N_S} \left(a_{mp} - b^S_{mp} \cosh(k_{mp} h_s)\right) \frac{k_{mp} \mathrm{J}_m(k_{mp}\rho_j)}{\sinh(k_{mp} h_s)} = \begin{cases} \sum_{p=1}^{N_D} b^D_{mp} \xi_{mp} \tanh(\xi_{mp}(h_d - h_s)) \mathcal{C}_m(\xi_{mp}\rho_j, \vartheta_{mp}), & \rho_j \in (r_1, r_2), \\ 0, & \rho_j \notin (r_1, r_2). \end{cases} \tag{6}$$

To close the system, the continuity condition (5a) gives

$$\sum_{p=1}^{N_S} b^S_{mp} \mathrm{J}_m(k_{mp}\rho_j) = \sum_{p=1}^{N_D} b^D_{mp} \mathcal{C}_m(\xi_{mp}\rho_j, \vartheta_{mp}), \quad \forall \rho_j \in (r_1, r_2). \tag{7}$$

The system of equations (6)–(7) may be expressed as a system of size $(N_S + N_D)$ and inverted for b^S_{mp} and b^D_{mp}. Equation (2) then determines the Dirichlet-to-Neumann map

$$\partial_z \phi^S_{mp}(0) = k_{mp}\left(a_{mp} \coth(k_{mp} h_s) - b^S_{mp} \operatorname{cosech}(k_{mp} h_s)\right).$$