

# Supplementary material

## Moment theories for a $d$ -dimensional dilute granular gas of Maxwell molecules

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(Received xx; revised xx; accepted xx)

This supplementary material provides the detailed derivation of the system of Grad  $(d+3)(d^2+6d+2)/3!$ -moment equations—which corresponds to the system of the Grad 29-moment (G29) equations in three dimensions—for  $d$ -dimensional inelastic Maxwell molecules (IMM). Recall that I shall refer to the number of moments with 29 (the corresponding number in three dimensions) for simplicity.

### 1. Derivation of the 29 moment equations

The transport equation for the property  $\psi$  is obtained by multiplying the Boltzmann equation (2.1) with  $\psi$  and integrating the resulting equation over the velocity space  $\mathbf{c}$ . After some algebra, the transport equation for the property  $\psi$  reads

$$\frac{D}{Dt} \int \psi f \, d\mathbf{c} + \frac{\partial}{\partial x_i} \int \psi C_i f \, d\mathbf{c} + \frac{\partial v_i}{\partial x_i} \int \psi f \, d\mathbf{c} - \int \left( \frac{D\psi}{Dt} + C_i \frac{\partial \psi}{\partial x_i} + F_i \frac{\partial \psi}{\partial c_i} \right) f \, d\mathbf{c} = \mathcal{P}(\psi). \quad (1.1)$$

The typical form of  $\psi$  is  $m C^{2a} C_{\langle i_1} C_{i_2} \dots C_{i_n \rangle}$ , where  $a, n \in \mathbb{N}_0$ , and the angle brackets around the indices denote the symmetric and traceless part of the corresponding quantity (see appendix A of the main paper for its definition). Thus, a general moment of the velocity distribution function is defined as

$$u_{i_1 i_2 \dots i_n}^a = m \int C^{2a} C_{\langle i_1} C_{i_2} \dots C_{i_n \rangle} f \, d\mathbf{c}, \quad a, n \in \mathbb{N}_0. \quad (1.2)$$

The moment equations for the 29 field variables, namely  $n$ ,  $v_i$ ,  $T$ ,  $\sigma_{ij}$ ,  $q_i$ ,  $u_{ijk}^0$ ,  $u^2$ ,  $u_{ij}^1$ ,  $u_i^2$ , are derived as follows.

#### 1.1. Mass balance equation

Taking  $\psi = 1$  in the transfer equation (1.1), one obtains the mass balance equation

$$\boxed{\frac{Dn}{Dt} + n \frac{\partial v_i}{\partial x_i} = 0} \quad (1.3)$$

The mass balance equation (1.3) can also be written as

$$\boxed{\frac{D\rho}{Dt} + \rho \frac{\partial v_i}{\partial x_i} = 0} \quad (1.4)$$

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### 1.2. Momentum balance equation

The momentum balance equation is obtained by taking  $\psi = c_i$  in the transfer equation (1.1). Taking  $\psi = c_i$ , the transfer equation (1.1) changes to

$$\frac{D(nv_i)}{Dt} + \frac{\partial}{\partial x_j} \int (C_i - v_i) C_j f \, d\mathbf{c} + nv_i \frac{\partial v_j}{\partial x_j} - \int F_j \frac{\partial c_i}{\partial c_j} f \, d\mathbf{c} = 0$$

or

$$n \frac{Dv_i}{Dt} + v_i \left( \frac{Dn}{Dt} + n \frac{\partial v_j}{\partial x_j} \right) + \frac{1}{m} \frac{\partial}{\partial x_j} \left( m \int C_i C_j f \, d\mathbf{c} \right) - F_j \delta_{ij} \times n = 0.$$

Using the mass balance equation (1.3) and the identity

$$m \int C_i C_j f \, d\mathbf{c} = m \int \left( C_{\langle i} C_{j \rangle} + \frac{1}{d} C^2 \delta_{ij} \right) f \, d\mathbf{c} = \sigma_{ij} + n T \delta_{ij} = \sigma_{ij} + \rho \theta \delta_{ij}, \quad (1.5)$$

the above equation simplifies to

$$n \frac{Dv_i}{Dt} + \frac{1}{m} \frac{\partial}{\partial x_j} (\sigma_{ij} + n T \delta_{ij}) - n F_i = 0.$$

Therefore the momentum balance equation reads

$$\boxed{\frac{Dv_i}{Dt} + \frac{1}{m n} \left[ \frac{\partial \sigma_{ij}}{\partial x_j} + \frac{\partial (n T)}{\partial x_i} \right] - F_i = 0} \quad (1.6)$$

The momentum balance equation (1.6) can also be written as

$$\boxed{\frac{Dv_i}{Dt} + \frac{1}{\rho} \left( \frac{\partial \sigma_{ij}}{\partial x_j} + \rho \frac{\partial \theta}{\partial x_i} + \theta \frac{\partial \rho}{\partial x_i} \right) - F_i = 0} \quad (1.7)$$

### 1.3. Energy balance equation

The energy balance equation is obtained by taking  $\psi = \frac{1}{d} m C^2$  in the transfer equation (1.1). Taking  $\psi = \frac{1}{d} m C^2$ , the transfer equation (1.1) changes to

$$\begin{aligned} & \frac{D(nT)}{Dt} + \frac{1}{d} \frac{\partial}{\partial x_i} \left( m \int C^2 C_i f \, d\mathbf{c} \right) + n T \frac{\partial v_i}{\partial x_i} \\ & - \frac{2}{d} m \int \left( -C_i \frac{Dv_i}{Dt} - C_i C_j \frac{\partial v_i}{\partial x_j} + F_i C_j \delta_{ij} \right) f \, d\mathbf{c} = \frac{1}{d} \mathcal{P}^1. \end{aligned}$$

Rearranging the terms, the above equation simplifies to

$$n \frac{DT}{Dt} + \frac{2}{d} \frac{\partial q_i}{\partial x_i} + T \left( \frac{Dn}{Dt} + n \frac{\partial v_i}{\partial x_i} \right) + \frac{2}{d} \frac{\partial v_i}{\partial x_j} m \int C_i C_j f \, d\mathbf{c} = \frac{1}{d} \mathcal{P}^1.$$

Using the mass balance equation (1.3) and identity (1.5), one obtains

$$n \frac{DT}{Dt} + \frac{2}{d} \frac{\partial q_i}{\partial x_i} + \frac{2}{d} \frac{\partial v_i}{\partial x_j} (\sigma_{ij} + n T \delta_{ij}) = \frac{1}{d} \mathcal{P}^1.$$

Therefore the energy balance equation reads

$$\boxed{\frac{DT}{Dt} + \frac{2}{d} \frac{1}{n} \left( \frac{\partial q_i}{\partial x_i} + \sigma_{ij} \frac{\partial v_i}{\partial x_j} + n T \frac{\partial v_i}{\partial x_i} \right) = -\zeta T} \quad (1.8)$$

where  $\zeta = -\frac{1}{dnT}\mathcal{P}^1$ . The energy balance equation (1.8) can also be written as

$$\boxed{\frac{D\theta}{Dt} + \frac{2}{d} \frac{1}{\rho} \left( \frac{\partial q_i}{\partial x_i} + \sigma_{ij} \frac{\partial v_i}{\partial x_j} + \rho \theta \frac{\partial v_i}{\partial x_i} \right) = \frac{1}{d} \frac{1}{\rho} \mathcal{P}^1 = -\zeta \theta} \quad (1.9)$$

#### 1.4. Stress balance equation

The stress balance equation is obtained by taking  $\psi = mC_{\langle i}C_{j \rangle}$  in the transfer equation (1.1). Taking  $\psi = mC_{\langle i}C_{j \rangle}$ , the transfer equation (1.1) changes to

$$\begin{aligned} \frac{D\sigma_{ij}}{Dt} + \frac{\partial}{\partial x_k} \left( m \int C_{\langle i}C_{j \rangle} C_k f \, d\mathbf{c} \right) + \sigma_{ij} \frac{\partial v_k}{\partial x_k} \\ - \int \left( -2C_{\langle i} \frac{Dv_{j \rangle}}{Dt} - 2C_k C_{\langle i} \frac{\partial v_{j \rangle}}{\partial x_k} + 2F_k C_{\langle i} \delta_{j \rangle k} \right) f \, d\mathbf{c} = \mathcal{P}_{ij}^0. \end{aligned}$$

Using (A 3),

$$\begin{aligned} m \int C_{\langle i}C_{j \rangle} C_k f \, d\mathbf{c} &= m \int C_{\langle i}C_{j \rangle} C_k f \, d\mathbf{c} + \frac{1}{d+2} m \int C^2 \left[ C_i \delta_{jk} + C_j \delta_{ik} - \frac{2}{d} C_k \delta_{ij} \right] f \, d\mathbf{c} \\ &= u_{ijk}^0 + \frac{2}{d+2} \left( q_i \delta_{jk} + q_j \delta_{ik} - \frac{2}{d} q_k \delta_{ij} \right) \\ &= u_{ijk}^0 + \frac{2}{d+2} \left( q_i \delta_{jk} + q_j \delta_{ik} - \frac{2}{d} q_l \delta_{lk} \delta_{ij} \right) \\ &= u_{ijk}^0 + \frac{4}{d+2} q_{\langle i} \delta_{j \rangle k}. \end{aligned} \quad (1.10)$$

Hence, the stress balance equation simplifies to

$$\begin{aligned} \frac{D\sigma_{ij}}{Dt} + \frac{\partial u_{ijk}^0}{\partial x_k} + \frac{2}{d+2} \left( \frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i} - \frac{2}{d} \frac{\partial q_k}{\partial x_k} \delta_{ij} \right) + \sigma_{ij} \frac{\partial v_k}{\partial x_k} \\ + \int C_k \left( C_i \frac{\partial v_j}{\partial x_k} + C_j \frac{\partial v_i}{\partial x_k} - \frac{2}{d} C_l \frac{\partial v_l}{\partial x_k} \delta_{ij} \right) f \, d\mathbf{c} = \mathcal{P}_{ij}^0. \end{aligned}$$

Using identity (1.5),

$$\begin{aligned} \frac{D\sigma_{ij}}{Dt} + \frac{\partial u_{ijk}^0}{\partial x_k} + \frac{2}{d+2} \left( \frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i} - \frac{2}{d} \frac{\partial q_k}{\partial x_k} \right) + \sigma_{ij} \frac{\partial v_k}{\partial x_k} \\ + \left[ (\sigma_{ki} + nT\delta_{ki}) \frac{\partial v_j}{\partial x_k} + (\sigma_{kj} + nT\delta_{kj}) \frac{\partial v_i}{\partial x_k} - \frac{2}{d} (\sigma_{kl} + nT\delta_{kl}) \frac{\partial v_l}{\partial x_k} \delta_{ij} \right] = \mathcal{P}_{ij}^0. \end{aligned}$$

On tidying up the terms, the stress balance equation reads

$$\boxed{\frac{D\sigma_{ij}}{Dt} + \frac{\partial u_{ijk}^0}{\partial x_k} + \frac{4}{d+2} \frac{\partial q_{\langle i}}{\partial x_{j \rangle}} + \sigma_{ij} \frac{\partial v_k}{\partial x_k} + 2\sigma_{k\langle i} \frac{\partial v_{j \rangle}}{\partial x_k} + 2nT \frac{\partial v_{\langle i}}{\partial x_{j \rangle}} = \mathcal{P}_{ij}^0} \quad (1.11)$$

The stress balance equation (1.11) can also be written as

$$\boxed{\frac{D\sigma_{ij}}{Dt} + \frac{\partial u_{ijk}^0}{\partial x_k} + \frac{4}{d+2} \frac{\partial q_{\langle i}}{\partial x_{j \rangle}} + \sigma_{ij} \frac{\partial v_k}{\partial x_k} + 2\sigma_{k\langle i} \frac{\partial v_{j \rangle}}{\partial x_k} + 2\rho\theta \frac{\partial v_{\langle i}}{\partial x_{j \rangle}} = \mathcal{P}_{ij}^0} \quad (1.12)$$

## 1.5. Heat flux balance equation

The heat flux balance equation is obtained by taking  $\psi = \frac{1}{2}mC^2C_i$  in the transfer equation (1.1). Taking  $\psi = \frac{1}{2}mC^2C_i$ , the transfer equation (1.1) changes to

$$\begin{aligned} \frac{Dq_i}{Dt} + \frac{\partial}{\partial x_j} \left( \frac{1}{2}m \int C^2 C_i C_j f \, d\mathbf{c} \right) + q_i \frac{\partial v_j}{\partial x_j} - \frac{1}{2}m \int \left[ -C^2 \frac{Dv_i}{Dt} - 2C_i C_j \frac{Dv_j}{Dt} \right. \\ \left. + C_j \left( -C^2 \frac{\partial v_i}{\partial x_j} - 2C_i C_k \frac{\partial v_k}{\partial x_j} \right) + F_k (C^2 \delta_{ik} + 2C_i C_j \delta_{jk}) \right] f \, d\mathbf{c} = \frac{1}{2}\mathcal{P}_i^1. \end{aligned}$$

Using the identity

$$m \int C^2 C_i C_j f \, d\mathbf{c} = m \int \left( C^2 C_{\langle i} C_{j \rangle} + \frac{1}{d} C^4 \delta_{ij} \right) f \, d\mathbf{c} = u_{ij}^1 + \frac{1}{d} u^2 \delta_{ij}, \quad (1.13)$$

the heat flux balance equation simplifies to

$$\begin{aligned} \frac{Dq_i}{Dt} + \frac{1}{2} \frac{\partial u_{ij}^1}{\partial x_j} + \frac{1}{2d} \frac{\partial u^2}{\partial x_i} + q_i \frac{\partial v_j}{\partial x_j} + \frac{d}{2} \rho \theta \left( \frac{Dv_i}{Dt} - F_i \right) + \left( \frac{Dv_j}{Dt} - F_j \right) m \int C_i C_j f \, d\mathbf{c} \\ + \frac{\partial v_j}{\partial x_k} \left( m \int C_i C_j C_k f \, d\mathbf{c} \right) + q_j \frac{\partial v_i}{\partial x_j} = \frac{1}{2}\mathcal{P}_i^1. \end{aligned}$$

Using the momentum balance equation (1.6), identity (1.5) and the identity

$$\begin{aligned} m \int C_i C_j C_k f \, d\mathbf{c} &= m \int C_{\langle i} C_j C_{k \rangle} f \, d\mathbf{c} + \frac{1}{d+2} m \int C^2 (C_i \delta_{jk} + C_j \delta_{ik} + C_k \delta_{ij}) f \, d\mathbf{c} \\ &= u_{ijk}^0 + \frac{2}{d+2} (q_i \delta_{jk} + q_j \delta_{ik} + q_k \delta_{ij}), \end{aligned} \quad (1.14)$$

the heat flux balance equation can be written as

$$\begin{aligned} \frac{Dq_i}{Dt} + \frac{1}{2} \frac{\partial u_{ij}^1}{\partial x_j} + \frac{1}{2d} \frac{\partial u^2}{\partial x_i} + q_i \frac{\partial v_j}{\partial x_j} - \frac{d}{2} \theta \left( \frac{\partial \sigma_{ij}}{\partial x_j} + \rho \frac{\partial \theta}{\partial x_i} + \theta \frac{\partial \rho}{\partial x_i} \right) \\ - \frac{1}{\rho} \left( \frac{\partial \sigma_{jk}}{\partial x_k} + \rho \frac{\partial \theta}{\partial x_j} + \theta \frac{\partial \rho}{\partial x_j} \right) (\sigma_{ij} + \rho \theta \delta_{ij}) \\ + u_{ijk}^0 \frac{\partial v_j}{\partial x_k} + \frac{2}{d+2} \left( q_i \frac{\partial v_j}{\partial x_j} + q_j \frac{\partial v_i}{\partial x_i} + q_j \frac{\partial v_i}{\partial x_j} \right) + q_j \frac{\partial v_i}{\partial x_j} = \frac{1}{2}\mathcal{P}_i^1. \end{aligned}$$

On tidying up the terms, the heat flux balance equation reads

$$\begin{aligned} \frac{Dq_i}{Dt} + \frac{1}{2} \frac{\partial u_{ij}^1}{\partial x_j} + \frac{1}{2d} \frac{\partial u^2}{\partial x_i} - \frac{d+2}{2} \theta \left( \frac{\partial \sigma_{ij}}{\partial x_j} + \rho \frac{\partial \theta}{\partial x_i} + \theta \frac{\partial \rho}{\partial x_i} \right) - \frac{\sigma_{ij}}{\rho} \left( \frac{\partial \sigma_{jk}}{\partial x_k} + \theta \frac{\partial \rho}{\partial x_j} \right) \\ - \sigma_{ij} \frac{\partial \theta}{\partial x_j} + u_{ijk}^0 \frac{\partial v_j}{\partial x_k} + \frac{d+4}{d+2} q_i \frac{\partial v_j}{\partial x_j} + \frac{d+4}{d+2} q_j \frac{\partial v_i}{\partial x_j} + \frac{2}{d+2} q_j \frac{\partial v_j}{\partial x_i} = \frac{1}{2}\mathcal{P}_i^1 \end{aligned}$$

(1.15)

1.6.  $u_{ijk}^0$  balance equation

The balance equation for  $u_{ijk}^0$  is obtained by taking  $\psi = mC_{\langle i}C_jC_k\rangle$  in the transfer equation (1.1). Taking  $\psi = mC_{\langle i}C_jC_k\rangle$ , the transfer equation (1.1) changes to

$$\begin{aligned} \frac{Du_{ijk}^0}{Dt} + \frac{\partial}{\partial x_l} \left( m \int C_{\langle i}C_jC_k\rangle C_l f d\mathbf{c} \right) + u_{ijk}^0 \frac{\partial v_l}{\partial x_l} \\ - m \int \left( \frac{D}{Dt} + C_l \frac{\partial}{\partial x_l} + F_l \frac{\partial}{\partial c_l} \right) C_{\langle i}C_jC_k\rangle f d\mathbf{c} = \mathcal{P}_{ijk}^0. \end{aligned} \quad (1.16)$$

The integrals in the above equation are computed separately in appendix B. On using (B1), (B2), (B3) and (B4), the balance equation for  $u_{ijk}^0$  simplifies to

$$\begin{aligned} \frac{Du_{ijk}^0}{Dt} + \frac{\partial u_{ijkl}^0}{\partial x_l} + \frac{3}{d+4} \frac{\partial u_{\langle ij}^1}{\partial x_{k\rangle}} + u_{ijk}^0 \frac{\partial v_l}{\partial x_l} \\ + 3\sigma_{\langle ij} \left( \frac{Dv_{k\rangle}}{Dt} - F_{k\rangle} \right) + 3u_{l\langle ij}^0 \frac{\partial v_{k\rangle}}{\partial x_l} + \frac{12}{d+2} q_{\langle i} \frac{\partial v_j}{\partial x_{k\rangle}} = \mathcal{P}_{ijk}^0. \end{aligned}$$

Using the momentum balance equation (1.6), the balance equation for  $u_{ijk}^0$  reads

$$\boxed{\begin{aligned} \frac{Du_{ijk}^0}{Dt} + \frac{\partial u_{ijkl}^0}{\partial x_l} + \frac{3}{d+4} \frac{\partial u_{\langle ij}^1}{\partial x_{k\rangle}} + u_{ijk}^0 \frac{\partial v_l}{\partial x_l} \\ - 3 \frac{\sigma_{\langle ij}}{\rho} \left( \frac{\partial \sigma_{k\rangle l}}{\partial x_l} + \rho \frac{\partial \theta}{\partial x_{k\rangle}} + \theta \frac{\partial \rho}{\partial x_{k\rangle}} \right) + 3u_{l\langle ij}^0 \frac{\partial v_{k\rangle}}{\partial x_l} + \frac{12}{d+2} q_{\langle i} \frac{\partial v_j}{\partial x_{k\rangle}} = \mathcal{P}_{ijk}^0 \end{aligned}} \quad (1.17)$$

1.7.  $u^2$  balance equation

The balance equation for  $u^2$  is obtained by taking  $\psi = mC^4$  in the transfer equation (1.1). Taking  $\psi = mC^4$ , the transfer equation (1.1) changes to

$$\frac{Du^2}{Dt} + \frac{\partial u_i^2}{\partial x_i} + u^2 \frac{\partial v_i}{\partial x_i} - m \int \left( -4C^2 C_i \frac{Dv_i}{Dt} - 4C^2 C_i C_j \frac{\partial v_i}{\partial x_j} + 4C^2 C_i F_j \delta_{ij} \right) f d\mathbf{c} = \mathcal{P}^2.$$

Finally, on using identity (1.13), the  $u^2$  balance equation simplifies to

$$\frac{Du^2}{Dt} + \frac{\partial u_i^2}{\partial x_i} + u^2 \frac{\partial v_i}{\partial x_i} + 8q_i \left( \frac{Dv_i}{Dt} - F_i \right) + 4 \left( u_{ij}^1 + \frac{1}{d} u^2 \delta_{ij} \right) \frac{\partial v_i}{\partial x_j} = \mathcal{P}^2.$$

Using the momentum balance equation (1.6), the  $u^2$  balance equation reads

$$\boxed{\frac{Du^2}{Dt} + \frac{\partial u_i^2}{\partial x_i} - 8 \frac{q_i}{\rho} \left( \frac{\partial \sigma_{ij}}{\partial x_j} + \rho \frac{\partial \theta}{\partial x_i} + \theta \frac{\partial \rho}{\partial x_i} \right) + 4u_{ij}^1 \frac{\partial v_i}{\partial x_j} + \frac{d+4}{d} u^2 \frac{\partial v_i}{\partial x_i} = \mathcal{P}^2} \quad (1.18)$$

On taking  $d = 3$ , the above equation matches with the first equation on page 79 of [Struchtrup \(2005\)](#) for  $a = 2$ .

1.8.  $u_{ij}^1$  balance equation

The balance equation for  $u_{ij}^1$  is obtained by taking  $\psi = mC^2C_{\langle i}C_{j\rangle}$  in the transfer equation (1.1). Taking  $\psi = mC^2C_{\langle i}C_{j\rangle}$ , the transfer equation (1.1) changes to

$$\begin{aligned} \frac{Du_{ij}^1}{Dt} + \frac{\partial}{\partial x_k} \left( m \int C^2 C_{\langle i} C_{j\rangle} C_k f \, d\mathbf{c} \right) + u_{ij}^1 \frac{\partial v_k}{\partial x_k} \\ - m \int \left( \frac{D}{Dt} + C_k \frac{\partial}{\partial x_k} + F_k \frac{\partial}{\partial C_k} \right) C^2 C_{\langle i} C_{j\rangle} f \, d\mathbf{c} = \mathcal{P}_{ij}^1. \end{aligned} \quad (1.19)$$

The integrals in the above equation are computed separately in appendix C. On using (C1), (C2), (C3) and (C4), the balance equation for  $u_{ij}^1$  simplifies to

$$\begin{aligned} \frac{Du_{ij}^1}{Dt} + \frac{\partial u_{ijk}^1}{\partial x_k} + \frac{2}{d+2} \frac{\partial u_{\langle i}^2}{\partial x_{j\rangle}} + u_{ij}^1 \frac{\partial v_k}{\partial x_k} + 2u_{ijk}^0 \left( \frac{Dv_k}{Dt} - F_k \right) + \frac{4(d+4)}{d+2} q_{\langle i} \left( \frac{Dv_{j\rangle}}{Dt} - F_{j\rangle} \right) \\ + 2u_{ijkl}^0 \frac{\partial v_k}{\partial x_l} + \frac{6}{d+4} u_{\langle ij}^1 \frac{\partial v_{k\rangle}}{\partial x_k} + \frac{4}{d+2} u_{k\langle i}^1 \frac{\partial v_{j\rangle}}{\partial x_k} + \frac{2(d+4)}{d(d+2)} u^2 \frac{\partial v_{\langle i}}{\partial x_{j\rangle}} = \mathcal{P}_{ij}^1. \end{aligned}$$

Using the momentum balance equation (1.6), the  $u_{ij}^1$  balance equation reads

$$\begin{aligned} \frac{Du_{ij}^1}{Dt} + \frac{\partial u_{ijk}^1}{\partial x_k} + \frac{2}{d+2} \frac{\partial u_{\langle i}^2}{\partial x_{j\rangle}} + u_{ij}^1 \frac{\partial v_k}{\partial x_k} - 2 \frac{u_{ijk}^0}{\rho} \left( \frac{\partial \sigma_{kl}}{\partial x_l} + \rho \frac{\partial \theta}{\partial x_k} + \theta \frac{\partial \rho}{\partial x_k} \right) \\ - \frac{4(d+4)}{d+2} \frac{q_{\langle i}}{\rho} \left( \frac{\partial \sigma_{j\rangle k}}{\partial x_k} + \rho \frac{\partial \theta}{\partial x_{j\rangle}} + \theta \frac{\partial \rho}{\partial x_{j\rangle}} \right) + 2u_{ijkl}^0 \frac{\partial v_k}{\partial x_l} + \frac{6}{d+4} u_{\langle ij}^1 \frac{\partial v_{k\rangle}}{\partial x_k} \\ + \frac{4}{d+2} u_{k\langle i}^1 \frac{\partial v_{j\rangle}}{\partial x_k} + 2u_{k\langle i}^1 \frac{\partial v_{j\rangle}}{\partial x_k} + \frac{2(d+4)}{d(d+2)} u^2 \frac{\partial v_{\langle i}}{\partial x_{j\rangle}} = \mathcal{P}_{ij}^1 \end{aligned} \quad (1.20)$$

On taking  $d = 3$ , the above equation matches with equation (5.16) of Struchtrup (2005) for  $a = 1$ .

1.9.  $u_i^2$  balance equation

The balance equation for  $u_i^2$  is obtained by taking  $\psi = mC^4C_i$  in the transfer equation (1.1). Taking  $\psi = mC^4C_i$ , the transfer equation (1.1) changes to

$$\begin{aligned} \frac{Du_i^2}{Dt} + \frac{\partial}{\partial x_j} \left( m \int C^4 C_i C_j f \, d\mathbf{c} \right) + u_i^2 \frac{\partial v_j}{\partial x_j} - m \int \left[ -C^4 \frac{Dv_i}{Dt} - 4C^2 C_i C_j \frac{Dv_j}{Dt} \right. \\ \left. + C_j \left( -C^4 \frac{\partial v_i}{\partial x_j} - 4C^2 C_i C_k \frac{\partial v_k}{\partial x_j} \right) + F_k (C^4 \delta_{ik} + 4C^2 C_i C_j \delta_{jk}) \right] f \, d\mathbf{c} = \mathcal{P}_i^2. \end{aligned}$$

Using identities (A1) and (A2), one obtains the relations:

$$m \int C^4 C_i C_j f \, d\mathbf{c} = m \int \left( C^4 C_{\langle i} C_{j\rangle} + \frac{1}{d} C^6 \delta_{ij} \right) f \, d\mathbf{c} = u_{ij}^2 + \frac{1}{d} u^3 \delta_{ij}, \quad (1.21)$$

$$\begin{aligned} m \int C^2 C_i C_j C_k f \, d\mathbf{c} &= m \int C^2 C_{\langle i} C_{j\rangle} C_k f \, d\mathbf{c} + \frac{1}{d+2} m \int C^4 (C_i \delta_{jk} + C_j \delta_{ik} + C_k \delta_{ij}) f \, d\mathbf{c} \\ &= u_{ijk}^1 + \frac{1}{d+2} (u_i^2 \delta_{jk} + u_j^2 \delta_{ik} + u_k^2 \delta_{ij}), \end{aligned} \quad (1.22)$$

which simplify the balance equation for  $u_i^2$  to

$$\begin{aligned} \frac{Du_i^2}{Dt} + \frac{\partial u_{ij}^2}{\partial x_j} + \frac{1}{d} \frac{\partial u^3}{\partial x_i} + u_i^2 \frac{\partial v_j}{\partial x_j} + u^2 \left( \frac{Dv_i}{Dt} - F_i \right) + 4 \left( \frac{Dv_j}{Dt} - F_j \right) \left( u_{ij}^1 + \frac{1}{d} u^2 \delta_{ij} \right) \\ + u_j^2 \frac{\partial v_i}{\partial x_j} + 4 \frac{\partial v_k}{\partial x_j} \left[ u_{ijk}^1 + \frac{1}{d+2} (u_i^2 \delta_{jk} + u_j^2 \delta_{ik} + u_k^2 \delta_{ij}) \right] = \mathcal{P}_i^2. \end{aligned}$$

Using the momentum balance equation (1.7) reduces to

$$\begin{aligned} \frac{Du_i^2}{Dt} + \frac{\partial u_{ij}^2}{\partial x_j} + \frac{1}{d} \frac{\partial u^3}{\partial x_i} + u_i^2 \frac{\partial v_j}{\partial x_j} - \frac{d+4}{d} \frac{u^2}{\rho} \left( \frac{\partial \sigma_{ij}}{\partial x_j} + \rho \frac{\partial \theta}{\partial x_i} + \theta \frac{\partial \rho}{\partial x_i} \right) \\ - 4 \frac{u_{ij}^1}{\rho} \left( \frac{\partial \sigma_{jk}}{\partial x_k} + \rho \frac{\partial \theta}{\partial x_j} + \theta \frac{\partial \rho}{\partial x_j} \right) + u_j^2 \frac{\partial v_i}{\partial x_j} \\ + 4 u_{ijk}^1 \frac{\partial v_k}{\partial x_j} + \frac{4}{d+2} \left( u_i^2 \frac{\partial v_j}{\partial x_j} + u_j^2 \frac{\partial v_i}{\partial x_j} + u_j^2 \frac{\partial v_j}{\partial x_i} \right) = \mathcal{P}_i^2. \end{aligned}$$

On tidying up the terms, the balance equation for  $u_i^2$  reads

$$\boxed{\begin{aligned} \frac{Du_i^2}{Dt} + \frac{\partial u_{ij}^2}{\partial x_j} + \frac{1}{d} \frac{\partial u^3}{\partial x_i} - \frac{d+4}{d} \frac{u^2}{\rho} \left( \frac{\partial \sigma_{ij}}{\partial x_j} + \rho \frac{\partial \theta}{\partial x_i} + \theta \frac{\partial \rho}{\partial x_i} \right) \\ - 4 \frac{u_{ij}^1}{\rho} \left( \frac{\partial \sigma_{jk}}{\partial x_k} + \rho \frac{\partial \theta}{\partial x_j} + \theta \frac{\partial \rho}{\partial x_j} \right) + 4 u_{ijk}^1 \frac{\partial v_k}{\partial x_j} \\ + \frac{d+6}{d+2} u_i^2 \frac{\partial v_j}{\partial x_j} + \frac{d+6}{d+2} u_j^2 \frac{\partial v_i}{\partial x_j} + \frac{4}{d+2} u_j^2 \frac{\partial v_j}{\partial x_i} = \mathcal{P}_i^2 \end{aligned}} \quad (1.23)$$

On taking  $d = 3$ , the above equation matches with equation (5.15) on page 80 of [Struchtrup \(2005\)](#) for  $a = 2$ . Note that  $u^3$  with the Maxwellian distribution function is  $105\rho\theta^3$  for  $d = 3$ .

## 2. Moment equations in new variables

Recall that the new variables

$$\left. \begin{aligned} m_{ijk} &:= u_{ijk}^0, & \Delta &:= \frac{u^2}{d(d+2)\rho\theta^2} - 1, \\ R_{ij} &:= u_{ij}^1 - (d+4)\theta\sigma_{ij}, & \varphi_i &:= u_i^2 - 4(d+4)\theta q_i \end{aligned} \right\} \quad (2.1)$$

were introduced in the moment equations so that the new variables in (2.1) vanish with any Grad distribution function based on the 29 or less moments (see § 3.2, 3.3 of the main paper). Therefore the moment equations derived above are expressed in the new variables as follows.

The mass, momentum and energy balance equations (1.3), (1.6), (1.8) remain the same while the other moment equations in new variables are obtained as follows.

### 2.1. Stress balance equation

The stress balance equation (1.11) in new variables reads

$$\boxed{\frac{D\sigma_{ij}}{Dt} + \frac{\partial m_{ijk}}{\partial x_k} + \frac{4}{d+2} \frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + \sigma_{ij} \frac{\partial v_k}{\partial x_k} + 2\sigma_{k\langle i} \frac{\partial v_{j\rangle}}{\partial x_k} + 2nT \frac{\partial v_{\langle i}}{\partial x_{j\rangle}} = \mathcal{P}_{ij}^0} \quad (2.2)$$

## 2.2. Heat flux balance equation

The heat flux balance equation (1.15) in new variables reads

$$\begin{aligned} \frac{Dq_i}{Dt} + \left( \frac{1}{2} \frac{\partial R_{ij}}{\partial x_j} + \frac{d+4}{2} \theta \frac{\partial \sigma_{ij}}{\partial x_j} + \frac{d+4}{2} \sigma_{ij} \frac{\partial \theta}{\partial x_j} \right) \\ + \frac{d+2}{2} \left[ \rho \theta^2 \frac{\partial \Delta}{\partial x_i} + (1+\Delta) \left( \theta^2 \frac{\partial \rho}{\partial x_i} + 2\rho \theta \frac{\partial \theta}{\partial x_i} \right) \right] \\ - \frac{d+2}{2} \theta \left( \frac{\partial \sigma_{ij}}{\partial x_j} + \rho \frac{\partial \theta}{\partial x_i} + \theta \frac{\partial \rho}{\partial x_i} \right) - \frac{\sigma_{ij}}{\rho} \left( \frac{\partial \sigma_{jk}}{\partial x_k} + \theta \frac{\partial \rho}{\partial x_j} \right) \\ - \sigma_{ij} \frac{\partial \theta}{\partial x_j} + m_{ijk} \frac{\partial v_j}{\partial x_k} + \frac{d+4}{d+2} q_i \frac{\partial v_j}{\partial x_j} + \frac{d+4}{d+2} q_j \frac{\partial v_i}{\partial x_j} + \frac{2}{d+2} q_j \frac{\partial v_j}{\partial x_i} = \frac{1}{2} \mathcal{P}_i^1. \end{aligned}$$

On tidying up the terms, the heat flux balance equation in new variables reads

$$\boxed{\begin{aligned} \frac{Dq_i}{Dt} + \frac{1}{2} \frac{\partial R_{ij}}{\partial x_j} + \frac{d+2}{2} \left[ \rho \theta^2 \frac{\partial \Delta}{\partial x_i} + \Delta \theta^2 \frac{\partial \rho}{\partial x_i} + (1+2\Delta) \rho \theta \frac{\partial \theta}{\partial x_i} + \sigma_{ij} \frac{\partial \theta}{\partial x_j} \right] \\ + \theta \frac{\partial \sigma_{ij}}{\partial x_j} - \frac{\sigma_{ij}}{\rho} \left( \frac{\partial \sigma_{jk}}{\partial x_k} - \theta \frac{\partial \rho}{\partial x_j} \right) + m_{ijk} \frac{\partial v_j}{\partial x_k} \\ + \frac{d+4}{d+2} q_i \frac{\partial v_j}{\partial x_j} + \frac{d+4}{d+2} q_j \frac{\partial v_i}{\partial x_j} + \frac{2}{d+2} q_j \frac{\partial v_j}{\partial x_i} = \frac{1}{2} \mathcal{P}_i^1 \end{aligned}} \quad (2.3)$$

2.3.  $m_{ijk}$  balance equation

The  $m_{ijk}$  balance equation follows from the  $u_{ijk}^0$  balance equation (1.17), which on inserting the new variables (2.1) reads

$$\begin{aligned} \frac{Dm_{ijk}}{Dt} + \frac{\partial u_{ijkl}^0}{\partial x_l} + \left( \frac{3}{d+4} \frac{\partial R_{\langle ij}}{\partial x_{\rangle}} + 3\theta \frac{\partial \sigma_{\langle ij}}{\partial x_{\rangle}} \right) + 3\sigma_{\langle ij} \frac{\partial \theta}{\partial x_{\rangle}} + m_{ijk} \frac{\partial v_l}{\partial x_l} \\ - 3 \frac{\sigma_{\langle ij}}{\rho} \left( \frac{\partial \sigma_{k\rangle l}}{\partial x_l} + \rho \frac{\partial \theta}{\partial x_{\rangle}} + \theta \frac{\partial \rho}{\partial x_{\rangle}} \right) + 3m_{l\langle ij} \frac{\partial v_{k\rangle}}{\partial x_l} + \frac{12}{d+2} q_{\langle i} \frac{\partial v_{j\rangle}}{\partial x_{\rangle}} = \mathcal{P}_{ijk}^0. \end{aligned}$$

On tidying up the terms, the  $m_{ijk}$  balance equation in new variables reads

$$\boxed{\begin{aligned} \frac{Dm_{ijk}}{Dt} + \frac{\partial u_{ijkl}^0}{\partial x_l} + \frac{3}{d+4} \frac{\partial R_{\langle ij}}{\partial x_{\rangle}} + 3\theta \frac{\partial \sigma_{\langle ij}}{\partial x_{\rangle}} - 3 \frac{\sigma_{\langle ij}}{\rho} \left( \frac{\partial \sigma_{k\rangle l}}{\partial x_l} + \theta \frac{\partial \rho}{\partial x_{\rangle}} \right) \\ + m_{ijk} \frac{\partial v_l}{\partial x_l} + 3m_{l\langle ij} \frac{\partial v_{k\rangle}}{\partial x_l} + \frac{12}{d+2} q_{\langle i} \frac{\partial v_{j\rangle}}{\partial x_{\rangle}} = \mathcal{P}_{ijk}^0 \end{aligned}} \quad (2.4)$$

2.4.  $\Delta$  balance equation

The  $\Delta$  balance equation follows from the  $u^2$  balance equation (1.18), which on inserting the new variables (2.1) reads

$$\begin{aligned} d(d+2) \left[ \rho \theta^2 \frac{D\Delta}{Dt} + (1+\Delta) \left( \theta^2 \frac{D\rho}{Dt} + 2\rho \theta \frac{D\theta}{Dt} \right) \right] \\ + \left[ \frac{\partial \varphi_i}{\partial x_i} + 4(d+4)\theta \frac{\partial q_i}{\partial x_i} + 4(d+4)q_i \frac{\partial \theta}{\partial x_i} \right] - 8 \frac{q_i}{\rho} \left( \frac{\partial \sigma_{ij}}{\partial x_j} + \rho \frac{\partial \theta}{\partial x_i} + \theta \frac{\partial \rho}{\partial x_i} \right) \\ + 4[R_{ij} + (d+4)\theta \sigma_{ij}] \frac{\partial v_i}{\partial x_j} + (d+2)(d+4)\rho \theta^2 (1+\Delta) \frac{\partial v_i}{\partial x_i} = \mathcal{P}^2 \end{aligned}$$



or

$$\begin{aligned} \frac{D\Delta}{Dt} + (1+\Delta)\frac{1}{\rho}\frac{D\rho}{Dt} + \frac{2(1+\Delta)}{\theta}\frac{D\theta}{Dt} \\ + \frac{1}{d(d+2)}\frac{1}{\rho\theta^2} \left[ \frac{\partial\varphi_i}{\partial x_i} + 4(d+4)\theta\frac{\partial q_i}{\partial x_i} + 4(d+2)q_i\frac{\partial\theta}{\partial x_i} - 8\frac{q_i}{\rho} \left( \frac{\partial\sigma_{ij}}{\partial x_j} + \theta\frac{\partial\rho}{\partial x_i} \right) \right. \\ \left. + 4R_{ij}\frac{\partial v_i}{\partial x_j} \right] + \frac{4(d+4)}{d(d+2)}\frac{\sigma_{ij}}{\rho\theta}\frac{\partial v_i}{\partial x_j} + \frac{d+4}{d}(1+\Delta)\frac{\partial v_i}{\partial x_i} = \frac{1}{d(d+2)}\frac{1}{\rho\theta^2}\mathcal{P}^2 \end{aligned}$$

or

$$\begin{aligned} \frac{D\Delta}{Dt} + \frac{4(d+4)}{d(d+2)}\frac{1}{\rho\theta} \left( \frac{\partial q_i}{\partial x_i} + \sigma_{ij}\frac{\partial v_i}{\partial x_j} \right) \\ + (1+\Delta)\frac{1}{\rho} \left( \frac{D\rho}{Dt} + \rho\frac{\partial v_i}{\partial x_i} \right) + \frac{2(1+\Delta)}{\theta} \left[ \frac{D\theta}{Dt} + \frac{2}{d}\theta\frac{\partial v_i}{\partial x_i} \right] \\ + \frac{1}{d(d+2)}\frac{1}{\rho\theta^2} \left[ \frac{\partial\varphi_i}{\partial x_i} + 4(d+2)q_i\frac{\partial\theta}{\partial x_i} - 8\frac{q_i}{\rho} \left( \frac{\partial\sigma_{ij}}{\partial x_j} + \theta\frac{\partial\rho}{\partial x_i} \right) + 4R_{ij}\frac{\partial v_i}{\partial x_j} \right] \\ = \frac{1}{d(d+2)}\frac{1}{\rho\theta^2}\mathcal{P}^2. \end{aligned}$$

Using the mass balance equation (1.4) and replacing the time derivative  $\theta$  using the energy balance equation (1.9), one obtains

$$\begin{aligned} \frac{D\Delta}{Dt} + \frac{4(d+4)}{d(d+2)}\frac{1}{\rho\theta} \left( \frac{\partial q_i}{\partial x_i} + \sigma_{ij}\frac{\partial v_i}{\partial x_j} \right) \\ + \frac{2(1+\Delta)}{\theta} \left[ \frac{1}{d}\frac{1}{\rho}\mathcal{P}^1 - \frac{2}{d}\frac{1}{\rho} \left( \frac{\partial q_i}{\partial x_i} + \sigma_{ij}\frac{\partial v_i}{\partial x_j} + \rho\theta\frac{\partial v_i}{\partial x_i} \right) + \frac{2}{d}\theta\frac{\partial v_i}{\partial x_i} \right] \\ + \frac{1}{d(d+2)}\frac{1}{\rho\theta^2} \left[ \frac{\partial\varphi_i}{\partial x_i} + 4(d+2)q_i\frac{\partial\theta}{\partial x_i} - 8\frac{q_i}{\rho} \left( \frac{\partial\sigma_{ij}}{\partial x_j} + \theta\frac{\partial\rho}{\partial x_i} \right) + 4R_{ij}\frac{\partial v_i}{\partial x_j} \right] \\ = \frac{1}{d(d+2)}\frac{1}{\rho\theta^2}\mathcal{P}^2 \end{aligned}$$

or

$$\begin{aligned} \frac{D\Delta}{Dt} + \frac{4(d+4)}{d(d+2)}\frac{1}{\rho\theta} \left( \frac{\partial q_i}{\partial x_i} + \sigma_{ij}\frac{\partial v_i}{\partial x_j} \right) - \frac{4(1+\Delta)}{d}\frac{1}{\rho\theta} \left( \frac{\partial q_i}{\partial x_i} + \sigma_{ij}\frac{\partial v_i}{\partial x_j} \right) \\ + \frac{1}{d(d+2)}\frac{1}{\rho\theta^2} \left[ \frac{\partial\varphi_i}{\partial x_i} + 4(d+2)q_i\frac{\partial\theta}{\partial x_i} - 8\frac{q_i}{\rho} \left( \frac{\partial\sigma_{ij}}{\partial x_j} + \theta\frac{\partial\rho}{\partial x_i} \right) + 4R_{ij}\frac{\partial v_i}{\partial x_j} \right] \\ = \frac{1}{d(d+2)}\frac{1}{\rho\theta^2}\mathcal{P}^2 - \frac{2(1+\Delta)}{d}\frac{1}{\rho\theta}\mathcal{P}^1. \end{aligned}$$

On tidying up the terms, the  $\Delta$  balance equation in new variables reads

$$\boxed{\begin{aligned} \frac{D\Delta}{Dt} + \frac{8}{d(d+2)}\frac{1}{\rho\theta} \left( 1 - \frac{d+2}{2}\Delta \right) \left( \frac{\partial q_i}{\partial x_i} + \sigma_{ij}\frac{\partial v_i}{\partial x_j} \right) \\ + \frac{1}{d(d+2)}\frac{1}{\rho\theta^2} \left[ \frac{\partial\varphi_i}{\partial x_i} + 4(d+2)q_i\frac{\partial\theta}{\partial x_i} - 8\frac{q_i}{\rho} \left( \frac{\partial\sigma_{ij}}{\partial x_j} + \theta\frac{\partial\rho}{\partial x_i} \right) + 4R_{ij}\frac{\partial v_i}{\partial x_j} \right] \\ = \frac{1}{d(d+2)}\frac{1}{\rho\theta^2} \left[ \mathcal{P}^2 - 2(d+2)(1+\Delta)\theta\mathcal{P}^1 \right], \end{aligned}} \quad (2.5)$$

2.5.  $R_{ij}$  balance equation

The  $R_{ij}$  balance equation follows from the  $u_{ij}^1$  balance equation (1.20), which on inserting the new variables (2.1) reads

$$\begin{aligned}
& \frac{DR_{ij}}{Dt} + (d+4)\theta \frac{D\sigma_{ij}}{Dt} + (d+4)\sigma_{ij} \frac{D\theta}{Dt} + \frac{\partial u_{ijk}^1}{\partial x_k} \\
& + \frac{2}{d+2} \frac{\partial \varphi_{\langle i}}{\partial x_{j\rangle}} + \frac{8(d+4)}{d+2} \theta \frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + \frac{8(d+4)}{d+2} q_{\langle i} \frac{\partial \theta}{\partial x_{j\rangle}} + [R_{ij} + (d+4)\theta \sigma_{ij}] \frac{\partial v_k}{\partial x_k} \\
& - 2 \frac{m_{ijk}}{\rho} \left( \frac{\partial \sigma_{kl}}{\partial x_l} + \rho \frac{\partial \theta}{\partial x_k} + \theta \frac{\partial \rho}{\partial x_k} \right) - \frac{4(d+4)}{d+2} \frac{q_{\langle i}}{\rho} \left( \frac{\partial \sigma_{j\rangle k}}{\partial x_k} + \rho \frac{\partial \theta}{\partial x_{j\rangle}} + \theta \frac{\partial \rho}{\partial x_{j\rangle}} \right) \\
& + 2u_{ijkl}^0 \frac{\partial v_k}{\partial x_l} + \frac{6}{d+4} R_{\langle ij} \frac{\partial v_{k\rangle}}{\partial x_k} + 6\theta \sigma_{ij} \frac{\partial v_{k\rangle}}{\partial x_k} + \frac{4}{d+2} R_{k\langle i} \frac{\partial v_k}{\partial x_{j\rangle}} + \frac{4(d+4)}{d+2} \theta \sigma_{k\langle i} \frac{\partial v_k}{\partial x_{j\rangle}} \\
& + 2R_{k\langle i} \frac{\partial v_{j\rangle}}{\partial x_k} + 2(d+4)\theta \sigma_{k\langle i} \frac{\partial v_{j\rangle}}{\partial x_k} + 2(d+4)(1+\Delta)\rho\theta^2 \frac{\partial v_{\langle i}}{\partial x_{j\rangle}} = \mathcal{P}_{ij}^1.
\end{aligned}$$

Replacing the time derivatives of  $\theta$  and  $\sigma_{ij}$  using the energy balance equation (1.9) and the stress balance equation (2.2), one has

$$\begin{aligned}
& \frac{DR_{ij}}{Dt} - (d+4)\theta \left[ \frac{\partial m_{ijk}}{\partial x_k} + \frac{4}{d+2} \frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + \sigma_{ij} \frac{\partial v_k}{\partial x_k} + 2\sigma_{k\langle i} \frac{\partial v_{j\rangle}}{\partial x_k} + 2\rho\theta \frac{\partial v_{\langle i}}{\partial x_{j\rangle}} - \mathcal{P}_{ij}^0 \right] \\
& - (d+4)\sigma_{ij} \left[ \frac{2}{d} \frac{1}{\rho} \left( \frac{\partial q_k}{\partial x_k} + \sigma_{kl} \frac{\partial v_k}{\partial x_l} + \rho\theta \frac{\partial v_k}{\partial x_k} \right) - \frac{1}{d} \frac{1}{\rho} \mathcal{P}^1 \right] \\
& + \frac{\partial u_{ijk}^1}{\partial x_k} + \frac{2}{d+2} \frac{\partial \varphi_{\langle i}}{\partial x_{j\rangle}} + \frac{8(d+4)}{d+2} \theta \frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + \frac{4(d+4)}{d+2} q_{\langle i} \frac{\partial \theta}{\partial x_{j\rangle}} + [R_{ij} + (d+4)\theta \sigma_{ij}] \frac{\partial v_k}{\partial x_k} \\
& - 2 \frac{m_{ijk}}{\rho} \left( \frac{\partial \sigma_{kl}}{\partial x_l} + \rho \frac{\partial \theta}{\partial x_k} + \theta \frac{\partial \rho}{\partial x_k} \right) - \frac{4(d+4)}{d+2} \frac{q_{\langle i}}{\rho} \left( \frac{\partial \sigma_{j\rangle k}}{\partial x_k} + \theta \frac{\partial \rho}{\partial x_{j\rangle}} \right) \\
& + 2u_{ijkl}^0 \frac{\partial v_k}{\partial x_l} + \frac{6}{d+4} R_{\langle ij} \frac{\partial v_{k\rangle}}{\partial x_k} + 6\theta \sigma_{\langle ij} \frac{\partial v_{k\rangle}}{\partial x_k} + \frac{4}{d+2} R_{k\langle i} \frac{\partial v_k}{\partial x_{j\rangle}} + \frac{4(d+4)}{d+2} \theta \sigma_{k\langle i} \frac{\partial v_k}{\partial x_{j\rangle}} \\
& + 2R_{k\langle i} \frac{\partial v_{j\rangle}}{\partial x_k} + 2(d+4)\theta \sigma_{k\langle i} \frac{\partial v_{j\rangle}}{\partial x_k} + 2(d+4)(1+\Delta)\rho\theta^2 \frac{\partial v_{\langle i}}{\partial x_{j\rangle}} = \mathcal{P}_{ij}^1
\end{aligned}$$

or

$$\begin{aligned}
& \frac{DR_{ij}}{Dt} + \frac{2}{d+2} \frac{\partial \varphi_{\langle i}}{\partial x_{j\rangle}} + \frac{4(d+4)}{d+2} \theta \frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + \frac{4(d+4)}{d+2} q_{\langle i} \frac{\partial \theta}{\partial x_{j\rangle}} - \frac{4(d+4)}{d+2} \frac{\theta}{\rho} q_{\langle i} \frac{\partial \rho}{\partial x_{j\rangle}} \\
& - \frac{4(d+4)}{d+2} \frac{q_{\langle i}}{\rho} \frac{\partial \sigma_{j\rangle k}}{\partial x_k} + \frac{4(d+4)}{d+2} \theta \sigma_{k\langle i} \frac{\partial v_k}{\partial x_{j\rangle}} + 6\theta \sigma_{\langle ij} \frac{\partial v_{k\rangle}}{\partial x_k} - \frac{2(d+4)}{d} \theta \sigma_{ij} \frac{\partial v_k}{\partial x_k} \\
& - \frac{2(d+4)}{d} \frac{\sigma_{ij}}{\rho} \frac{\partial q_k}{\partial x_k} - \frac{2(d+4)}{d} \frac{\sigma_{ij}}{\rho} \sigma_{kl} \frac{\partial v_k}{\partial x_l} - (d+4)\theta \frac{\partial m_{ijk}}{\partial x_k} \\
& - 2 \frac{m_{ijk}}{\rho} \left( \frac{\partial \sigma_{kl}}{\partial x_l} + \rho \frac{\partial \theta}{\partial x_k} + \theta \frac{\partial \rho}{\partial x_k} \right) + \frac{\partial u_{ijk}^1}{\partial x_k} + 2u_{ijkl}^0 \frac{\partial v_k}{\partial x_l} \\
& + \frac{6}{d+4} R_{\langle ij} \frac{\partial v_{k\rangle}}{\partial x_k} + \frac{4}{d+2} R_{k\langle i} \frac{\partial v_k}{\partial x_{j\rangle}} + 2R_{k\langle i} \frac{\partial v_{j\rangle}}{\partial x_k} + R_{ij} \frac{\partial v_k}{\partial x_k} + 2(d+4)\Delta\rho\theta^2 \frac{\partial v_{\langle i}}{\partial x_{j\rangle}} \\
& = \mathcal{P}_{ij}^1 - (d+4)\theta \mathcal{P}_{ij}^0 - \frac{d+4}{d} \frac{\sigma_{ij}}{\rho} \mathcal{P}^1. \tag{2.6}
\end{aligned}$$

For further simplification, one can use the following relation stemming from the definition of the traceless parts of rank two and three tensors (see equations (A1) and (A2) of the main paper):

$$\begin{aligned}
& \sigma_{\langle ij} \frac{\partial v_k \rangle}{\partial x_k} \\
&= \frac{1}{3} \left( \sigma_{ij} \frac{\partial v_k}{\partial x_k} + \sigma_{ki} \frac{\partial v_j}{\partial x_k} + \sigma_{kj} \frac{\partial v_i}{\partial x_k} \right) - \frac{2}{3(d+2)} \left( \sigma_{kl} \frac{\partial v_l}{\partial x_k} \delta_{ij} + \sigma_{li} \frac{\partial v_l}{\partial x_k} \delta_{jk} + \sigma_{lj} \frac{\partial v_l}{\partial x_k} \delta_{ik} \right) \\
&= \frac{1}{3} \left( \sigma_{ij} \frac{\partial v_k}{\partial x_k} + 2\sigma_{k\langle i} \frac{\partial v_{j\rangle}}{\partial x_k} + \frac{2}{d} \sigma_{kl} \frac{\partial v_l}{\partial x_k} \delta_{ij} \right) - \frac{2}{3(d+2)} \left( \sigma_{kl} \frac{\partial v_l}{\partial x_k} \delta_{ij} + \sigma_{li} \frac{\partial v_l}{\partial x_j} + \sigma_{lj} \frac{\partial v_l}{\partial x_i} \right) \\
&= \frac{1}{3} \left( \sigma_{ij} \frac{\partial v_k}{\partial x_k} + 2\sigma_{k\langle i} \frac{\partial v_{j\rangle}}{\partial x_k} + \frac{2}{d} \sigma_{kl} \frac{\partial v_l}{\partial x_k} \delta_{ij} \right) \\
&\quad - \frac{2}{3(d+2)} \left( \sigma_{kl} \frac{\partial v_l}{\partial x_k} \delta_{ij} + 2\sigma_{l\langle i} \frac{\partial v_{j\rangle}}{\partial x_k} + \frac{2}{d} \sigma_{lk} \frac{\partial v_l}{\partial x_k} \delta_{ij} \right)
\end{aligned}$$

which on tidying up the terms reduces to

$$\sigma_{\langle ij} \frac{\partial v_k \rangle}{\partial x_k} = \frac{1}{3} \sigma_{ij} \frac{\partial v_k}{\partial x_k} + \frac{2}{3} \sigma_{k\langle i} \frac{\partial v_{j\rangle}}{\partial x_k} - \frac{4}{3(d+2)} \sigma_{k\langle i} \frac{\partial v_k \rangle}{\partial x_j}. \quad (2.7)$$

Using relation (2.7), the following terms in (2.6) can be rewritten as

$$\begin{aligned}
& \frac{4(d+4)}{d+2} \theta \sigma_{k\langle i} \frac{\partial v_k \rangle}{\partial x_j} + 6\theta \sigma_{\langle ij} \frac{\partial v_k \rangle}{\partial x_k} - \frac{2(d+4)}{d} \theta \sigma_{ij} \frac{\partial v_k}{\partial x_k} \\
&= 4\theta \sigma_{k\langle i} \frac{\partial v_k \rangle}{\partial x_j} + 4\theta \sigma_{k\langle i} \frac{\partial v_{j\rangle}}{\partial x_k} - \frac{8}{d} \theta \sigma_{ij} \frac{\partial v_k}{\partial x_k}.
\end{aligned}$$

Consequently, the  $R_{ij}$  balance equation reduces to

$$\begin{aligned}
& \frac{DR_{ij}}{Dt} + \frac{2}{d+2} \frac{\partial \varphi_{\langle i}}{\partial x_{j\rangle}} + \frac{4(d+4)}{d+2} \theta \frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + \frac{4(d+4)}{d+2} q_{\langle i} \frac{\partial \theta}{\partial x_{j\rangle}} - \frac{4(d+4)}{d+2} \frac{\theta}{\rho} q_{\langle i} \frac{\partial \rho}{\partial x_{j\rangle}} \\
& - \frac{4(d+4)}{d+2} \frac{q_{\langle i}}{\rho} \frac{\partial \sigma_{j\rangle k}}{\partial x_k} + 4\theta \sigma_{k\langle i} \frac{\partial v_k \rangle}{\partial x_{j\rangle}} + 4\theta \sigma_{k\langle i} \frac{\partial v_{j\rangle}}{\partial x_k} - \frac{8}{d} \theta \sigma_{ij} \frac{\partial v_k}{\partial x_k} - \frac{2(d+4)}{d} \frac{\sigma_{ij}}{\rho} \frac{\partial q_k}{\partial x_k} \\
& - \frac{2(d+4)}{d} \frac{\sigma_{ij} \sigma_{kl}}{\rho} \frac{\partial v_k}{\partial x_l} - (d+4) \theta \frac{\partial m_{ijk}}{\partial x_k} - 2 \frac{m_{ijk}}{\rho} \left( \frac{\partial \sigma_{kl}}{\partial x_l} + \rho \frac{\partial \theta}{\partial x_k} + \theta \frac{\partial \rho}{\partial x_k} \right) + \frac{\partial u_{ijk}^1}{\partial x_k} \\
& + 2u_{ijkl}^0 \frac{\partial v_k}{\partial x_l} + \frac{6}{d+4} R_{\langle ij} \frac{\partial v_k \rangle}{\partial x_k} + \frac{4}{d+2} R_{k\langle i} \frac{\partial v_k \rangle}{\partial x_{j\rangle}} + 2R_{k\langle i} \frac{\partial v_{j\rangle}}{\partial x_k} + R_{ij} \frac{\partial v_k}{\partial x_k} \\
& + 2(d+4) \Delta \rho \theta^2 \frac{\partial v_{\langle i}}{\partial x_{j\rangle}} = \mathcal{P}_{ij}^1 - (d+4) \theta \mathcal{P}_{ij}^0 - \frac{d+4}{d} \frac{\sigma_{ij}}{\rho} \mathcal{P}^1.
\end{aligned}$$

Note that this equation matches with eq. (7.3) of [Struchtrup \(2005\)](#). However, I shall further exploit a similar relation as (2.7) for the following term

$$R_{\langle ij} \frac{\partial v_k \rangle}{\partial x_k} = \frac{1}{3} R_{ij} \frac{\partial v_k}{\partial x_k} + \frac{2}{3} R_{k\langle i} \frac{\partial v_{j\rangle}}{\partial x_k} - \frac{4}{3(d+2)} R_{k\langle i} \frac{\partial v_k \rangle}{\partial x_{j\rangle}} \quad (2.8)$$

to obtain

$$\begin{aligned}
& \frac{DR_{ij}}{Dt} + \frac{2}{d+2} \frac{\partial \varphi_{\langle i}}{\partial x_j} + \frac{4(d+4)}{d+2} \theta \frac{\partial q_{\langle i}}{\partial x_j} + \frac{4(d+4)}{d+2} q_{\langle i} \frac{\partial \theta}{\partial x_j} - \frac{4(d+4)}{d+2} \frac{\theta}{\rho} q_{\langle i} \frac{\partial \rho}{\partial x_j} \\
& - \frac{4(d+4)}{d+2} \frac{q_{\langle i}}{\rho} \frac{\partial \sigma_{j\rangle k}}{\partial x_k} + 4\theta \sigma_{k\langle i} \frac{\partial v_k}{\partial x_j} + 4\theta \sigma_{k\langle i} \frac{\partial v_j}{\partial x_k} - \frac{8}{d} \theta \sigma_{ij} \frac{\partial v_k}{\partial x_k} - \frac{2(d+4)}{d} \frac{\sigma_{ij}}{\rho} \frac{\partial q_k}{\partial x_k} \\
& - \frac{2(d+4)}{d} \frac{\sigma_{ij} \sigma_{kl}}{\rho} \frac{\partial v_k}{\partial x_l} - (d+4)\theta \frac{\partial m_{ijk}}{\partial x_k} - 2 \frac{m_{ijk}}{\rho} \left( \frac{\partial \sigma_{kl}}{\partial x_l} + \rho \frac{\partial \theta}{\partial x_k} + \theta \frac{\partial \rho}{\partial x_k} \right) + \frac{\partial u_{ijk}^1}{\partial x_k} \\
& + 2u_{ijkl}^0 \frac{\partial v_k}{\partial x_l} + \frac{6}{d+4} \left( \frac{1}{3} R_{ij} \frac{\partial v_k}{\partial x_k} + \frac{2}{3} R_{k\langle i} \frac{\partial v_j \rangle}{\partial x_k} - \frac{4}{3(d+2)} R_{k\langle i} \frac{\partial v_k \rangle}{\partial x_j} \right) + \frac{4}{d+2} R_{k\langle i} \frac{\partial v_k}{\partial x_j} \\
& + 2R_{k\langle i} \frac{\partial v_j \rangle}{\partial x_k} + R_{ij} \frac{\partial v_k}{\partial x_k} + 2(d+4)\Delta \rho \theta^2 \frac{\partial v_{\langle i}}{\partial x_j} = \mathcal{P}_{ij}^1 - (d+4)\theta \mathcal{P}_{ij}^0 - \frac{d+4}{d} \frac{\sigma_{ij}}{\rho} \mathcal{P}^1.
\end{aligned}$$

On tidying up the terms, the  $R_{ij}$  balance equation reads

$$\begin{aligned}
& \frac{DR_{ij}}{Dt} + \frac{2}{d+2} \frac{\partial \varphi_{\langle i}}{\partial x_j} + \frac{4(d+4)}{d+2} \left( \theta \frac{\partial q_{\langle i}}{\partial x_j} + q_{\langle i} \frac{\partial \theta}{\partial x_j} - \frac{q_{\langle i}}{\rho} \frac{\partial \sigma_{j\rangle k}}{\partial x_k} - \frac{\theta}{\rho} q_{\langle i} \frac{\partial \rho}{\partial x_j} \right) \\
& + 4\theta \sigma_{k\langle i} \frac{\partial v_k}{\partial x_j} + 4\theta \sigma_{k\langle i} \frac{\partial v_j \rangle}{\partial x_k} - \frac{8}{d} \theta \sigma_{ij} \frac{\partial v_k}{\partial x_k} - \frac{2(d+4)}{d} \frac{\sigma_{ij}}{\rho} \left( \frac{\partial q_k}{\partial x_k} + \sigma_{kl} \frac{\partial v_k}{\partial x_l} \right) \\
& + \frac{\partial u_{ijk}^1}{\partial x_k} - (d+4)\theta \frac{\partial m_{ijk}}{\partial x_k} - 2 \frac{m_{ijk}}{\rho} \left( \frac{\partial \sigma_{kl}}{\partial x_l} + \rho \frac{\partial \theta}{\partial x_k} + \theta \frac{\partial \rho}{\partial x_k} \right) + 2u_{ijkl}^0 \frac{\partial v_k}{\partial x_l} \\
& + \frac{d+6}{d+4} \left( R_{ij} \frac{\partial v_k}{\partial x_k} + 2R_{k\langle i} \frac{\partial v_j \rangle}{\partial x_k} \right) + \frac{4}{d+4} R_{k\langle i} \frac{\partial v_k}{\partial x_j} + 2(d+4)\Delta \rho \theta^2 \frac{\partial v_{\langle i}}{\partial x_j} \\
& = \mathcal{P}_{ij}^1 - (d+4)\theta \mathcal{P}_{ij}^0 - \frac{d+4}{d} \frac{\sigma_{ij}}{\rho} \mathcal{P}^1,
\end{aligned} \tag{2.9}$$

## 2.6. $\varphi_i$ balance equation

The  $\varphi_i$  balance equation follows from the  $u_i^2$  balance equation (1.23), which on inserting the new variables (2.1) reads

$$\begin{aligned}
& \frac{D\varphi_i}{Dt} + 4(d+4)\theta \frac{Dq_i}{Dt} + 4(d+4)q_i \frac{D\theta}{Dt} + \frac{\partial u_{ij}^2}{\partial x_j} + \frac{1}{d} \frac{\partial u^3}{\partial x_i} \\
& - \frac{d+4}{d} \frac{d(d+2)\rho \theta^2 (1+\Delta)}{\rho} \left( \frac{\partial \sigma_{ij}}{\partial x_j} + \rho \frac{\partial \theta}{\partial x_i} + \theta \frac{\partial \rho}{\partial x_i} \right) \\
& - 4 \frac{R_{ij} + (d+4)\theta \sigma_{ij}}{\rho} \left( \frac{\partial \sigma_{jk}}{\partial x_k} + \rho \frac{\partial \theta}{\partial x_j} + \theta \frac{\partial \rho}{\partial x_j} \right) \\
& + 4u_{ijk}^1 \frac{\partial v_k}{\partial x_j} + \frac{d+6}{d+2} [\varphi_i + 4(d+4)\theta q_i] \frac{\partial v_j}{\partial x_j} + \frac{d+6}{d+2} [\varphi_j + 4(d+4)\theta q_j] \frac{\partial v_i}{\partial x_j} \\
& + \frac{4}{d+2} [\varphi_j + 4(d+4)\theta q_j] \frac{\partial v_j}{\partial x_i} = \mathcal{P}_i^2.
\end{aligned}$$

Replacing the time derivatives of  $\theta$  and  $q_i$  using the energy balance equation (1.9) and the heat flux balance equation (2.3), one has

$$\begin{aligned}
\frac{D\varphi_i}{Dt} - 4(d+4)\theta \left[ \frac{1}{2} \frac{\partial R_{ij}}{\partial x_j} + \frac{d+2}{2} \left\{ \rho\theta^2 \frac{\partial \Delta}{\partial x_i} + \Delta\theta^2 \frac{\partial \rho}{\partial x_i} + (1+2\Delta)\rho\theta \frac{\partial \theta}{\partial x_i} + \sigma_{ij} \frac{\partial \theta}{\partial x_j} \right\} \right. \\
+ \theta \frac{\partial \sigma_{ij}}{\partial x_j} - \frac{\sigma_{ij}}{\rho} \frac{\partial \sigma_{jk}}{\partial x_k} - \frac{\sigma_{ij}}{\rho} \theta \frac{\partial \rho}{\partial x_j} + m_{ijk} \frac{\partial v_j}{\partial x_k} + \frac{d+4}{d+2} q_i \frac{\partial v_j}{\partial x_j} + \frac{d+4}{d+2} q_j \frac{\partial v_i}{\partial x_j} \\
+ \frac{2}{d+2} q_j \frac{\partial v_j}{\partial x_i} - \frac{1}{2} \mathcal{P}_i^1 \left. \right] + 4(d+4)q_i \left[ \frac{1}{d} \frac{1}{\rho} \mathcal{P}^1 - \frac{2}{d} \frac{1}{\rho} \left( \frac{\partial q_j}{\partial x_j} + \sigma_{jk} \frac{\partial v_j}{\partial x_k} + \rho\theta \frac{\partial v_j}{\partial x_j} \right) \right] \\
+ \frac{\partial u_{ij}^2}{\partial x_j} + \frac{1}{d} \frac{\partial u^3}{\partial x_i} - (d+2)(d+4)(1+\Delta)\theta^2 \left( \frac{\partial \sigma_{ij}}{\partial x_j} + \rho \frac{\partial \theta}{\partial x_i} + \theta \frac{\partial \rho}{\partial x_i} \right) \\
- 4 \frac{R_{ij} + (d+4)\theta \sigma_{ij}}{\rho} \left( \frac{\partial \sigma_{jk}}{\partial x_k} + \rho \frac{\partial \theta}{\partial x_j} + \theta \frac{\partial \rho}{\partial x_j} \right) \\
+ 4u_{ijk}^1 \frac{\partial v_k}{\partial x_j} + 4(d+4)\theta \left[ \frac{d+6}{d+2} q_i \frac{\partial v_j}{\partial x_j} + \frac{d+6}{d+2} q_j \frac{\partial v_i}{\partial x_j} + \frac{4}{d+2} q_j \frac{\partial v_j}{\partial x_i} \right] \\
+ \frac{d+6}{d+2} \varphi_i \frac{\partial v_j}{\partial x_j} + \frac{d+6}{d+2} \varphi_j \frac{\partial v_i}{\partial x_j} + \frac{4}{d+2} \varphi_j \frac{\partial v_j}{\partial x_i} = \mathcal{P}_i^2.
\end{aligned}$$

On tidying up the terms, the  $\varphi_i$  balance equation reads

$$\begin{aligned}
\frac{D\varphi_i}{Dt} - \frac{8(d+4)}{d} \frac{q_i}{\rho} \left( \frac{\partial q_j}{\partial x_j} + \sigma_{jk} \frac{\partial v_j}{\partial x_k} + \rho\theta \frac{\partial v_j}{\partial x_j} \right) + \frac{\partial u_{ij}^2}{\partial x_j} + \frac{1}{d} \frac{\partial u^3}{\partial x_i} - 2(d+4)\theta \frac{\partial R_{ij}}{\partial x_j} \\
- 4R_{ij} \frac{\partial \theta}{\partial x_j} - (d+4)[(d+6) + (d+2)\Delta]\theta^2 \frac{\partial \sigma_{ij}}{\partial x_j} - 2(d+4)^2 \theta \sigma_{ij} \frac{\partial \theta}{\partial x_j} \\
- (d+2)(d+4) \left[ 2\rho\theta^3 \frac{\partial \Delta}{\partial x_i} + (1+3\Delta)\theta^3 \frac{\partial \rho}{\partial x_i} + (3+5\Delta)\rho\theta^2 \frac{\partial \theta}{\partial x_i} \right] \\
- 4 \frac{R_{ij}}{\rho} \left( \frac{\partial \sigma_{jk}}{\partial x_k} + \theta \frac{\partial \rho}{\partial x_j} \right) + 4u_{ijk}^1 \frac{\partial v_j}{\partial x_k} - 4(d+4)\theta m_{ijk} \frac{\partial v_j}{\partial x_k} \\
+ \frac{8(d+4)}{d+2} \theta \left( q_i \frac{\partial v_j}{\partial x_j} + q_j \frac{\partial v_i}{\partial x_j} + q_j \frac{\partial v_j}{\partial x_i} \right) + \frac{d+6}{d+2} \varphi_i \frac{\partial v_j}{\partial x_j} + \frac{d+6}{d+2} \varphi_j \frac{\partial v_i}{\partial x_j} \\
+ \frac{4}{d+2} \varphi_j \frac{\partial v_j}{\partial x_i} = \mathcal{P}_i^2 - 2(d+4)\theta \mathcal{P}_i^1 - \frac{4(d+4)}{d} \frac{q_i}{\rho} \mathcal{P}^1
\end{aligned} \tag{2.10}$$

### 3. Grad closure for the 29 moment equations

Clearly, the system of 29 moment equations in new variables (eqs. (1.3), (1.6), (1.8), (2.2), (2.3), (2.4), (2.5), (2.9) and (2.10)) derived above is not closed since it contains the additional unknowns  $u_{ijk}^0$ ,  $u_{ijk}^1$ ,  $u_{ij}^2$  and  $u^3$ . The system is closed using the G29 distribution function which reads

$$\begin{aligned}
f_{|G29} = f_M \left[ 1 + \frac{1}{2} \frac{\sigma_{ij} C_i C_j}{\rho\theta^2} + \frac{q_i C_i}{\rho\theta^2} \left( \frac{1}{d+2} \frac{C^2}{\theta} - 1 \right) + \frac{1}{6} \frac{m_{ijk} C_i C_j C_k}{\rho\theta^3} \right. \\
+ \frac{d(d+2)\Delta}{8} \left( 1 - \frac{2}{d} \frac{C^2}{\theta} + \frac{1}{d(d+2)} \frac{C^4}{\theta^2} \right) + \frac{1}{4} \frac{R_{ij} C_i C_j}{\rho\theta^3} \left( \frac{1}{d+4} \frac{C^2}{\theta} - 1 \right) \\
\left. + \frac{1}{8} \frac{\varphi_i C_i}{\rho\theta^3} \left( 1 - \frac{2}{d+2} \frac{C^2}{\theta} + \frac{1}{(d+2)(d+4)} \frac{C^4}{\theta^2} \right) \right], \tag{3.1}
\end{aligned}$$

where

$$f_M \equiv f_M(t, \mathbf{x}, \mathbf{c}) = n \left( \frac{1}{2\pi\theta} \right)^{d/2} \exp \left( -\frac{C^2}{2\theta} \right) \quad (3.2)$$

is the Maxwellian distribution function (see appendix D of the main paper for the derivation of the G29 distribution function).

### 3.1. Closure for $u_{ijkl}^0$

Ignoring the vanishing integrals and using identities (A 11) and (A 13), it turns out that

$$\boxed{u_{ijkl}^0|_{\text{G29}} = m \int C_{\langle i} C_j C_k C_l \rangle f|_{\text{G29}} d\mathbf{c} = 0} \quad (3.3)$$

### 3.2. Closure for $u_{ijk}^1$

Ignoring the vanishing integrals and using identities (A 7), (A 8) and (A 11), it turns out that

$$\begin{aligned} u_{ijk}^1|_{\text{G29}} &= m \int C^2 C_{\langle i} C_j C_k \rangle f|_{\text{G29}} d\mathbf{c} \\ &= \frac{1}{6} \frac{m_{lrs}}{\rho\theta^3} m \int C^2 C_{\langle i} C_j C_k \rangle C_l C_r C_s f_M d\mathbf{c} \\ &= \frac{1}{6} \frac{m_{lrs}}{\rho\theta^3} m \frac{\rho}{m} \frac{1}{(2\pi\theta)^{d/2}} \int C^2 \left[ C_i C_j C_k - \frac{1}{d+2} C^2 (C_i \delta_{jk} + C_j \delta_{ik} + C_k \delta_{ij}) \right] \\ &\quad \times C_l C_r C_s e^{-C^2/(2\theta)} d\mathbf{c} \\ &= \frac{1}{6} \frac{m_{lrs}}{\theta^3} \frac{1}{(2\pi\theta)^{d/2}} \left[ \frac{15}{d(d+2)(d+4)} \delta_{(ij} \delta_{kl} \delta_{rs)} \right. \\ &\quad \left. - \frac{3}{d(d+2)} \left\{ \delta_{(il} \delta_{rs)} \delta_{jk} + \delta_{(jl} \delta_{rs)} \delta_{ik} + \delta_{(kl} \delta_{rs)} \delta_{ij} \right\} \right] \int C^8 e^{-C^2/(2\theta)} d\mathbf{C} \\ &= \frac{1}{6} \frac{1}{\theta^3} \frac{1}{(2\pi\theta)^{d/2}} \frac{6 m_{ijk}}{d(d+2)(d+4)} \int C^8 e^{-C^2/(2\theta)} d\mathbf{C} \\ &= \frac{1}{d(d+2)(d+4)} \frac{m_{ijk}}{\theta^3} \frac{1}{(2\pi\theta)^{d/2}} \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \int_0^\infty C^{d+7} e^{-C^2/(2\theta)} dC \\ &= \frac{1}{d(d+2)(d+4)} \frac{m_{ijk}}{\theta^3} \frac{1}{(2\theta)^{d/2}} \frac{2}{\Gamma(\frac{d}{2})} \times 2^{(d/2)+3} \theta^{(d/2)+4} \Gamma\left(\frac{d}{2} + 4\right) \\ &= \frac{1}{d(d+2)(d+4)} \theta m_{ijk} (d+6)(d+4)(d+2)d \end{aligned}$$

or

$$\boxed{u_{ijk}^1|_{\text{G29}} = (d+6) \theta m_{ijk}} \quad (3.4)$$

3.3. Closure for  $u_{ij}^2$ 

Ignoring the vanishing integrals and using identities (A 6), (A 7) and (A 10), it turns out that

$$\begin{aligned}
& u_{ij|G29}^2 \\
&= m \int C^4 C_{\langle i} C_{j \rangle} f_{|G29} d\mathbf{c} \\
&= m \int C^4 C_{\langle i} C_{j \rangle} f_M \left[ \frac{1}{2} \frac{\sigma_{kl} C_k C_l}{\rho \theta^2} + \frac{1}{4} \frac{R_{kl} C_k C_l}{\rho \theta^3} \left( \frac{1}{d+4} \frac{C^2}{\theta} - 1 \right) \right] d\mathbf{c} \\
&= m \int C^4 C_{\langle i} C_{j \rangle} C_k C_l f_M \left[ \frac{1}{2} \frac{\sigma_{kl}}{\rho \theta^2} + \frac{1}{4} \frac{R_{kl}}{\rho \theta^3} \left( \frac{1}{d+4} \frac{C^2}{\theta} - 1 \right) \right] d\mathbf{c} \\
&= m \frac{\rho}{m} \frac{1}{(2\pi\theta)^{d/2}} \int C^4 \left( C_i C_j - \frac{1}{d} C^2 \delta_{ij} \right) C_k C_l e^{-C^2/(2\theta)} \\
&\quad \times \left[ \frac{1}{2} \frac{\sigma_{kl}}{\rho \theta^2} + \frac{1}{4} \frac{R_{kl}}{\rho \theta^3} \left( \frac{1}{d+4} \frac{C^2}{\theta} - 1 \right) \right] d\mathbf{c} \\
&= \frac{1}{2} \frac{\sigma_{kl}}{\theta^2} \frac{1}{(2\pi\theta)^{d/2}} \left[ \frac{1}{d(d+2)} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) - \frac{1}{d^2} \delta_{ij} \delta_{kl} \right] \int C^8 e^{-C^2/(2\theta)} d\mathbf{C} \\
&\quad + \frac{1}{4} \frac{R_{kl}}{\theta^3} \frac{1}{(2\pi\theta)^{d/2}} \left[ \frac{1}{d(d+2)} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) - \frac{1}{d^2} \delta_{ij} \delta_{kl} \right] \\
&\quad \times \left( \frac{1}{d+4} \frac{1}{\theta} \int C^{10} e^{-C^2/(2\theta)} d\mathbf{C} - \int C^8 e^{-C^2/(2\theta)} d\mathbf{C} \right) \\
&= \frac{1}{2} \frac{2\sigma_{ij}}{d(d+2)\theta^2} \frac{1}{(2\pi\theta)^{d/2}} \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \int_0^\infty C^{d+7} e^{-C^2/(2\theta)} dC \\
&\quad + \frac{1}{4} \frac{2R_{ij}}{d(d+2)\theta^3} \frac{1}{(2\pi\theta)^{d/2}} \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \\
&\quad \times \left( \frac{1}{d+4} \frac{1}{\theta} \int_0^\infty C^{d+9} e^{-C^2/(2\theta)} dC - \int_0^\infty C^{d+7} e^{-C^2/(2\theta)} dC \right) \\
&= \frac{1}{d(d+2)} \frac{\sigma_{ij}}{\theta^2} \frac{1}{(2\theta)^{d/2}} \frac{2}{\Gamma(\frac{d}{2})} \times 2^{(d/2)+3} \theta^{(d/2)+4} \Gamma\left(\frac{d}{2} + 4\right) \\
&\quad + \frac{1}{2d(d+2)} \frac{R_{ij}}{\theta^3} \frac{1}{(2\theta)^{d/2}} \frac{2}{\Gamma(\frac{d}{2})} \\
&\quad \times \left[ \frac{1}{d+4} \frac{1}{\theta} \times 2^{(d/2)+4} \theta^{(d/2)+5} \Gamma\left(\frac{d}{2} + 5\right) - 2^{(d/2)+3} \theta^{(d/2)+4} \Gamma\left(\frac{d}{2} + 4\right) \right] \\
&= \frac{1}{d(d+2)} \theta^2 \sigma_{ij} \times (d+6)(d+4)(d+2)d + \frac{1}{2d(d+2)} \theta R_{ij} \times 4(d+6)(d+2)d \\
&= (d+6)(d+4)\theta^2 \sigma_{ij} + 2(d+6)\theta R_{ij}
\end{aligned}$$

or

$$\boxed{u_{ij|G29}^2 = (d+6)\theta[2R_{ij} + (d+4)\theta\sigma_{ij}]} \quad (3.5)$$

3.4. Closure for  $u^3$ 

Ignoring the vanishing integrals and using identity (A 10), it turns out that

$$\begin{aligned}
 u_{|G29}^3 &= m \int C^6 f_{|G29} d\mathbf{c} \\
 &= m \int C^6 f_M \left[ 1 + \frac{d(d+2)\Delta}{8} \left( 1 - \frac{2}{d} \frac{C^2}{\theta} + \frac{1}{d(d+2)} \frac{C^4}{\theta^2} \right) \right] d\mathbf{c} \\
 &= m \frac{\rho}{m} \frac{1}{(2\pi\theta)^{d/2}} \int C^6 e^{-C^2/(2\theta)} \left[ 1 + \frac{d(d+2)\Delta}{8} \left( 1 - \frac{2}{d} \frac{C^2}{\theta} + \frac{1}{d(d+2)} \frac{C^4}{\theta^2} \right) \right] d\mathbf{C} \\
 &= \frac{\rho}{(2\pi\theta)^{d/2}} \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \int_0^\infty C^{d+5} e^{-C^2/(2\theta)} \left[ 1 + \frac{d(d+2)\Delta}{8} \left( 1 - \frac{2}{d} \frac{C^2}{\theta} + \frac{1}{d(d+2)} \frac{C^4}{\theta^2} \right) \right] dC \\
 &= \frac{\rho}{(2\theta)^{d/2}} \frac{2}{\Gamma\left(\frac{d}{2}\right)} \left[ \left( 1 + \frac{d(d+2)\Delta}{8} \right) \times 2^{(d/2)+2} \theta^{(d/2)+3} \Gamma\left(\frac{d}{2} + 3\right) \right. \\
 &\quad \left. - \frac{(d+2)\Delta}{4\theta} \times 2^{(d/2)+3} \theta^{(d/2)+4} \Gamma\left(\frac{d}{2} + 4\right) + \frac{\Delta}{8\theta^2} \times 2^{(d/2)+4} \theta^{(d/2)+5} \Gamma\left(\frac{d}{2} + 5\right) \right] \\
 &= \rho \theta \frac{2^3 \Gamma\left(\frac{d}{2} + 3\right)}{\Gamma\left(\frac{d}{2}\right)} \left[ \left( 1 + \frac{d(d+2)\Delta}{8} \right) \theta^2 - \frac{(d+2)\theta^2 \Delta}{4} (d+6) + \frac{\theta^2 \Delta}{8} (d+6)(d+8) \right] \\
 &= \rho \theta^3 (d+4)(d+2)d \left[ 1 + \frac{\Delta}{8} \left\{ d(d+2) - 2(d+2)(d+6) + (d+6)(d+8) \right\} \right] \\
 &= \rho \theta^3 (d+4)(d+2)d \left( 1 + \frac{\Delta}{8} \times 24 \right)
 \end{aligned}$$

or

$$u_{|G29}^3 = d(d+2)(d+4)(1+3\Delta)\rho\theta^3 \quad (3.6)$$

#### 4. The system of the 29 moment equations with closed left-hand sides

The left-hand sides of 29 moment equations in new variables (eqs. (1.3), (1.6), (1.8), (2.2), (2.3), (2.4), (2.5), (2.9) and (2.10)) derived above are closed with closures (3.3), (3.4), (3.5) and (3.6), which were obtained with the G29 distribution function (3.1). Note that the mass, momentum, energy, stress, heat flux and  $\Delta$  balance equations (eqs. (1.3), (1.6), (1.8), (2.2), (2.3) and (2.5)) will not change as they do not contain the additional unknowns. On the other hand, the left-hand sides the  $m_{ijk}$ ,  $R_{ij}$  and  $\varphi_i$  balance equations (eqs. (2.4), (2.9) and (2.10)) need to be closed using the G29 closures (3.3), (3.4), (3.5) and (3.6).

##### 4.1. $m_{ijk}$ balance equation with the closed left-hand side

The  $m_{ijk}$  balance equation (2.4) with the closed left-hand side reads

$$\begin{aligned}
 &\left[ \frac{Dm_{ijk}}{Dt} + \frac{3}{d+4} \frac{\partial R_{\langle ij}}{\partial x_{k\rangle}} + 3\theta \frac{\partial \sigma_{\langle ij}}{\partial x_{k\rangle}} - 3 \frac{\sigma_{\langle ij}}{\rho} \left( \frac{\partial \sigma_{k\rangle l}}{\partial x_l} + \theta \frac{\partial \rho}{\partial x_{k\rangle}} \right) \right. \\
 &\quad \left. + m_{ijk} \frac{\partial v_l}{\partial x_l} + 3m_{l\langle ij} \frac{\partial v_{k\rangle}}{\partial x_l} + \frac{12}{d+2} q_{\langle i} \frac{\partial v_{j\rangle}}{\partial x_{k\rangle}} \right] = \mathcal{P}_{ijk}^0
 \end{aligned} \quad (4.1)$$



4.2.  $R_{ij}$  balance equation with the closed left-hand side

The  $R_{ij}$  balance equation (2.9) on substituting closures (3.3)–(3.6) becomes

$$\begin{aligned}
\frac{DR_{ij}}{Dt} &+ \frac{2}{d+2} \frac{\partial \varphi_{\langle i}}{\partial x_{j\rangle}} + \frac{4(d+4)}{d+2} \left( \theta \frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + q_{\langle i} \frac{\partial \theta}{\partial x_{j\rangle}} - \frac{q_{\langle i}}{\rho} \frac{\partial \sigma_{j\rangle k}}{\partial x_k} - \frac{\theta}{\rho} q_{\langle i} \frac{\partial \rho}{\partial x_{j\rangle}} \right) \\
&+ 4\theta \sigma_{k\langle i} \frac{\partial v_k}{\partial x_{j\rangle}} + 4\theta \sigma_{k\langle i} \frac{\partial v_{j\rangle}}{\partial x_k} - \frac{8}{d} \theta \sigma_{ij} \frac{\partial v_k}{\partial x_k} - \frac{2(d+4)}{d} \frac{\sigma_{ij}}{\rho} \left( \frac{\partial q_k}{\partial x_k} + \sigma_{kl} \frac{\partial v_k}{\partial x_l} \right) \\
&+ (d+6) \left( \theta \frac{\partial m_{ijk}}{\partial x_k} + m_{ijk} \frac{\partial \theta}{\partial x_k} \right) - (d+4) \theta \frac{\partial m_{ijk}}{\partial x_k} \\
&- 2 \frac{m_{ijk}}{\rho} \left( \frac{\partial \sigma_{kl}}{\partial x_l} + \rho \frac{\partial \theta}{\partial x_k} + \theta \frac{\partial \rho}{\partial x_k} \right) + \frac{d+6}{d+4} \left( R_{ij} \frac{\partial v_k}{\partial x_k} + 2R_{k\langle i} \frac{\partial v_{j\rangle}}{\partial x_k} \right) \\
&+ \frac{4}{d+4} R_{k\langle i} \frac{\partial v_k}{\partial x_{j\rangle}} + 2(d+4) \Delta \rho \theta^2 \frac{\partial v_{\langle i}}{\partial x_{j\rangle}} = \mathcal{P}_{ij}^1 - (d+4) \theta \mathcal{P}_{ij}^0 - \frac{d+4}{d} \frac{\sigma_{ij}}{\rho} \mathcal{P}^1,
\end{aligned}$$

which on tidying up the terms give the  $R_{ij}$  balance equation with the closed left-hand side:

$$\begin{aligned}
&\frac{DR_{ij}}{Dt} + \frac{2}{d+2} \frac{\partial \varphi_{\langle i}}{\partial x_{j\rangle}} + \frac{4(d+4)}{d+2} \left( \theta \frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + q_{\langle i} \frac{\partial \theta}{\partial x_{j\rangle}} - \frac{q_{\langle i}}{\rho} \frac{\partial \sigma_{j\rangle k}}{\partial x_k} - \frac{\theta}{\rho} q_{\langle i} \frac{\partial \rho}{\partial x_{j\rangle}} \right) \\
&+ 4\theta \sigma_{k\langle i} \frac{\partial v_k}{\partial x_{j\rangle}} + 4\theta \sigma_{k\langle i} \frac{\partial v_{j\rangle}}{\partial x_k} - \frac{8}{d} \theta \sigma_{ij} \frac{\partial v_k}{\partial x_k} - \frac{2(d+4)}{d} \frac{\sigma_{ij}}{\rho} \left( \frac{\partial q_k}{\partial x_k} + \sigma_{kl} \frac{\partial v_k}{\partial x_l} \right) \\
&+ (d+4) m_{ijk} \frac{\partial \theta}{\partial x_k} + 2\theta \frac{\partial m_{ijk}}{\partial x_k} - 2 \frac{m_{ijk}}{\rho} \left( \frac{\partial \sigma_{kl}}{\partial x_l} + \theta \frac{\partial \rho}{\partial x_k} \right) \\
&+ \frac{d+6}{d+4} \left( R_{ij} \frac{\partial v_k}{\partial x_k} + 2R_{k\langle i} \frac{\partial v_{j\rangle}}{\partial x_k} \right) + \frac{4}{d+4} R_{k\langle i} \frac{\partial v_k}{\partial x_{j\rangle}} + 2(d+4) \Delta \rho \theta^2 \frac{\partial v_{\langle i}}{\partial x_{j\rangle}} \\
&= \mathcal{P}_{ij}^1 - (d+4) \theta \mathcal{P}_{ij}^0 - \frac{d+4}{d} \frac{\sigma_{ij}}{\rho} \mathcal{P}^1,
\end{aligned} \tag{4.2}$$

4.3.  $\varphi_i$  balance equation with the closed left-hand side

The  $\varphi_i$  balance equation (2.10) on substituting closures (3.3)–(3.6) becomes

$$\begin{aligned}
\frac{D\varphi_i}{Dt} &- \frac{8(d+4)}{d} \frac{q_i}{\rho} \left( \frac{\partial q_j}{\partial x_j} + \sigma_{jk} \frac{\partial v_j}{\partial x_k} + \rho \theta \frac{\partial v_j}{\partial x_j} \right) \\
&+ 2(d+6) \left( \theta \frac{\partial R_{ij}}{\partial x_j} + R_{ij} \frac{\partial \theta}{\partial x_j} \right) + (d+4)(d+6) \left( \theta^2 \frac{\partial \sigma_{ij}}{\partial x_j} + 2\theta \sigma_{ij} \frac{\partial \theta}{\partial x_j} \right) \\
&+ (d+2)(d+4) \left[ 3\rho \theta^3 \frac{\partial \Delta}{\partial x_i} + (1+3\Delta) \left( \theta^3 \frac{\partial \rho}{\partial x_i} + 3\rho \theta^2 \frac{\partial \theta}{\partial x_i} \right) \right] - 2(d+4) \theta \frac{\partial R_{ij}}{\partial x_j} \\
&- 4R_{ij} \frac{\partial \theta}{\partial x_j} - (d+4) [(d+6) + (d+2)\Delta] \theta^2 \frac{\partial \sigma_{ij}}{\partial x_j} - 2(d+4) \theta^2 \sigma_{ij} \frac{\partial \theta}{\partial x_j} \\
&- (d+2)(d+4) \left[ 2\rho \theta^3 \frac{\partial \Delta}{\partial x_i} + (1+3\Delta) \theta^3 \frac{\partial \rho}{\partial x_i} + (3+5\Delta) \rho \theta^2 \frac{\partial \theta}{\partial x_i} \right] \\
&- 4 \frac{R_{ij}}{\rho} \left( \frac{\partial \sigma_{jk}}{\partial x_k} + \theta \frac{\partial \rho}{\partial x_j} \right) + 4(d+6) \theta m_{ijk} \frac{\partial v_j}{\partial x_k} - 4(d+4) \theta m_{ijk} \frac{\partial v_j}{\partial x_k} \\
&+ \frac{8(d+4)}{d+2} \theta \left( q_i \frac{\partial v_j}{\partial x_j} + q_j \frac{\partial v_i}{\partial x_j} + q_j \frac{\partial v_j}{\partial x_i} \right) + \frac{d+6}{d+2} \varphi_i \frac{\partial v_j}{\partial x_j} + \frac{d+6}{d+2} \varphi_j \frac{\partial v_i}{\partial x_j}
\end{aligned}$$

$$+ \frac{4}{d+2} \varphi_j \frac{\partial v_j}{\partial x_i} = \mathcal{P}_i^2 - 2(d+4)\theta \mathcal{P}_i^1 - \frac{4(d+4)}{d} \frac{q_i}{\rho} \mathcal{P}^1,$$

which on tidying up the terms give the  $\varphi_i$  balance equation with the closed left-hand side:

$$\boxed{\begin{aligned} & \frac{D\varphi_i}{Dt} - \frac{8(d+4)}{d} \frac{q_i}{\rho} \left( \frac{\partial q_j}{\partial x_j} + \sigma_{jk} \frac{\partial v_j}{\partial x_k} + \rho \theta \frac{\partial v_j}{\partial x_j} \right) + 4\theta \frac{\partial R_{ij}}{\partial x_j} \\ & + (d+2)(d+4)\theta^2 \left[ \rho \theta \frac{\partial \Delta}{\partial x_i} + 4\Delta \rho \frac{\partial \theta}{\partial x_i} - \Delta \frac{\partial \sigma_{ij}}{\partial x_j} \right] \\ & - 4 \frac{R_{ij}}{\rho} \left( \frac{\partial \sigma_{jk}}{\partial x_k} + \theta \frac{\partial \rho}{\partial x_j} \right) + 2(d+4) R_{ij} \frac{\partial \theta}{\partial x_j} + 4(d+4) \theta \sigma_{ij} \frac{\partial \theta}{\partial x_j} + 8\theta m_{ijk} \frac{\partial v_j}{\partial x_k} \\ & + \frac{8(d+4)}{d+2} \theta \left( q_i \frac{\partial v_j}{\partial x_j} + q_j \frac{\partial v_i}{\partial x_j} + q_j \frac{\partial v_j}{\partial x_i} \right) + \frac{d+6}{d+2} \varphi_i \frac{\partial v_j}{\partial x_j} + \frac{d+6}{d+2} \varphi_j \frac{\partial v_i}{\partial x_j} \\ & + \frac{4}{d+2} \varphi_j \frac{\partial v_j}{\partial x_i} = \mathcal{P}_i^2 - 2(d+4)\theta \mathcal{P}_i^1 - \frac{4(d+4)}{d} \frac{q_i}{\rho} \mathcal{P}^1 \end{aligned}}$$

(4.3)

Thus the 29 moment equations (having the closed left-hand sides, which were closed using the G29 closures (3.3)–(3.6)) consists of equations (1.3), (1.6), (1.8), (2.2), (2.3), (4.1), (2.5), (4.2) and (4.3).

Finally, the collisional production terms on the right-hand sides of these moment equations are computed for inelastic Maxwell molecules (IMM), which are given in equations (3.11)–(3.17) of the main paper. Inserting the collisional production terms into the aforementioned system of the 29 moment equations (with the closed left-hand sides), one obtains the system of the G29 equations for IMM, which are given in (3.22)–(3.30) of the main paper.

## Appendix A. Some identities

$$C_i C_j = C_{\langle i} C_{j \rangle} + \frac{1}{d} C^2 \delta_{ij}, \quad (\text{A } 1)$$

$$C_i C_j C_k = C_{\langle i} C_j C_{k \rangle} + \frac{1}{d+2} C^2 (C_i \delta_{jk} + C_j \delta_{ik} + C_k \delta_{ij}), \quad (\text{A } 2)$$

$$C_{\langle i} C_{j \rangle} C_k = C_i C_j C_k - \frac{1}{d} C^2 C_k \delta_{ij} = C_{\langle i} C_j C_{k \rangle} + \frac{1}{d+2} C^2 \left[ C_i \delta_{jk} + C_j \delta_{ik} - \frac{2}{d} C_k \delta_{ij} \right]. \quad (\text{A } 3)$$

$$\begin{aligned} C_{\langle i} C_j C_{k \rangle} C_l &= \left[ C_i C_j C_k - \frac{1}{d+2} C^2 (C_i \delta_{jk} + C_j \delta_{ik} + C_k \delta_{ij}) \right] C_l \\ &= -\frac{1}{d+2} C^2 (C_i C_l \delta_{jk} + C_j C_l \delta_{ik} + C_k C_l \delta_{ij}) + C_i C_j C_k C_l \\ &= -\frac{1}{d+2} C^2 (C_i C_l \delta_{jk} + C_j C_l \delta_{ik} + C_k C_l \delta_{ij}) + C_{\langle i} C_j C_k C_{l \rangle} \\ &\quad + \frac{1}{d+4} C^2 [C_i C_j \delta_{kl} + C_i C_k \delta_{jl} + C_i C_l \delta_{jk} + C_j C_k \delta_{il} + C_j C_l \delta_{ik} + C_k C_l \delta_{ij}] \\ &\quad - \frac{1}{(d+2)(d+4)} C^4 (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \end{aligned}$$

On tidying up the terms the above equation reduces to

$$\begin{aligned}
C_{\langle i} C_j C_k \rangle C_l &= C_{\langle i} C_j C_k C_l \rangle + \frac{1}{d+4} C^2 \left( C_i C_j \delta_{kl} + C_i C_k \delta_{jl} + C_j C_k \delta_{il} \right) \\
&\quad - \frac{2}{(d+2)(d+4)} C^2 \left( C_i C_l \delta_{jk} + C_j C_l \delta_{ik} + C_k C_l \delta_{ij} \right) \\
&\quad - \frac{1}{(d+2)(d+4)} C^4 \left( \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \\
&= C_{\langle i} C_j C_k C_l \rangle + \frac{1}{d+4} C^2 \left( C_{\langle i} C_j \rangle \delta_{kl} + C_{\langle i} C_k \rangle \delta_{jl} + C_{\langle j} C_k \rangle \delta_{il} \right) \\
&\quad - \frac{2}{(d+2)(d+4)} C^2 \left( C_{\langle i} C_l \rangle \delta_{jk} + C_{\langle j} C_l \rangle \delta_{ik} + C_{\langle k} C_l \rangle \delta_{ij} \right). \quad (\text{A } 4)
\end{aligned}$$

#### A.1. Some integral identities

The integrals of type

$$\mathcal{J}_{i_1 i_2 \dots i_n} = \int C_{i_1} C_{i_2} \dots C_{i_n} h(C) d\mathbf{C}, \quad (\text{A } 5)$$

where  $h(C)$  is an even function, vanish when  $n$  is odd while for an even  $n$ , they are evaluated as follows.

- The tensorial form of the integrand of the integral  $\mathcal{J}_{ij}$  suggests that it should have the following form

$$\mathcal{J}_{ij} = \int C_i C_j h(C) d\mathbf{C} = \gamma_1 \delta_{ij} \int C^2 h(C) d\mathbf{C},$$

where the coefficients  $\gamma_1$  is unknown and computed as follows. Multiplying the above equation with  $\delta_{ij}$ , one gets

$$\int C^2 h(C) d\mathbf{C} = \gamma_1 d \int C^2 h(C) d\mathbf{C},$$

which gives  $\gamma_1 = 1/d$ . Hence,

$$\boxed{\mathcal{J}_{ij} = \int C_i C_j h(C) d\mathbf{C} = \frac{1}{d} \delta_{ij} \int C^2 h(C) d\mathbf{C}}, \quad (\text{A } 6)$$

- The tensorial form of the integrand of the integral  $\mathcal{J}_{ijkl}$  suggests that it should have the following form

$$\mathcal{J}_{ijkl} = \int C_i C_j C_k C_l h(C) d\mathbf{C} = \gamma_2 (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \int C^4 h(C) d\mathbf{C},$$

where the coefficients  $\gamma_2$  is unknown and computed as follows. Multiplying the above equation with  $\delta_{ij} \delta_{kl}$ , one gets

$$\int C^4 h(C) d\mathbf{C} = \gamma_2 (d^2 + d + d) \int C^4 h(C) d\mathbf{C},$$

which gives  $\gamma_2 = 1/[d(d+2)]$ . Hence,

$$\boxed{\mathcal{J}_{ijkl} = \int C_i C_j C_k C_l h(C) d\mathbf{C} = \frac{1}{d(d+2)} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \int C^4 h(C) d\mathbf{C}} \quad (\text{A } 7)$$

• The tensorial form of the integrand of the integral  $\mathcal{J}_{ijklrs}$  suggests that it should have the following form

$$\mathcal{J}_{ijklrs} = \int C_i C_j C_k C_l C_r C_s h(C) d\mathbf{C} = 15\gamma_3 \delta_{(ij} \delta_{kl} \delta_{rs)} \int C^6 h(C) d\mathbf{C},$$

where the coefficients  $\gamma_3$  is unknown and computed as follows. Multiplying the above equation with  $\delta_{ij} \delta_{kl} \delta_{rs}$ , one gets

$$\int C^6 h(C) d\mathbf{C} = \gamma_3 (d^3 + 6d^2 + 8d) \int C^6 h(C) d\mathbf{C},$$

which gives  $\gamma_2 = 1/[d(d+2)(d+4)]$ . Hence,

$$\mathcal{J}_{ijklrs} = \int C_i C_j C_k C_l C_r C_s h(C) d\mathbf{C} = \frac{15}{d(d+2)(d+4)} \delta_{(ij} \delta_{kl} \delta_{rs)} \int C^6 h(C) d\mathbf{C} \quad (\text{A } 8)$$

• Continuing in a similar way, for an even  $n$ , one finds that

$$\begin{aligned} \mathcal{J}_{i_1 i_2 \dots i_n} &= \int C_{i_1} C_{i_2} \dots C_{i_n} h(C) d\mathbf{C} \\ &= \frac{n!}{2^{n/2} (n/2)!} \frac{1}{\prod_{j=0}^{(n/2)-1} (d+2j)} \delta_{(i_1 i_2} \delta_{i_3 i_4} \dots \delta_{i_{n-1} i_n)} \int C^n h(C) d\mathbf{C} \end{aligned} \quad (\text{A } 9)$$

• Using (A 6),

$$\begin{aligned} \int C_{\langle i} C_{j \rangle} h(C) d\mathbf{C} &= \int C_i C_j h(C) d\mathbf{C} - \frac{1}{d} \delta_{ij} \int C^2 h(C) d\mathbf{C} \\ &= \frac{1}{d} \delta_{ij} \int C^2 h(C) d\mathbf{C} - \frac{1}{d} \delta_{ij} \int C^2 h(C) d\mathbf{C} \end{aligned}$$

or

$$\int C_{\langle i} C_{j \rangle} h(C) d\mathbf{C} = 0 \quad (\text{A } 10)$$

• Using (A 6) and (A 7),

$$\begin{aligned} &\int C_i C_{\langle j} C_k C_{l \rangle} h(C) d\mathbf{C} \\ &= \int C_i C_j C_k C_l h(C) d\mathbf{C} - \frac{1}{d+2} \int C_i C^2 (C_j \delta_{kl} + C_k \delta_{jl} + C_l \delta_{jk}) h(C) d\mathbf{C} \\ &= \frac{1}{d(d+2)} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \int C^4 h(C) d\mathbf{C} \\ &\quad - \frac{1}{d+2} \times \frac{1}{d} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \int C^4 h(C) d\mathbf{C} \end{aligned}$$

or

$$\int C_i C_{\langle j} C_k C_{l \rangle} h(C) d\mathbf{C} = 0 \quad (\text{A } 11)$$

- Using eqs. (A 6) and (A 7),

$$\begin{aligned}
& \int C_{\langle i} C_j C_k C_l \rangle h(C) d\mathbf{C} \\
&= \int \left[ C_i C_j C_k C_l - \frac{1}{d+4} C^2 \left( C_i C_j \delta_{kl} + C_i C_k \delta_{jl} + C_i C_l \delta_{jk} + C_j C_k \delta_{il} \right. \right. \\
&\quad \left. \left. + C_j C_l \delta_{ik} + C_k C_l \delta_{ij} \right) + \frac{1}{(d+2)(d+4)} C^4 \left( \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \right] h(C) d\mathbf{C} \\
&= \left[ \frac{1}{d(d+2)} - \frac{1}{d+4} \frac{2}{d} + \frac{1}{(d+2)(d+4)} \right] (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \int C^4 h(C) d\mathbf{C}
\end{aligned}$$

or

$$\boxed{\int C_{\langle i} C_j C_k C_l \rangle h(C) d\mathbf{C} = 0} \quad (\text{A } 12)$$

- Using eqs. (A 6), (A 7) and (A 8),

$$\begin{aligned}
& \int C_{\langle i} C_j C_k C_l \rangle C_r C_s h(C) d\mathbf{C} \\
&= \int \left[ C_i C_j C_k C_l - \frac{1}{d+4} C^2 \left( C_i C_j \delta_{kl} + C_i C_k \delta_{jl} + C_i C_l \delta_{jk} + C_j C_k \delta_{il} \right. \right. \\
&\quad \left. \left. + C_j C_l \delta_{ik} + C_k C_l \delta_{ij} \right) + \frac{1}{(d+2)(d+4)} C^4 \left( \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \right] C_r C_s h(C) d\mathbf{C} \\
&= \left[ \frac{15}{d(d+2)(d+4)} \delta_{(ij} \delta_{kl} \delta_{rs)} - \frac{3}{d(d+2)(d+4)} \left\{ \delta_{(ij} \delta_{rs)} \delta_{kl} + \delta_{(ik} \delta_{rs)} \delta_{jl} + \delta_{(il} \delta_{rs)} \delta_{jk} \right. \right. \\
&\quad \left. \left. + \delta_{(jk} \delta_{rs)} \delta_{il} + \delta_{(jl} \delta_{rs)} \delta_{ik} + \delta_{(kl} \delta_{rs)} \delta_{ij} \right\} + \frac{3}{d(d+2)(d+4)} \delta_{(ij} \delta_{kl} \delta_{rs)} \right] \int C^6 h(C) d\mathbf{C}
\end{aligned}$$

or

$$\boxed{\int C_{\langle i} C_j C_k C_l \rangle C_r C_s h(C) d\mathbf{C} = 0} \quad (\text{A } 13)$$

## Appendix B. Few terms in the $u_{ijk}^0$ balance equation (1.16)

### B.1. The term $m \int C_{\langle i} C_j C_k \rangle C_l f d\mathbf{c}$

Using identity (A 4), this term turns out to be

$$\begin{aligned}
& m \int C_{\langle i} C_j C_k \rangle C_l f d\mathbf{c} \\
&= u_{ijkl}^0 + \frac{1}{d+4} \left( u_{ij}^1 \delta_{kl} + u_{ik}^1 \delta_{jl} + u_{jk}^1 \delta_{il} \right) - \frac{2}{(d+2)(d+4)} \left( u_{il}^1 \delta_{jk} + u_{jl}^1 \delta_{ik} + u_{kl}^1 \delta_{ij} \right) \\
&= u_{ijkl}^0 + \frac{1}{d+4} \left[ u_{ij}^1 \delta_{kl} + u_{ik}^1 \delta_{jl} + u_{jk}^1 \delta_{il} - \frac{2}{d+2} \left( u_{il}^1 \delta_{jk} + u_{jl}^1 \delta_{ik} + u_{kl}^1 \delta_{ij} \right) \right].
\end{aligned}$$

Therefore

$$\boxed{m \int C_{\langle i} C_j C_k \rangle C_l f d\mathbf{c} = u_{ijkl}^0 + \frac{3}{d+4} u_{\langle ij}^1 \delta_{k \rangle l}} \quad (\text{B } 1)$$

B.2. The term  $m \int \frac{D}{Dt} C_{\langle i} C_j C_k \rangle f \, d\mathbf{c}$

Let us first evaluate the following.

$$\begin{aligned} \frac{D}{Dt} C_{\langle i} C_j C_k \rangle &= \frac{D}{Dt} C_i C_j C_k - \frac{1}{d+2} \frac{D}{Dt} C^2 \left( C_i \delta_{jk} + C_j \delta_{ik} + C_k \delta_{ij} \right) \\ &= - \left( C_i C_j \frac{Dv_k}{Dt} + C_i C_k \frac{Dv_j}{Dt} + C_j C_k \frac{Dv_i}{Dt} \right) \\ &\quad + \frac{1}{d+2} \left[ \left( C^2 \frac{Dv_i}{Dt} + 2C_i C_l \frac{Dv_l}{Dt} \right) \delta_{jk} \right. \\ &\quad \left. + \left( C^2 \frac{Dv_j}{Dt} + 2C_j C_l \frac{Dv_l}{Dt} \right) \delta_{ik} + \left( C^2 \frac{Dv_k}{Dt} + 2C_k C_l \frac{Dv_l}{Dt} \right) \delta_{ij} \right]. \end{aligned}$$

Using the above equation and (1.5), one obtains

$$\begin{aligned} m \int \frac{D}{Dt} C_{\langle i} C_j C_k \rangle f \, d\mathbf{c} &= - \left[ (\sigma_{ij} + \rho\theta\delta_{ij}) \frac{Dv_k}{Dt} + (\sigma_{ik} + \rho\theta\delta_{ik}) \frac{Dv_j}{Dt} + (\sigma_{jk} + \rho\theta\delta_{jk}) \frac{Dv_i}{Dt} \right] \\ &\quad + \frac{1}{d+2} \left[ \left\{ d\rho\theta \frac{Dv_i}{Dt} + 2(\sigma_{il} + \rho\theta\delta_{il}) \frac{Dv_l}{Dt} \right\} \delta_{jk} \right. \\ &\quad \left. + \left\{ d\rho\theta \frac{Dv_j}{Dt} + 2(\sigma_{jl} + \rho\theta\delta_{jl}) \frac{Dv_l}{Dt} \right\} \delta_{ik} + \left\{ d\rho\theta \frac{Dv_k}{Dt} + 2(\sigma_{kl} + \rho\theta\delta_{kl}) \frac{Dv_l}{Dt} \right\} \delta_{ij} \right] \\ &= - \left[ \sigma_{ij} \frac{Dv_k}{Dt} + \sigma_{ik} \frac{Dv_j}{Dt} + \sigma_{jk} \frac{Dv_i}{Dt} \right. \\ &\quad \left. - \frac{2}{d+2} \left( \sigma_{il} \frac{Dv_l}{Dt} \delta_{jk} + \sigma_{jl} \frac{Dv_l}{Dt} \delta_{ik} + \sigma_{kl} \frac{Dv_l}{Dt} \delta_{ij} \right) \right]. \end{aligned}$$

Therefore

$$\boxed{m \int \frac{D}{Dt} C_{\langle i} C_j C_k \rangle f \, d\mathbf{c} = -3\sigma_{\langle ij} \frac{Dv_k \rangle}{Dt}} \quad (\text{B } 2)$$

B.3. The term  $m \int C_l \frac{\partial}{\partial x_l} C_{\langle i} C_j C_k \rangle f \, d\mathbf{c}$

Let us first evaluate the following.

$$\begin{aligned} C_l \frac{\partial}{\partial x_l} C_{\langle i} C_j C_k \rangle &= C_l \frac{\partial}{\partial x_l} C_i C_j C_k - \frac{1}{d+2} C_l \frac{\partial}{\partial x_l} C^2 \left( C_i \delta_{jk} + C_j \delta_{ik} + C_k \delta_{ij} \right) \\ &= -C_l \left( C_i C_j \frac{\partial v_k}{\partial x_l} + C_i C_k \frac{\partial v_j}{\partial x_l} + C_j C_k \frac{\partial v_i}{\partial x_l} \right) \\ &\quad + \frac{1}{d+2} C_l \left[ \left( C^2 \frac{\partial v_i}{\partial x_l} + 2C_i C_r \frac{\partial v_r}{\partial x_l} \right) \delta_{jk} \right. \\ &\quad \left. + \left( C^2 \frac{\partial v_j}{\partial x_l} + 2C_j C_r \frac{\partial v_r}{\partial x_l} \right) \delta_{ik} + \left( C^2 \frac{\partial v_k}{\partial x_l} + 2C_k C_r \frac{\partial v_r}{\partial x_l} \right) \delta_{ij} \right]. \end{aligned}$$

Using the above equation and (1.14), one obtains

$$\begin{aligned}
& m \int C_l \frac{\partial}{\partial x_l} C_{\langle i} C_j C_k \rangle f \, d\mathbf{c} \\
&= - \left[ u_{ijl}^0 + \frac{2}{d+2} (q_i \delta_{jl} + q_j \delta_{il} + q_l \delta_{ij}) \right] \frac{\partial v_k}{\partial x_l} - \left[ u_{ikl}^0 + \frac{2}{d+2} (q_i \delta_{kl} + q_k \delta_{il} + q_l \delta_{ik}) \right] \frac{\partial v_j}{\partial x_l} \\
&\quad - \left[ u_{jkl}^0 + \frac{2}{d+2} (q_j \delta_{kl} + q_k \delta_{jl} + q_l \delta_{jk}) \right] \frac{\partial v_i}{\partial x_l} + \frac{2}{d+2} q_l \left[ \frac{\partial v_i}{\partial x_l} \delta_{jk} + \frac{\partial v_j}{\partial x_l} \delta_{ik} + \frac{\partial v_k}{\partial x_l} \delta_{ij} \right] \\
&\quad + \frac{2}{d+2} \left[ u_{irl}^0 + \frac{2}{d+2} (q_i \delta_{rl} + q_r \delta_{il} + q_l \delta_{ir}) \right] \frac{\partial v_r}{\partial x_l} \delta_{jk} \\
&\quad + \frac{2}{d+2} \left[ u_{jrl}^0 + \frac{2}{d+2} (q_j \delta_{rl} + q_r \delta_{jl} + q_l \delta_{jr}) \right] \frac{\partial v_r}{\partial x_l} \delta_{ik} \\
&\quad + \frac{2}{d+2} \left[ u_{krl}^0 + \frac{2}{d+2} (q_k \delta_{rl} + q_r \delta_{kl} + q_l \delta_{kr}) \right] \frac{\partial v_r}{\partial x_l} \delta_{ij} \\
&= - \left[ u_{lij}^0 \frac{\partial v_k}{\partial x_l} + u_{lik}^0 \frac{\partial v_j}{\partial x_l} + u_{ljk}^0 \frac{\partial v_i}{\partial x_l} - \frac{2}{d+2} \left( u_{lir}^0 \frac{\partial v_r}{\partial x_l} \delta_{jk} + u_{ljr}^0 \frac{\partial v_r}{\partial x_l} \delta_{ik} + u_{lkr}^0 \frac{\partial v_r}{\partial x_l} \delta_{ij} \right) \right] \\
&\quad - \frac{2}{d+2} \left( q_i \frac{\partial v_k}{\partial x_j} + q_j \frac{\partial v_k}{\partial x_i} + q_l \frac{\partial v_k}{\partial x_l} \delta_{ij} \right) - \frac{2}{d+2} \left( q_i \frac{\partial v_j}{\partial x_k} + q_k \frac{\partial v_j}{\partial x_i} + q_l \frac{\partial v_j}{\partial x_l} \delta_{ik} \right) \\
&\quad - \frac{2}{d+2} \left( q_j \frac{\partial v_i}{\partial x_k} + q_k \frac{\partial v_i}{\partial x_j} + q_l \frac{\partial v_i}{\partial x_l} \delta_{jk} \right) + \frac{2}{d+2} q_l \left( \frac{\partial v_i}{\partial x_l} \delta_{jk} + \frac{\partial v_j}{\partial x_l} \delta_{ik} + \frac{\partial v_k}{\partial x_l} \delta_{ij} \right) \\
&\quad + \frac{4}{(d+2)^2} \left( q_i \frac{\partial v_l}{\partial x_l} + q_r \frac{\partial v_r}{\partial x_i} + q_l \frac{\partial v_i}{\partial x_l} \right) \delta_{jk} + \frac{4}{(d+2)^2} \left( q_j \frac{\partial v_l}{\partial x_l} + q_r \frac{\partial v_r}{\partial x_j} + q_l \frac{\partial v_j}{\partial x_l} \right) \delta_{ik} \\
&\quad + \frac{4}{(d+2)^2} \left( q_k \frac{\partial v_l}{\partial x_l} + q_r \frac{\partial v_r}{\partial x_k} + q_l \frac{\partial v_k}{\partial x_l} \right) \delta_{ij} \\
&= -3u_{lij}^0 \frac{\partial v_k}{\partial x_l} - \frac{2}{d+2} \left[ \left( q_i \frac{\partial v_j}{\partial x_k} + q_i \frac{\partial v_k}{\partial x_j} + q_j \frac{\partial v_i}{\partial x_k} + q_j \frac{\partial v_k}{\partial x_i} + q_k \frac{\partial v_i}{\partial x_j} + q_k \frac{\partial v_j}{\partial x_i} \right) \right. \\
&\quad \left. - \frac{2}{d+2} \left\{ \left( q_i \frac{\partial v_l}{\partial x_l} + q_l \frac{\partial v_i}{\partial x_l} + q_l \frac{\partial v_l}{\partial x_i} \right) \delta_{jk} \right. \right. \\
&\quad \left. \left. + \left( q_j \frac{\partial v_l}{\partial x_l} + q_l \frac{\partial v_l}{\partial x_j} + q_l \frac{\partial v_j}{\partial x_l} \right) \delta_{ik} + \left( q_k \frac{\partial v_l}{\partial x_l} + q_l \frac{\partial v_l}{\partial x_k} + q_l \frac{\partial v_k}{\partial x_l} \right) \delta_{ij} \right\} \right].
\end{aligned}$$

Therefore

$$\boxed{m \int C_l \frac{\partial}{\partial x_l} C_{\langle i} C_j C_k \rangle f \, d\mathbf{c} = -3u_{lij}^0 \frac{\partial v_k}{\partial x_l} - \frac{12}{d+2} q_{\langle i} \frac{\partial v_j}{\partial x_k} } \quad (\text{B } 3)$$

#### B.4. The term $m \int F_l \frac{\partial}{\partial c_l} C_{\langle i} C_j C_k \rangle f \, d\mathbf{c}$

Let us first evaluate the following.

$$\begin{aligned}
F_l \frac{\partial}{\partial c_l} C_{\langle i} C_j C_k \rangle &= F_l \frac{\partial}{\partial c_l} (C_i C_j C_k) - \frac{1}{d+2} F_l \frac{\partial}{\partial c_l} \left[ C^2 (C_i \delta_{jk} + C_j \delta_{ik} + C_k \delta_{ij}) \right] \\
&= F_l (C_i C_j \delta_{kl} + C_i C_k \delta_{jl} + C_j C_k \delta_{il}) + \frac{1}{d+2} F_l \left[ (C^2 \delta_{il} + 2C_i C_r \delta_{rl}) \delta_{jk} \right. \\
&\quad \left. + (C^2 \delta_{jl} + 2C_j C_r \delta_{rl}) \delta_{ik} + (C^2 \delta_{kl} + 2C_k C_r \delta_{rl}) \delta_{ij} \right]
\end{aligned}$$

or

$$F_l \frac{\partial}{\partial c_l} C_{\langle i} C_j C_k \rangle = (C_i C_j F_k + C_i C_k F_j + C_j C_k F_i) + \frac{1}{d+2} \left[ (C^2 F_i + 2C_i C_r F_r) \delta_{jk} + (C^2 F_j + 2C_j C_r F_r) \delta_{ik} + (C^2 F_k + 2C_k C_r F_r) \delta_{ij} \right].$$

Using the above equation and (1.5), one obtains

$$\begin{aligned} & m \int F_l \frac{\partial}{\partial c_l} C_{\langle i} C_j C_k \rangle f \, d\mathbf{c} \\ &= (\sigma_{ij} + \rho \theta \delta_{ij}) F_k + (\sigma_{ik} + \rho \theta \delta_{ik}) F_j + (\sigma_{jk} + \rho \theta \delta_{jk}) F_i \\ &\quad - \frac{d}{d+2} \rho \theta (F_i \delta_{jk} + F_j \delta_{ik} + F_k \delta_{ij}) \\ &\quad - \frac{2}{d+2} \left[ (\sigma_{ir} + \rho \theta \delta_{ir}) F_r \delta_{jk} + (\sigma_{jr} + \rho \theta \delta_{jr}) F_r \delta_{ik} + (\sigma_{kr} + \rho \theta \delta_{kr}) F_r \delta_{ij} \right] \\ &= \sigma_{ij} F_k + \sigma_{ik} F_j + \sigma_{jk} F_i - \frac{2}{d+2} \left[ \sigma_{ir} F_r \delta_{jk} + \sigma_{jr} F_r \delta_{ik} + \sigma_{kr} F_r \delta_{ij} \right]. \end{aligned}$$

Therefore

$$\boxed{m \int F_l \frac{\partial}{\partial c_l} C_{\langle i} C_j C_k \rangle f \, d\mathbf{c} = 3\sigma_{\langle ij} F_{k \rangle}. \quad (\text{B } 4)}$$

## Appendix C. Few terms in the $u_{ij}^1$ balance equation (1.19)

### C.1. The term $m \int C^2 C_{\langle i} C_j \rangle C_k f \, d\mathbf{c}$

Using identity (A 3), this term turns out to be

$$\begin{aligned} & m \int C^2 C_{\langle i} C_j \rangle C_k f \, d\mathbf{c} \\ &= m \int C^2 C_{\langle i} C_j C_k \rangle f \, d\mathbf{c} + \frac{1}{d+2} m \int C^4 \left( C_i \delta_{jk} + C_j \delta_{ik} - \frac{2}{d} C_k \delta_{ij} \right) f \, d\mathbf{c} \\ &= u_{ijk}^1 + \frac{1}{d+2} \left( u_i^2 \delta_{jk} + u_j^2 \delta_{ik} - \frac{2}{d} u_k^2 \delta_{ij} \right). \end{aligned}$$

Therefore

$$\boxed{m \int C^2 C_{\langle i} C_j \rangle C_k f \, d\mathbf{c} = u_{ijk}^1 + \frac{2}{d+2} u_{\langle i}^2 \delta_{j \rangle k} \quad (\text{C } 1)}$$

### C.2. The term $m \int \frac{D}{Dt} C^2 C_{\langle i} C_j \rangle f \, d\mathbf{c}$

Let us first evaluate the following.

$$\begin{aligned} \frac{D}{Dt} C^2 C_{\langle i} C_j \rangle &= \frac{D}{Dt} (C^2 C_i C_j) - \frac{1}{d} \frac{D}{Dt} C^4 \delta_{ij} \\ &= - \left( 2C_i C_j C_k \frac{Dv_k}{Dt} + C^2 C_i \frac{Dv_j}{Dt} + C^2 C_j \frac{Dv_i}{Dt} \right) + \frac{4}{d} C^2 C_k \frac{Dv_k}{Dt} \delta_{ij}. \end{aligned}$$



Using the above equation and (1.14), one obtains

$$\begin{aligned}
 m \int \frac{D}{Dt} C^2 C_{\langle i} C_{j \rangle} f \, d\mathbf{c} &= -2 \left[ u_{ijk}^0 + \frac{2}{d+2} (q_i \delta_{jk} + q_j \delta_{ik} + q_k \delta_{ij}) \right] \frac{Dv_k}{Dt} \\
 &\quad - 2q_i \frac{Dv_j}{Dt} - 2q_j \frac{Dv_i}{Dt} + \frac{8}{d} q_k \frac{Dv_k}{Dt} \delta_{ij} \\
 &= -2u_{ijk}^0 \frac{Dv_k}{Dt} - \frac{4}{d+2} \left( q_i \frac{Dv_j}{Dt} + q_j \frac{Dv_i}{Dt} + q_k \frac{Dv_k}{Dt} \delta_{ij} \right) \\
 &\quad - 2q_i \frac{Dv_j}{Dt} - 2q_j \frac{Dv_i}{Dt} + \frac{8}{d} q_k \frac{Dv_k}{Dt} \delta_{ij} \\
 &= -2u_{ijk}^0 \frac{Dv_k}{Dt} - \frac{2(d+4)}{d+2} \left( q_i \frac{Dv_j}{Dt} + q_j \frac{Dv_i}{Dt} - \frac{2}{d} q_k \frac{Dv_k}{Dt} \delta_{ij} \right) \\
 &= -2u_{ijk}^0 \frac{Dv_k}{Dt} - \frac{4(d+4)}{d+2} q_{\langle i} \frac{Dv_{j \rangle}}{Dt}.
 \end{aligned}$$

Therefore

$$\boxed{m \int \frac{D}{Dt} C^2 C_{\langle i} C_{j \rangle} f \, d\mathbf{c} = -2u_{ijk}^0 \frac{Dv_k}{Dt} - \frac{4(d+4)}{d+2} q_{\langle i} \frac{Dv_{j \rangle}}{Dt}} \quad (\text{C } 2)$$

C.3. The term  $m \int C_k \frac{\partial}{\partial x_k} C^2 C_{\langle i} C_{j \rangle} f \, d\mathbf{c}$

Let us first evaluate the following.

$$\begin{aligned}
 &C_k \frac{\partial}{\partial x_k} C^2 C_{\langle i} C_{j \rangle} \\
 &= C_k \frac{\partial}{\partial x_k} (C^2 C_i C_j) - \frac{1}{d} C_k \frac{\partial}{\partial x_k} C^4 \delta_{ij} \\
 &= -C_k \left( 2C_i C_j C_l \frac{\partial v_l}{\partial x_k} + C^2 C_i \frac{\partial v_j}{\partial x_k} + C^2 C_j \frac{\partial v_i}{\partial x_k} \right) + \frac{4}{d} C^2 C_k C_l \frac{\partial v_l}{\partial x_k} \delta_{ij} \\
 &= -2 \left[ C_{\langle i} C_j C_k C_{l \rangle} + \frac{1}{d+4} C^2 (C_i C_j \delta_{kl} + C_i C_k \delta_{jl} + C_i C_l \delta_{jk} + C_j C_k \delta_{il} \right. \\
 &\quad \left. + C_j C_l \delta_{ik} + C_k C_l \delta_{ij}) - \frac{1}{(d+2)(d+4)} C^4 (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \right] \frac{\partial v_l}{\partial x_k} \\
 &\quad - \left( C^2 C_i C_k \frac{\partial v_j}{\partial x_k} + C^2 C_j C_k \frac{\partial v_i}{\partial x_k} \right) + \frac{4}{d} C^2 C_k C_l \frac{\partial v_l}{\partial x_k} \delta_{ij}.
 \end{aligned}$$

Using the above equation and (1.13), one obtains

$$\begin{aligned}
 &m \int C_k \frac{\partial}{\partial x_k} C^2 C_{\langle i} C_{j \rangle} f \, d\mathbf{c} \\
 &= -2u_{ijkl}^0 \frac{\partial v_l}{\partial x_k} - \frac{2}{d+4} \left[ \left( u_{ij}^1 + \frac{1}{d} u^2 \delta_{ij} \right) \delta_{kl} + \left( u_{ik}^1 + \frac{1}{d} u^2 \delta_{ik} \right) \delta_{jl} \right. \\
 &\quad \left. + \left( u_{il}^1 + \frac{1}{d} u^2 \delta_{il} \right) \delta_{jk} + \left( u_{jk}^1 + \frac{1}{d} u^2 \delta_{jk} \right) \delta_{il} + \left( u_{jl}^1 + \frac{1}{d} u^2 \delta_{jl} \right) \delta_{ik} \right. \\
 &\quad \left. + \left( u_{kl}^1 + \frac{1}{d} u^2 \delta_{kl} \right) \delta_{ij} \right] \frac{\partial v_l}{\partial x_k} + \frac{2}{(d+2)(d+4)} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) u^2 \frac{\partial v_l}{\partial x_k} \\
 &\quad - \left( u_{ik}^1 + \frac{1}{d} u^2 \delta_{ik} \right) \frac{\partial v_j}{\partial x_k} - \left( u_{jk}^1 + \frac{1}{d} u^2 \delta_{jk} \right) \frac{\partial v_i}{\partial x_k} + \frac{4}{d} \left( u_{kl}^1 + \frac{1}{d} u^2 \delta_{kl} \right) \frac{\partial v_l}{\partial x_k} \delta_{ij}
 \end{aligned}$$

or

$$\begin{aligned}
& m \int C_k \frac{\partial}{\partial x_k} C^2 C_{\langle i} C_{j \rangle} f \, d\mathbf{c} \\
&= -2u_{ijk}^0 \frac{\partial v_l}{\partial x_k} - \frac{2}{d+4} \left[ u_{ij}^1 \delta_{kl} + u_{ik}^1 \delta_{jl} + u_{il}^1 \delta_{jk} + u_{jk}^1 \delta_{il} + u_{jl}^1 \delta_{ik} + u_{kl}^1 \delta_{ij} \right. \\
&\quad \left. + \frac{2}{d} u^2 \left( \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \right] \frac{\partial v_l}{\partial x_k} + \frac{2}{(d+2)(d+4)} \left( \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) u^2 \frac{\partial v_l}{\partial x_k} \\
&\quad - \left( u_{ik}^1 \frac{\partial v_j}{\partial x_k} + u_{jk}^1 \frac{\partial v_i}{\partial x_k} - \frac{4}{d} u_{kl}^1 \frac{\partial v_l}{\partial x_k} \delta_{ij} \right) - \frac{1}{d} u^2 \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} - \frac{4}{d} \frac{\partial v_k}{\partial x_k} \delta_{ij} \right) \\
&= -2u_{ijk}^0 \frac{\partial v_k}{\partial x_l} - \frac{2}{d+4} \left( u_{ij}^1 \frac{\partial v_k}{\partial x_k} + u_{ik}^1 \frac{\partial v_j}{\partial x_k} + u_{il}^1 \frac{\partial v_k}{\partial x_j} + u_{jk}^1 \frac{\partial v_i}{\partial x_k} + u_{jl}^1 \frac{\partial v_k}{\partial x_i} + u_{kl}^1 \delta_{ij} \frac{\partial v_l}{\partial x_k} \right) \\
&\quad - \left( u_{ik}^1 \frac{\partial v_j}{\partial x_k} + u_{jk}^1 \frac{\partial v_i}{\partial x_k} - \frac{4}{d} u_{kl}^1 \frac{\partial v_l}{\partial x_k} \delta_{ij} \right) - \frac{4}{d(d+4)} u^2 \left( \delta_{ij} \frac{\partial v_k}{\partial x_k} + \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) \\
&\quad + \frac{2}{(d+2)(d+4)} u^2 \left( \delta_{ij} \frac{\partial v_k}{\partial x_k} + \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) - \frac{1}{d} u^2 \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} - \frac{4}{d} \frac{\partial v_k}{\partial x_k} \delta_{ij} \right) \\
&= -2u_{ijk}^0 \frac{\partial v_k}{\partial x_l} - \frac{6}{d+4} u_{\langle ij}^1 \frac{\partial v_{k \rangle}}{\partial x_k} - \frac{4}{(d+2)(d+4)} \left( u_{ik}^1 \frac{\partial v_k}{\partial x_j} + u_{jk}^1 \frac{\partial v_k}{\partial x_i} + u_{kl}^1 \frac{\partial v_k}{\partial x_l} \delta_{ij} \right) \\
&\quad - \frac{2}{d+4} \left( u_{ik}^1 \frac{\partial v_k}{\partial x_j} + u_{jk}^1 \frac{\partial v_k}{\partial x_i} + u_{kl}^1 \delta_{ij} \frac{\partial v_l}{\partial x_k} \right) - \left( u_{ik}^1 \frac{\partial v_j}{\partial x_k} + u_{jk}^1 \frac{\partial v_i}{\partial x_k} - \frac{4}{d} u_{kl}^1 \frac{\partial v_l}{\partial x_k} \delta_{ij} \right) \\
&\quad - \frac{2}{d(d+2)} u^2 \left( \delta_{ij} \frac{\partial v_k}{\partial x_k} + \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) - \frac{1}{d} u^2 \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} - \frac{4}{d} \frac{\partial v_k}{\partial x_k} \delta_{ij} \right), \\
&= -2u_{ijk}^0 \frac{\partial v_k}{\partial x_l} - \frac{6}{d+4} u_{\langle ij}^1 \frac{\partial v_{k \rangle}}{\partial x_k} - \frac{2}{d+2} \left( u_{ik}^1 \frac{\partial v_k}{\partial x_j} + u_{jk}^1 \frac{\partial v_k}{\partial x_i} + u_{kl}^1 \frac{\partial v_l}{\partial x_k} \delta_{ij} \right) \\
&\quad - \left( u_{ik}^1 \frac{\partial v_j}{\partial x_k} + u_{jk}^1 \frac{\partial v_i}{\partial x_k} - \frac{4}{d} u_{kl}^1 \frac{\partial v_l}{\partial x_k} \delta_{ij} \right) - \frac{d+4}{d(d+2)} u^2 \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{d} \delta_{ij} \frac{\partial v_k}{\partial x_k} \right) \\
&= -2u_{ijk}^0 \frac{\partial v_k}{\partial x_l} - \frac{6}{d+4} u_{\langle ij}^1 \frac{\partial v_{k \rangle}}{\partial x_k} \\
&\quad - \frac{2}{d+2} \left[ u_{ki}^1 \frac{\partial v_k}{\partial x_j} + u_{kj}^1 \frac{\partial v_k}{\partial x_i} - \frac{2}{d} u_{kl}^1 \frac{\partial v_k}{\partial x_l} \delta_{ij} + \frac{d+2}{d} u_{kl}^1 \frac{\partial v_k}{\partial x_l} \delta_{ij} \right] \\
&\quad - \left( u_{ki}^1 \frac{\partial v_j}{\partial x_k} + u_{kj}^1 \frac{\partial v_i}{\partial x_k} - \frac{2}{d} u_{kl}^1 \frac{\partial v_l}{\partial x_k} \delta_{ij} \right) + \frac{2}{d} u_{kl}^1 \frac{\partial v_k}{\partial x_l} \delta_{ij} - \frac{2(d+4)}{d(d+2)} u^2 \frac{\partial v_{\langle i}}{\partial x_{j \rangle}}.
\end{aligned}$$

Therefore

$$\boxed{m \int C_k \frac{\partial}{\partial x_k} C^2 C_{\langle i} C_{j \rangle} f \, d\mathbf{c} = -2u_{ijk}^0 \frac{\partial v_k}{\partial x_l} - \frac{6}{d+4} u_{\langle ij}^1 \frac{\partial v_{k \rangle}}{\partial x_k} - \frac{4}{d+2} u_{k \langle i}^1 \frac{\partial v_{k \rangle}}{\partial x_{j \rangle}} - 2u_{k \langle i}^1 \frac{\partial v_{j \rangle}}{\partial x_k} - \frac{2(d+4)}{d(d+2)} u^2 \frac{\partial v_{\langle i}}{\partial x_{j \rangle}}} \quad (\text{C } 3)$$

C.4. The term  $m \int F_k \frac{\partial}{\partial c_k} C^2 C_{\langle i} C_{j \rangle} f d\mathbf{c}$

Let us first evaluate the following.

$$\begin{aligned}
 F_k \frac{\partial}{\partial c_k} C^2 C_{\langle i} C_{j \rangle} &= F_k \frac{\partial}{\partial c_k} (C^2 C_i C_j) - \frac{1}{d} F_k \frac{\partial}{\partial c_k} C^4 \delta_{ij} \\
 &= F_k (2C_i C_j C_l \delta_{lk} + C^2 C_i \delta_{jk} + C^2 C_j \delta_{ik}) - \frac{4}{d} C^2 C_l \delta_{lk} \times F_k \delta_{ij} \\
 &= 2C_i C_j C_k F_k + C^2 C_i F_j + C^2 C_j F_i - \frac{4}{d} C^2 C_k F_k \delta_{ij}.
 \end{aligned}$$

Using (A 2),

$$\begin{aligned}
 F_k \frac{\partial}{\partial c_k} C^2 C_{\langle i} C_{j \rangle} &= 2C_{\langle i} C_j C_k \rangle F_k + \frac{2}{d+2} C^2 (C_i \delta_{jk} + C_j \delta_{ik} + C_k \delta_{ij}) F_k \\
 &\quad + C^2 C_i F_j + C^2 C_j F_i - \frac{4}{d} C^2 C_k F_k \delta_{ij} \\
 &= 2C_{\langle i} C_j C_k \rangle F_k + \frac{d+4}{d+2} \left( C^2 C_i F_j + C^2 C_j F_i - \frac{2}{d} C^2 C_k F_k \delta_{ij} \right).
 \end{aligned}$$

Using the above equation,

$$m \int F_k \frac{\partial}{\partial c_k} C^2 C_{\langle i} C_{j \rangle} f d\mathbf{c} = 2u_{ijk}^0 F_k + \frac{2(d+4)}{d+2} \left( q_i F_j + q_j F_i - \frac{2}{d} q_k F_k \delta_{ij} \right)$$

or

$$\boxed{m \int F_k \frac{\partial}{\partial c_k} C^2 C_{\langle i} C_{j \rangle} f d\mathbf{c} = 2u_{ijk}^0 F_k + \frac{4(d+4)}{d+2} q_{\langle i} F_{j \rangle}} \quad (\text{C } 4)$$

## REFERENCES

STRUCHTRUP, H. 2005 *Macroscopic Transport Equations for Rarefied Gas Flows*. Berlin: Springer.