Supplementary Material to Burnett-order constitutive relations, second moment anisotropy and co-existing states in sheared dense gas-solid suspensions

Saikat Saha[1] and Meheboob Alam [1]†

[1]Jawaharlal Nehru Centre for Advanced Scientific Research, Jakkur P.O., Bangalore 560064, India

† Email address for correspondence: meheboob@jncasr.ac.in

Appendix A. Evaluation of collision integrals $\Theta_{\alpha\beta}$, $A_{\alpha\beta}$, $\widehat{E}_{\alpha\beta}$ and $\widehat{G}_{\alpha\beta}$

The collision integrals $\Theta_{\alpha\beta}$ (2.14), $A_{\alpha\beta}$ (2.16a), $\widehat{E}_{\alpha\beta}$ (2.16b) and $\widehat{G}_{\alpha\beta}$ (2.16c) appearing in the second-moment balance (2.24a-d) of the main text of Saha & Alam (2020) can be simplified to:

$$\left.\begin{aligned}
2\Theta_{xy} &= \frac{3(1+e)\rho\nu g_0 T}{\pi^{\frac{3}{2}}}\Big(\cos 2\phi \mathcal{J}^{30}_{012} - \sin 2\phi \mathcal{J}^{30}_{102}\Big)\\
\Theta_{xx} + \Theta_{yy} &= \frac{3(1+e)\rho\nu g_0 T}{\pi^{\frac{3}{2}}}\mathcal{J}^{30}_{002}\\
\Theta_{yy} - \Theta_{xx} &= \frac{3(1+e)\rho\nu g_0 T}{\pi^{\frac{3}{2}}}\Big(\mathcal{J}^{30}_{102}\cos 2\phi + \mathcal{I}^{30}_{012}\sin 2\phi\Big)
\end{aligned}\right\}, \tag{A 1}$$

$$\left.\begin{aligned}
A_{xx} &= -\frac{6(1-e^2)\rho\nu g_0 T^{\frac{3}{2}}}{\sigma\pi^{\frac{3}{2}}}\left[\frac{1}{2}\left(\mathcal{H}^{30}_{003} - \cos 2\phi \mathcal{H}^{30}_{103} - \sin 2\phi \mathcal{H}^{30}_{013}\right)\right]\\
A_{yy} &= -\frac{6(1-e^2)\rho\nu g_0 T^{\frac{3}{2}}}{\sigma\pi^{\frac{3}{2}}}\left[\frac{1}{2}\left(\mathcal{H}^{30}_{003} + \cos 2\phi \mathcal{H}^{30}_{103} + \sin 2\phi \mathcal{H}^{30}_{013}\right)\right]\\
A_{zz} &= -\frac{6(1-e^2)\rho\nu g_0 T^{\frac{3}{2}}}{\sigma\pi^{\frac{3}{2}}}\mathcal{H}^{12}_{003}\\
A_{xy} &= -\frac{6(1-e^2)\rho\nu g_0 T^{\frac{3}{2}}}{\sigma\pi^{\frac{3}{2}}}\left[\frac{1}{2}\left(\cos 2\phi \mathcal{H}^{30}_{013} - \sin 2\phi \mathcal{H}^{30}_{103}\right)\right]
\end{aligned}\right\}, \tag{A 2}$$

$$\left.\begin{aligned}
\widehat{E}_{xx} &= -(\widehat{E}_{yy} + \widehat{E}_{zz}),\\
\widehat{E}_{yy} &= -\frac{6(1+e)\rho\nu g_0 T^{\frac{3}{2}}}{\sigma\pi^{\frac{3}{2}}}\\
&\quad \times \Big[\eta\Big\{\mathcal{H}^{31}_{101} - \mathcal{H}^{32}_{011} + \cos 2\phi\Big(\mathcal{H}^{31}_{201} - \mathcal{H}^{31}_{021} + \mathcal{H}^{50}_{111}\Big) - \sin 2\phi\Big(\mathcal{H}^{30}_{201} + \mathcal{H}^{32}_{021}\Big)\Big\}\\
&\qquad + 3\lambda^2\Big\{\mathcal{H}^{32}_{001} + \cos 2\phi\Big(\mathcal{H}^{32}_{101} + \mathcal{H}^{31}_{011}\Big) + \sin 2\phi\Big(\mathcal{H}^{32}_{011} - \mathcal{H}^{31}_{101}\Big)\Big\}\Big]\\
\widehat{E}_{zz} &= -\frac{12(1+e)\rho\nu g_0 T^{\frac{3}{2}}}{\sigma\pi^{\frac{3}{2}}}\Big\{-\eta\Big(\mathcal{H}^{31}_{101} - \mathcal{H}^{32}_{011}\Big) - \lambda^2\Big(\mathcal{H}^{32}_{001}\Big)\Big\}\\
\widehat{E}_{xy} &= -\frac{12(1+e)\rho\nu g_0 T^{\frac{3}{2}}}{\sigma\pi^{\frac{3}{2}}}\Bigg[\frac{1}{2}\eta\Big\{\Big(2\mathcal{H}^{31}_{111} - \mathcal{H}^{32}_{021} - \mathcal{H}^{30}_{201}\Big)\cos 2\phi\\
&\qquad + \Big(\mathcal{H}^{31}_{021} + \mathcal{H}^{32}_{111} - \mathcal{H}^{30}_{111} - \mathcal{H}^{31}_{201}\Big)\sin 2\phi\Big\}\\
&\qquad + \frac{3}{2}\lambda^2\Big\{\Big(\mathcal{H}^{32}_{011} - \mathcal{H}^{31}_{101}\Big)\cos 2\phi - \Big(\mathcal{H}^{32}_{101} + \mathcal{H}^{31}_{011}\Big)\sin 2\phi\Big\}\Bigg]
\end{aligned}\right\} \tag{A 3}$$

$$\left.\begin{aligned}
\widehat{G}_{xx} &= -(\widehat{G}_{yy}+\widehat{G}_{zz})\\
\widehat{G}_{yy} &= -\frac{6(1+e)\rho\nu g_0 T^{\frac{3}{2}}}{\sigma\pi^{\frac{3}{2}}}2R\Big\{\mathcal{K}_{00}^{31}+\cos 2\phi\Big(\mathcal{K}_{10}^{31}+\mathcal{K}_{01}^{30}\Big)+\sin 2\phi\Big(\mathcal{K}_{01}^{31}-\mathcal{K}_{10}^{30}\Big)\Big\}\\
\widehat{G}_{zz} &= -\frac{6(1+e)\rho\nu g_0 T^{\frac{3}{2}}}{\sigma\pi^{\frac{3}{2}}}\Big\{-4R\mathcal{K}_{00}^{31}\Big\}\\
\widehat{G}_{xy} &= -\frac{6(1+e)\rho\nu g_0 T^{\frac{3}{2}}}{\sigma\pi^{\frac{3}{2}}}(-4R)\Big\{\frac{1}{2}\cos 2\phi\Big(\mathcal{K}_{01}^{31}-\mathcal{K}_{10}^{30}\Big)+\frac{1}{2}\sin 2\phi\Big(\mathcal{K}_{10}^{31}+\mathcal{K}_{01}^{30}\Big)\Big\}
\end{aligned}\right\} \tag{A 4}$$

In the above expressions, $\mathcal{H}_{\alpha\beta\gamma}^{\delta p}$, $\mathcal{J}_{\alpha\beta\gamma}^{\delta p}$ and $\mathcal{K}_{\alpha\beta}^{\delta p}$ represent tensorial integrals defined as

$$\begin{aligned}
\mathcal{H}_{\alpha\beta\gamma}^{\delta p}(\eta,R,\phi,\lambda) \equiv \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi} & \sin^{\alpha}2\theta\cos^{\beta}2\theta\sin^{\delta}\varphi\cos^{p}\varphi\\
&\times\ \big(1-\eta\sin^2\varphi\cos 2\theta+\lambda^2(3\sin^2\varphi-2)\big)^{\frac{\gamma}{2}}\,\mathfrak{F}\left(\chi\left[\eta,R,\phi,\lambda;\theta,\varphi\right]\right)d\varphi d\theta, \quad \text{(A 5)}\\
\mathcal{J}_{\alpha\beta\gamma}^{\delta p}(\eta,R,\phi,\lambda) \equiv \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi} & \sin^{\alpha}2\theta\cos^{\beta}2\theta\sin^{\delta}\varphi\cos^{p}\varphi\\
&\times\ \{1-\eta\sin^2\varphi\cos 2\theta+\lambda^2(3\sin^2\varphi-2)\}^{\frac{\gamma}{2}}\,\mathfrak{G}\left(\chi\left[\eta,R,\phi,\lambda;\theta,\varphi\right]\right)d\varphi d\theta, \quad \text{(A 6)}\\
\mathcal{K}_{\alpha\beta}^{\delta p}(\eta,R,\phi,\lambda) \equiv \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi} & \sin^{\alpha}2\theta\cos^{\beta}2\theta\sin^{\delta}\varphi\cos^{p}\varphi\left[(1-2\lambda^2)\{\sin(2\phi+2\theta)-\cos\varphi\right.\\
\times\ \cos(2\phi+2\theta)\}+\sin^2\varphi\{3\lambda^2 & \left.\sin(2\phi+2\theta)-\eta\sin 2\phi\}\right]\mathfrak{G}\left(\chi\left[\eta,R,\phi,\lambda;\theta,\varphi\right]\right)d\varphi d\theta, \quad \text{(A 7)}
\end{aligned}$$

where $\mathfrak{F}(\chi)$ and $\mathfrak{G}(\chi)$ are given by (2.18a) and (2.18b), respectively, and each integral is evaluated over $\theta \in (0, 2\pi)$ and $\varphi \in (0, \pi)$; see figure 1 for the definitions of θ and φ with reference to the contact vector $\mathbf{k}$. We follow an analytical approach to evaluate the above elliptic integrals following our previous works (Saha & Alam 2014, 2016) on sheared dry granular flows – the underlying methodology is identical to above works and is briefly discussed below.

Since both $\mathfrak{F}(\chi)$ and $\mathfrak{G}(\chi)$ are analytical functions of χ, we substitute (2.19) into (2.18a) and (2.18b) and use the power-series representations for the complementary error function and the exponential. This yields the following infinite series representation (Saha & Alam 2014) for $\mathfrak{F}(\chi)$ and $\mathfrak{G}(\chi)$:

$$\begin{aligned}
\mathfrak{F}(\eta,R,\phi,\lambda;\theta,\varphi) = -\sqrt{\pi}\Bigg[&\frac{3R\sin^2\varphi\cos(2\phi+2\theta)}{\{1-\eta\sin^2\varphi\cos 2\theta+\lambda^2(3\sin^2\varphi-2)\}^{\frac{1}{2}}}\\
&+\left\{\frac{2R\sin^2\varphi\cos(2\phi+2\theta)}{\{1-\eta\sin^2\varphi\cos 2\theta+\lambda^2(3\sin^2\varphi-2)\}^{\frac{1}{2}}}\right\}^3\Bigg]\\
+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{3}{(2n-1)(2n-3)}&\left[\frac{2R\sin^2\varphi\cos(2\phi+2\theta)}{\{1-\eta\sin^2\varphi\cos 2\theta+\lambda^2(3\sin^2\varphi-2)\}^{\frac{1}{2}}}\right]^{2n}, \quad \text{(A 8)}\\
\mathfrak{G}(\eta,R,\phi,\lambda;\theta,\varphi) = \sqrt{\pi}\Bigg[&\frac{1}{2}+\frac{4R^2\sin^4\varphi\cos^2(2\phi+2\theta)}{1-\eta\sin^2\varphi\cos 2\theta+\lambda^2(3\sin^2\varphi-2)}\Bigg]\\
+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{2}{(2n-1)(2n+1)}&\left[\frac{2R\sin^2\varphi\cos(2\phi+2\theta)}{\{1-\eta\sin^2\varphi\cos 2\theta+\lambda^2(3\sin^2\varphi-2)\}^{\frac{1}{2}}}\right]^{2n+1} \quad \text{(A 9)}
\end{aligned}$$

Substituting (A 8-A 9) into (A 5-A 7) and carrying out term-by-term integrations over

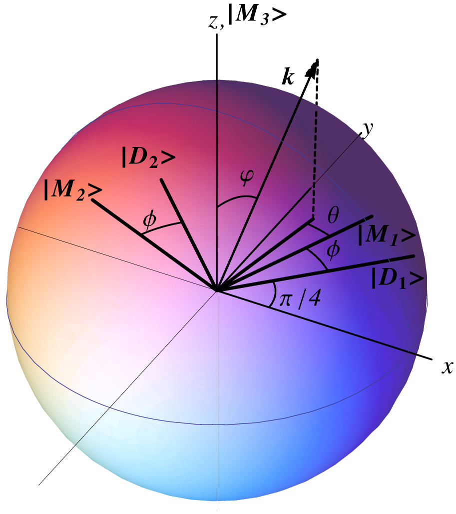

FIGURE 1. Sketch of the coordinate system (x, y, z), with the shear-field $[u_x(y) = \dot{\gamma} y]$ being applied on the (x, y)-plane and z is the mean vorticity direction; adapted from Saha & Alam (2016). The eigen-directions of the strain-rate tensor $\boldsymbol{D}$ are marked as $(|D_1\rangle, |D_2\rangle, |D_3\rangle)$ and that of the second moment tensor $\boldsymbol{M}$ as $(|M_1\rangle, |M_2\rangle, |M_3\rangle)$. Note that $\mathbf{k}$ represents the contact vector between two colliding particles, that makes an angle φ with z-direction and its projection on the (x, y) shear-plane makes an angle θ with $|M_1\rangle$ of $\boldsymbol{M}$. The *principle axes frame* (x', y', z') is given by the orthonormal triad of $(|M_1\rangle, |M_2\rangle, |M_3\rangle)$ which represents a rotation of the original (x, y, z) axes by an angle $\vartheta = \phi + \pi/4$ about the z-axis.

$\theta \in (0, 2\pi)$ and $\varphi \in (0, \pi)$, we obtain infinite series expressions for $\mathcal{H}^{\delta p}_{\alpha\beta\gamma}$, $\mathcal{J}^{\delta p}_{\alpha\beta\gamma}$ and $\mathcal{K}^{\delta p}_{\alpha\beta\gamma}$ as functions of η, $\sin\phi$, λ and R.

Truncating the infinite series representations of $\mathcal{H}^{\delta p}_{\alpha\beta\gamma}$, $\mathcal{J}^{\delta p}_{\alpha\beta\gamma}$ and $\mathcal{K}^{\delta p}_{\alpha\beta\gamma}$ at the Burnett order [i.e. the second-order in $O(\eta^i\lambda^j R^k \sin^l 2\phi$, with $i + j + k + l \leqslant 2)$], the following Burnett-order expressions are obtained:

$$\frac{1}{2}\Big(\cos 2\phi \mathcal{J}^{30}_{012} - \sin 2\phi \mathcal{J}^{30}_{102}\Big) = -\frac{4\pi}{15}\Big(8R + \sqrt{\pi}\eta\cos 2\phi\Big), \tag{A 10}$$

$$\mathcal{J}^{30}_{002} = \frac{4\pi}{105}\Big\{\sqrt{\pi}\Big(35 + 96R^2 + 14\lambda^2\Big) + 48R\eta\cos 2\phi\Big\}, \tag{A 11}$$

$$\big(\mathcal{J}^{30}_{102}\cos 2\phi + \mathcal{I}^{30}_{012}\sin 2\phi\big) = -\frac{8}{15}\pi^{\frac{3}{2}}\eta\sin 2\phi, \tag{A 12}$$

$$\left[\frac{1}{2}\left(\mathcal{H}^{30}_{003} - \cos 2\phi\mathcal{H}^{30}_{103} - \sin 2\phi\mathcal{H}^{30}_{013}\right)\right] = \frac{2\pi}{105}\Big(70 + 9\eta^2 + 42\lambda^2 + 288R^2 + 72\sqrt{\pi}\eta R\cos 2\phi + 42\eta\sin 2\phi\Big), \tag{A 13}$$

$$\left[\frac{1}{2}\left(\mathcal{H}^{30}_{003} + \cos 2\phi\mathcal{H}^{30}_{103} + \sin 2\phi\mathcal{H}^{30}_{013}\right)\right] = \frac{2\pi}{105}\Big(70 + 9\eta^2 + 42\lambda^2 + 288R^2 + 72\sqrt{\pi}\eta R\cos 2\phi - 42\eta\sin 2\phi\Big), \tag{A 14}$$

$$\mathcal{H}^{12}_{003} = \frac{2\pi}{105}\Big(70 + 3\eta^2 - 84\lambda^2 + 96R^2 + 24\sqrt{\pi}\eta R\cos 2\phi\Big), \tag{A 15}$$

$$\left[\frac{1}{2}\left(\cos 2\phi\mathcal{H}^{30}_{013} - \sin 2\phi\mathcal{H}^{30}_{103}\right)\right] = -\frac{4\pi}{5}\Big(2\sqrt{\pi}R + \eta\cos 2\phi\Big), \tag{A 16}$$

$$\widehat{E}_{yy} = -\frac{6(1+e)\rho\nu g_0 T^{\frac{3}{2}}}{\sigma\pi^{\frac{3}{2}}}\left[\frac{8\pi}{105}\left(\eta^2 + 21\lambda^2 + 6\sqrt{\pi}\eta R\cos 2\phi - 21\eta\sin 2\phi\right)\right], \tag{A 17}$$

$$\widehat{E}_{zz} = -\frac{12(1+e)\rho\nu g_0 T^{\frac{3}{2}}}{\sigma\pi^{\frac{3}{2}}}\left[-\frac{8\pi}{105}\left(\eta^2 + 21\lambda^2 + 6\sqrt{\pi}\eta R\cos 2\phi\right)\right], \tag{A 18}$$

$$\widehat{E}_{xy} = -\frac{12(1+e)\rho\nu g_0 T^{\frac{3}{2}}}{\sigma\pi^{\frac{3}{2}}}\left[-\frac{4\pi}{5}\eta\cos 2\phi\right], \tag{A 19}$$

$$2R\left\{\mathcal{K}_{00}^{31} + \cos 2\phi\left(\mathcal{K}_{10}^{31} + \mathcal{K}_{01}^{30}\right) + \sin 2\phi\left(\mathcal{K}_{01}^{31} - \mathcal{K}_{10}^{30}\right)\right\} = \frac{128\pi R^2}{105},\ \mathcal{K}_{00}^{31} = \frac{64\pi R}{105}, \tag{A 20}$$

$$\frac{1}{2}\cos 2\phi\left(\mathcal{K}_{01}^{31} - \mathcal{K}_{10}^{30}\right) + \frac{1}{2}\sin 2\phi\left(\mathcal{K}_{10}^{31} + \mathcal{K}_{01}^{30}\right) = \frac{2\pi}{105}\left[\sqrt{\pi}\left(21 + 32R^2\right) - 16\eta R\cos 2\phi\right]. \tag{A 21}$$

A.1. *Stress tensor up-to fourth-order*

The diagonal components of the dimensionless stress tensor, $P_{ii}^* = \tilde{P}_{ii}/\rho_p\nu U_R^2$, with $U_R = \dot{\gamma}\sigma/2$, correct up-to $\mathcal{O}(\eta^i\lambda^j R^k\sin^l 2\phi,\ i+j+k+l \leqslant 4)$, have following expressions:

$$\begin{aligned}\frac{P_{xx}^*}{T^*} &= (1+\lambda^2+\eta\sin 2\phi) + \frac{2(1+e)\nu g_0}{35}\Big[(35 + 96R^2 + 14\eta\sin 2\phi + 14\lambda^2)\\ &\quad + \frac{8}{\sqrt{\pi}}\eta R\cos 2\phi\Big\{3(66+5\eta^2-22\lambda^2) - 160R^2 - 22\eta\sin 2\phi\Big\}\Big],\end{aligned} \tag{A 22}$$

$$\begin{aligned}\frac{P_{yy}^*}{T^*} &= (1+\lambda^2-\eta\sin 2\phi) + \frac{2(1+e)\nu g_0}{35}\Big[(35 + 96R^2 - 14\eta\sin 2\phi + 14\lambda^2)\\ &\quad + \frac{8}{\sqrt{\pi}}\eta R\cos 2\phi\Big\{3(66+5\eta^2-22\lambda^2) - 160R^2 + 22\eta\sin 2\phi\Big\}\Big],\end{aligned} \tag{A 23}$$

$$\begin{aligned}\frac{P_{zz}^*}{T^*} &= (1-2\lambda^2) + \frac{2(1+e)\nu g_0}{35}\Big[(35 + 32R^2 - 28\lambda^2)\\ &\quad + \frac{8}{\sqrt{\pi}}\eta R\cos 2\phi(66+3\eta^2-32R^2)\Big].\end{aligned} \tag{A 24}$$

where $T^* = \tilde{T}/U_R^2$ is the dimensionless temperature.

To calculate the fourth-order viscosity, we need to retain terms up-to the fifth-order $\mathcal{O}(\eta^i\lambda^j R^k\sin^l 2\phi,\ i+j+k+l \leqslant 5)$ in the shear stress. The dimensionless shear stress, $P_{xy}^* = \tilde{P}_{xy}/\rho_p\nu U_R^2$, with $U_R = \dot{\gamma}\sigma/2$, at $\mathcal{O}(R^5)$, can be written as :

$$\begin{aligned}\frac{P_{xy}^*}{T^*} &= -\eta\cos 2\phi - \frac{4(1+e)\nu g_0}{105\sqrt{\pi}}\Big[21R\Big\{8 + \sqrt{\pi}\frac{\eta\cos 2\phi}{R}\Big\}\\ &\qquad + 4R^3\Big\{32 + 12\frac{\lambda^2}{R^2} - \frac{\eta^2}{R^2}(2+\cos 4\phi)\Big\}\\ &\quad + \frac{R^5}{143}\Big\{-5120 + 15\frac{\eta^2}{R^2}\left(64 - 5\frac{\eta^2}{R^2}\right)(3+2\cos 4\phi)\\ &\qquad + 52\frac{\lambda^2}{R^2}\left(-128 - 33\frac{\lambda^2}{R^2} + 12(2+\cos 4\phi)\frac{\eta^2}{R^2}\right)\Big\}\Big].\end{aligned} \tag{A 25}$$

Appendix B. Expressions for coefficients a_i in Eqn. (3.1)

The expressions for the coefficients of the temperature equation (3.1) are given by:

$$a_0 = 10000(1+e)^5\pi^{3/2}St_d^3\{63(1-e)\pi - 4(11+e)St_d\}^2(4+St_d^2)\nu^3 g_0^3 \tag{B 1}$$

$$\begin{aligned}
a_1 = &-2000(1+e)^2\pi St_d\nu g_0\Big(18522000(1-e)\pi^2 + 463050(1-e)\pi^3 St_d\,\{5-2(1+e)(1-3e)\nu g\}\\
&+29400\pi^2 St_d^2\{5(67-55e)+2(1+e)(23+4e-3e^2)\nu g\}\\
&+441\pi^2 St_d^3[125(13-9e)\pi - 50(1+e)(11-54e+27e^2)\pi\nu g\\
&+40(1+e)^2(1+3e)\{6(1-e)-\pi(1-3e)\}\nu^2 g_0^2 - 864(1-e)^2(3-e)(1+e)^4\nu^3 g_0^3]\\
&+56\pi St_d^4[1750(13-9e)\pi + 25(1+e)(277+54e-27e^2)\pi\nu g\\
&-20(1+e)^2(1+3e)(6(11+e)-(17+3e)\pi)\nu^2 g_0^2 + 648(1-e)(3-e)(1+e)^4(11+e)\nu^3 g_0^3]\\
&+12(1+e)^2 St_d^5\nu^2 g_0^2[2205(1-e)(1+3e)\pi^2\\
&-(3-e)(1+e)^2\{64(11+e)^2+3969(1-e)^2\pi^2\}\nu g]\\
&-336(1+e)^2(11+e)\pi St_d^6\nu^2 g_0^2\{5(1+3e)-9(1-e)(3-e)(1+e)^2\nu g\}\Big)
\end{aligned} \tag{B 2}$$

$$\begin{aligned}
a_2 = &-600(1+e)\sqrt{\pi}St_d\nu g_0\Big(54022500(1-e)(1+3e)\pi^3\\
&+444528000(1-e)(3-e)(1+e)^2\pi^2 St_d\nu g + 14700\pi^3 St_d^2\{175(11-9e)(1+3e)\\
&+5(1+e)(1829-381e-1773e^2+549e^3)\nu g + 126(1-3e)(1-e)(1+e)^3(-29+9e)\nu^2 g_0^2\}\\
&+8400(1+e)^2\pi^2 St_d^3\nu g_0\{5(11399-13078e+3071e^2)\\
&+14(1+e)(541-45e-133e^2+21e^3)\nu g\} + 147\pi^2 St_d^4[1875(13-9e)(1+3e)\pi\\
&+125(1+e)(1351+237e-999e^2+243e^3)\pi\nu g - 50(1+e)^2\{16(1+3e)^2\\
&+(1117-4836e-1566e^2+3636e^3-783e^4)\pi\}\nu^2 g_0^2\\
&+1440(3-e)(1+e)^4(1+3e)\{6(1-e)-\pi(1-3e)\}\nu^3 g_0^3\\
&-15552(1-e)^2(3-e)^2(1+e)^6\nu^4 g_0^4] + 28(1+e)^2\pi St_d^5\nu g_0[125(705-221e)(13-9e)\pi\\
&+50(1+e)(6417-17e-1689e^2+153e^3)\pi\nu g\\
&-320(3-e)(1+e)^2(1+3e)\{6(11+e)-(17+3e)\pi\}\nu^2 g_0^2\\
&+5184(1-e)(3-e)^2(1+e)^4(11+e)\nu^3 g_0^3] - 24(1+e)^2 St_d^6\nu^2 g_0^2[1225(1+3e)^2\pi^2\\
&-4410(3-e)(1-e)(1+e)^2(1+3e)\pi^2\nu g_0 + (3-e)^2(1+e)^4\{64(11+e)^2\\
&+3969(1-e)^2\pi^2\}\nu^2 g_0^2]\Big)
\end{aligned} \tag{B 3}$$

$$\begin{aligned}
a_3 = &+2520\pi\Big(96468750(1-e)\pi^3 + 91875\pi^2 St_d^2[25(13-9e)\pi + 80(1+e)(3-8e+6e^2)\pi\nu g_0\\
&-28(1-e)(1+e)^2\{36(3-e)(1+3e)+(1-3e)^2\pi\}\nu^2 g_0^2]\\
&-317520000(1-e)(3-e)^2(1+e)^4\pi St_d^3\nu^3 g_0^3\\
&+350(1+e)\pi^2 St_d^4\nu g_0[125(11-24e)(13-9e)\pi\\
&-25(1+e)\{12(1+3e)(937-1073e+252e^2)+(47-645e+1485e^2-567e^3)\pi\}\nu g_0\\
&-20(1+e)^2\{6(5333-2151e-4815e^2+2991e^3-462e^4)+7(1-3e)^2(1+3e)\pi\}\nu^2 g_0^2\\
&+12096(1-3e)(7-2e)(1-e)(3-e)(1+e)^4\nu^3 g_0^3]\\
&-800(3-e)(1+e)^4\pi St_d^5\nu^3 g_0^3[5(51851-59290e+13775e^2)\\
&+126(1+e)(5+3e)(53-32e+3e^2)\nu g_0]\\
&-42(1+e)^2\pi St_d^6\nu^2 g_0^2[625(13-9e)(19-6e)(1+3e)\pi\\
&+250(1+e)(1976+129e-1557e^2+615e^3-75e^4)\pi\nu g_0\\
&-100(3-e)(1+e)^2\{16(1+3e)^2+(523-2316e-882e^2+1692e^3-297e^4)\pi\}\nu^2 g_0^2
\end{aligned}$$

$$
\begin{aligned}
&+1440(3-e)^2(1+e)^4(1+3e)\{6(1-e)-(1-3e)\pi\}\nu^3 g_0^3 \\
&-10368(1-e)^2(3-e)^3(1+e)^6\nu^4 g_0^4] \\
&-8(3-e)(1+e)^4 St_d^7\nu^3 g_0^3[125(4709-4790e+977e^2)\pi \\
&+50(1+e)(3093+443e-1149e^2+45e^3)\pi\nu g_0 \\
&-160(3-e)(1+e)^2(1+3e)\{6(11+e)-(17+3e)\pi\}\nu^2 g_0^2 \\
&+1728(1-e)(3-e)^2(1+e)^4(11+e)\nu^3 g_0^3]\Big)
\end{aligned} \tag{B 4}
$$

$$
\begin{aligned}
a_4 = &+63(1+e)\pi^{3/2} St_d\nu g_0\Big(1543500000(59-23e)(1-e)\pi^2 \\
&+52500\pi St_d^2[25(14647-16830e+4919e^2)\pi \\
&+140(1+e)(2041-6273e+5999e^2-1383e^3)\pi\nu g_0 \\
&-588(1-e)(1+e)^2\{360(3-e)^2(1+3e)+(1-3e)^2(59-19e)\pi\}\nu^2 g_0^2] \\
&-20321280000(1-e)(3-e)^3(1+e)^4 St_d^3\nu^3 g_0^3-175\pi St_d^4[625(13-9e)(107+193e)\pi \\
&-500(1+e)(28181-94445e+71523e^2-14283e^3)\pi\nu g_0 \\
&+100(1+e)^2\{96(3-e)(1+3e)(4267-4873e+1134e^2) \\
&+(10223-137768e+356922e^2-219384e^3+36855e^4)\pi\}\nu^2 g_0^2 \\
&+5760(3-e)(1+e)^3\{6(1736-439e-1557e^2+831e^3-123e^4) \\
&+7(1-3e)^2(1+3e)\pi\}\nu^3 g_0^3-290304(27-7e)(1-3e)(1-e)(3-e)^2(1+e)^5\nu^4 g_0^4] \\
&-38400(3-e)^2(1+e)^4 St_d^5\nu^3 g_0^3\{5(34879-39854e+9151e^2) \\
&+42(1+e)(519+41e-191e^2+15e^3)\nu g_0\} \\
&-672(3-e)(1+e)^2 St_d^6\nu^2 g_0^2[625(1+3e)(382-391e+81e^2)\pi \\
&+375(1+e)(1303+394e-984e^2+198e^3-15e^4)\pi\nu g_0 \\
&-150(3-e)(1+e)^2\{16(1+3e)^2+(325-1476e-654e^2+1044e^3-135e^4)\pi\}\nu^2 g_0^2 \\
&+1440(3-e)^2(1+e)^4(1+3e)\{6(1-e)-(1-3e)\pi\}\nu^3 g_0^3 \\
&-7776(1-e)^2(3-e)^3(1+e)^6\nu^4 g_0^4]\Big)
\end{aligned} \tag{B 5}
$$

$$
\begin{aligned}
a_5 = &+3780(3-e)(1+e)^2\pi St_d^2\nu^2 g_0^2\Big(463050000(11-5e)(1-e)\pi^2 \\
&+4200\pi St_d^2[25(6405-8420e+3047e^2)\pi \\
&+140(1+e)(963-3040e+2881e^2-660e^3)\pi\nu g_0 \\
&-588(1-e)(1+e)^2\{120(3-e)^2(1+3e)+(29-9e)(1-3e)^2\pi\}\nu^2 g_0^2] \\
&-304819200(1-e)(3-e)^3(1+e)^4 St_d^3\nu^3 g_0^3-7\pi St_d^4[625(13-9e)(131+205e)\pi \\
&-500(1+e)(12821-45203e+33483e^2-6381e^3)\pi\nu g_0 \\
&+100(1+e)^2\{48(3-e)(1+3e)(2873-3281e+756e^2) \\
&+(5147-66416e+173322e^2-104688e^3+16443e^4)\pi\}\nu^2 g_0^2 \\
&+2880(3-e)(1+e)^3\{2(2556-235e-2253e^2+963e^3-135e^4) \\
&+7(1-3e)^2(1+3e)\pi\}\nu^3 g_0^3-145152(1-3e)(13-3e)(1-e)(3-e)^2(1+e)^5\nu^4 g_0^4] \\
&-768(3-e)^2(1+e)^4 St_d^5\nu^3 g_0^3\{5(8783-10054e+2279e^2) \\
&+42(1+e)(127+21e-55e^2+3e^3)\nu g_0\}\Big)
\end{aligned} \tag{B 6}
$$

$$
\begin{aligned}
a_6 = &+15876(3-e)^2(1+e)^3\sqrt{\pi}St_d^3\nu^3g_0^3\Big(132300000(17-9e)(1-e)\pi^2\\
&+St_d^2[5000(32639-50458e+21923e^2)\pi^2\\
&+252000(1+e)(605-1967e+1843e^2-417e^3)\pi^2\nu g_0\\
&-352800(1-e)(1+e)^2\pi\{180(3-e)^2(1+3e)+(57-17e)(1-3e)^2\pi\}\nu^2g_0^2]\\
&-34836480(1-e)(3-e)^3(1+e)^4St_d^3\nu^3g_0^3\\
&-\pi St_d^4[625(1951+1362e-1969e^2)\pi\\
&-500(1+e)(7757-28985e+20859e^2-3663e^3)\pi\nu g_0\\
&+100(1+e)^2\{96(3-e)(1+3e)(724-829e+189e^2)\\
&+(3455-42632e+112122e^2-66456e^3+9639e^4)\pi\}\nu^2g_0^2\\
&+1920(3-e)(1+e)^3\{3(1009+78e-864e^2+258e^3-33e^4)\\
&+7(1-3e)^2(1+3e)\pi\}\nu^3g_0^3-145152(1-3e)(1-e)(3-e)^2(5-e)(1+e)^5\nu^4g_0^4]\Big) \qquad (B\,7)
\end{aligned}
$$

$$
\begin{aligned}
a_7 = &\ 3048192(3-e)^3(1+e)^4St_d^4\nu^4g_0^4\Big(275625(47-29e)(1-e)\pi^2\\
&+5\pi St_d^2[25(3325-6284e+3211e^2)\pi+210(1+e)(427-1434e+1325e^2-294e^3)\pi\nu g_0\\
&-1764(1-e)(1+e)^2\{18(3-e)^2(1+3e)+(1-3e)^2(7-2e)\pi\}\nu^2g_0^2]\\
&-36288(1-e)(3-e)^3(1+e)^4St_d^3\nu^3g_0^3\Big) \qquad (B\,8)
\end{aligned}
$$

$$
\begin{aligned}
a_8 = &+2286144(1-e)(3-e)^4(1+e)^5\sqrt{\pi}St_d^5\nu^5g_0^5\Big(264600(43-31e)\pi\\
&+St_d^2[25(4069-5917e)\pi+420(1+e)(321-796e+219e^2)\pi\nu g_0\\
&-1764(1+e)^2\{24(3-e)^2(1+3e)+(1-3e)^2(11-3e)\pi\}\nu^2g_0^2]\Big) \qquad (B\,9)
\end{aligned}
$$

$$
a_9 = +725896442880(13-11e)(1-e)(3-e)^5(1+e)^6\pi St_d^6\nu^6g_0^6 \qquad (B\,10)
$$

$$
a_{10} = +1451792885760(3-e)^6(1-e)^2(1+e)^7\sqrt{\pi}St_d^7\nu^7g_0^7. \qquad (B\,11)
$$

Note that $a_{10}=0=a_9=a_8$ for $e=1$, leading to a seventh-order polynomial for the temperature equation (3.1).

B.1. *Coefficients in equation (3.2)*

The expressions for the coefficients of the quadratic polynomial (3.2) are given by

$$
\begin{aligned}
\mathfrak{a} = &\Big\{4375(403507877743-2246328611260e+3804527790618e^2\\
&-2609043261628e^3+637175564527e^4)+7000(1841694494389-7655913090100e\\
&+9936315870558e^2-5002578505972e^3+804447129877e^4)\nu g_0\\
&-1600(62781766140229-251533031265556e+383366453401950e^2\\
&-256170205851796e^3+62747567827717e^4)\nu^2g_0^2-5120(15714866428013\\
&-15498531439412e-53893773596274e^2+89513065956556e^3-35535888468883e^4)\nu^3g_0^3\\
&+16384(55881178513091-252823502621588e+422083141225338e^2\\
&-306989989576772e^3+81830578488731e^4)\nu^4g_0^4\Big\}, \qquad (B\,12)
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{b} = -\Big\{ & 625(18778586124871 - 207069149476948e + 405023838452178e^2 \\
& -286513612682692e^3 + 68321253421567e^4) - 4000(195746263524613 \\
& -832915077870466e + 1329037811913048e^2 - 934639544844094e^3 \\
& +243370589968483e^4)\nu g_0 - 3200(1463725651004753 - 5379389667615500e \\
& +7142510907700158e^2 - 4045727120864924e^3 + 816845472782489e^4)\nu^2 g_0^2 \\
& +10240(1346413690308923 - 5798356073502980e + 9330745323357066e^2 \\
& -6630820234286324e^3 + 1752390117615923e^4)\nu^3 g_0^3 \\
& +262144(16659939237511 - 30369558221428e - 11077909546134e^2 \\
& +44890628459564e^3 - 20186596004777e^4)\nu^4 g_0^4 \Big\},
\end{aligned}
\tag{B 13}
$$

$$
\begin{aligned}
\mathfrak{c} = 48\Big\{ & 1875(5292277916157 - 7198799909266e - 8983084338480e^2 \\
& +18332139150594e^3 - 7387862080253e^4) - 1000(653424923400545 \\
& -2603969210032424e + 3825404450727618e^2 - 2459579042010032e^3 \\
& +584400338366645e^4)\nu g_0 + 2400(449794922964867 - 2000985299639524e \\
& +3319509658053858e^2 - 2435905168315812e^3 + 667705483931683e^4)\nu^2 g_0^2 \\
& +7680(1001122119985027 - 3916700322857816e + 5710226196969174e^2 \\
& -3680145930163120e^3 + 885453528512687e^4)\nu^3 g_0^3 \\
& +4096(1-e)(1554567774124823 - 5288085618364101e \\
& +5986900650648069e^2 - 2247667963924055e^3)\nu^4 g_0^4 \Big\}.
\end{aligned}
\tag{B 14}
$$

Appendix C. Ignited state: analysis in the principal axes frame

The second-moment balance equation (4.1) in the principal-axes frame can be written as four independent equations (Alam, Saha & Gupta 2019):

$$
-4\eta\rho T\dot{\gamma}\cos 2\phi + 2\dot{\gamma}\left[(\Theta_{x'x'} - \Theta_{y'y'})\cos 2\phi - 2\Theta_{x'y'}\sin 2\phi\right] + \frac{6\rho T\dot{\gamma}}{St_d} = A_{\alpha'\alpha'}, \tag{C 1}
$$

$$
-4\eta\rho T\dot{\gamma}\cos 2\phi + 2\dot{\gamma}[(\Theta_{x'x'} - \Theta_{y'y'})\cos 2\phi - 2\Theta_{x'y'}\sin 2\phi] + \frac{12\rho T\dot{\gamma}\lambda^2}{St_d} = -3\widehat{\Gamma}_{z'z'}, \tag{C 2}
$$

$$
4\left(1+\lambda^2\right)\rho T\dot{\gamma}\cos 2\phi + 2\dot{\gamma}\left(\Theta_{x'x'} + \Theta_{y'y'}\right)\cos 2\phi - \frac{4\rho T\dot{\gamma}\eta}{St_d} = \Gamma_{x'x'} - \Gamma_{y'y'}, \tag{C 3}
$$

$$
2\rho T\dot{\gamma}\left[\eta - \left(1+\lambda^2\right)\sin 2\phi\right] - \left(\Theta_{x'x'} + \Theta_{y'y'}\right)\dot{\gamma}\sin 2\phi = \Gamma_{x'y'}. \tag{C 4}
$$

They represent (i) the trace of (4.1), with $A_{\alpha'\alpha'} = A_{x'x'} + A_{y'y'} + A_{z'z'}$, (ii) the z'-z' component of the deviatoric part of (4.1) (iii) the difference between the x'-x' and y'-y' components and (iv) the off-diagonal x'-y' component of (4.1). The expressions for various integrals $A_{\alpha'\beta'}$, $\widehat{E}_{\alpha'\beta'}$, $\widehat{G}_{\alpha'\beta'}$ and $\Theta_{\alpha'\beta'}$ appearing in (C 1-C 4) remain the same as those for the dry granular flow; these have been evaluated explicitly by Saha & Alam (2016).

Retaining terms up-to fourth-order $O(\eta^i\lambda^j R^k \sin^l 2\phi)$, $i+j+k+l \leqslant 4$ in the expressions of $A_{\alpha'\beta'}$, $\widehat{E}_{\alpha'\beta'}$, $\widehat{G}_{\alpha'\beta'}$ and $\Theta_{\alpha'\beta'}$, the above equations (C 1-C 4) simplify to the following set of four coupled nonlinear algebraic equations

$$
\left.\begin{array}{c}
5\sqrt{\pi}\eta R\cos 2\phi + 4(1+e)\nu g_0 R\big(\sqrt{\pi}\eta\cos 2\phi + 8R\big) \\
-\frac{3}{4}(1-e^2)\nu g_0\big(10+\eta^2+32R^2+8\sqrt{\pi}\eta R\cos 2\phi\big) - \frac{15\sqrt{\pi}R}{St_d} \\
-\frac{1}{28}\nu g_0 R^4\Big\{256 - 16(2+\cos 4\phi)(\frac{\eta}{R})^2 + 192(\frac{\lambda}{R})^2 - 6(\frac{\eta}{R})^2(\frac{\lambda}{R})^2 + \frac{3}{4}(\frac{\eta}{R})^4 + 64(\frac{\lambda}{R})^4\Big\} \\
+\frac{16}{21}(1+e)\nu g_0 R^4\Big\{32 - (2+\cos 4\phi)(\frac{\eta}{R})^2 + 12(\frac{\lambda}{R})^2\Big\} = 0,
\end{array}\right. \tag{C 5}
$$

$$
\begin{array}{c}
35\sqrt{\pi}\eta R\cos 2\phi + (1+e)\nu g_0\big[32(1+3e)R^2 - 3(3-e)(\eta^2+21\lambda^2) \\
-8\sqrt{\pi}(4-3e)\eta R\cos 2\phi\big] - \frac{210\sqrt{\pi}\lambda^2 R}{St_d} \\
+\frac{16}{33}(1+e)\nu g_0 R^4\Big\{32(5+3e) + 2(2+\cos 4\phi)(2-3e)(\frac{\eta}{R})^2 \\
-33(5-3e)(\frac{\lambda}{R})^2 - \frac{9}{32}(3-e)\big[(\frac{\eta}{R})^4 - 11(\frac{\eta}{R})^2(\frac{\lambda}{R})^2 - 66(\frac{\lambda}{R})^4\big]\Big\} = 0,
\end{array} \tag{C 6}
$$

$$
\begin{array}{c}
5\sqrt{\pi}R\cos 2\phi - (1+e)\nu g_0\big[3(3-e)\eta + 2\sqrt{\pi}(1-3e)R\cos 2\phi\big] - \frac{10\sqrt{\pi}\eta R}{St_d} \\
+5\sqrt{\pi}\lambda^2 R\cos 2\phi - (1+e)\nu g_0 R^3\Big\{\frac{8}{7}\sqrt{\pi}\big[(4-3e)(\frac{\lambda}{R})^2 - 8(1+e)\big]\cos 2\phi \\
+\frac{1}{42}(\frac{\eta}{R})\big[64(4-3e) - 32(5+3e)\cos 4\phi - 3(3-e)(\frac{\eta}{R})^2 + 36(3-e)(\frac{\lambda}{R})^2\big]\Big\} = 0,
\end{array} \tag{C 7}
$$

$$
\begin{array}{c}
5\big(\eta - \sin 2\phi\big) + 2(1+e)(1-3e)\nu g_0\sin 2\phi - 5\lambda^2\sin 2\phi \\
-\frac{8}{7}(1+e)\nu g_0 R^2\sin 2\phi\Big\{8(1+e) - (4-3e)(\frac{\lambda}{R})^2 + \frac{4}{3\sqrt{\pi}}(5+3e)\frac{\eta}{R}\cos 2\phi\Big\} = 0.
\end{array} \tag{C 8}
$$

Note that equations (C 5-C 6) contain only "even" order terms in $(\eta, \sin 2\phi, \lambda, R)$, with the neglected terms being of order six and beyond; on the other hand, equations (C 7-C 8) contain only "odd" order terms in $(\eta, \sin 2\phi, \lambda, R)$, with the neglected terms being of order five and beyond. Therefore, equations (C 5-C 8) belong to the "super-super-Burnett" orders since they incorporate all terms up-to "quartic-order" in the shear-rate $(R \sim \dot{\gamma})$.

C.1. *Exact solution at Burnett order for the whole range of density*

Removing the terms within the second-brackets in (C 5-C 8), we obtain the Burnett-order equations as given in (4.5). The latter equations admit an exact solution, given by (4.10a) and (4.11), as discussed in §4.1.

C.2. *Perturbation solutions beyond Burnett order*

The super-Burnett and super-super-Burnett order equations [(C 5-C 8)] are solved using a regular perturbation expansion around the exact Burnett-order solution (4.10a) and (4.11):

$$
\left.\begin{array}{rcl}
\eta &=& \eta^{(2)} + \varepsilon\eta^{(3)} + \varepsilon^2\eta^{(4)} \\
\lambda &=& \lambda^{(2)} + \varepsilon\lambda^{(3)} + \varepsilon^2\lambda^{(4)} \\
R &=& R^{(2)} + \varepsilon R^{(3)} + \varepsilon^2 R^{(4)} \\
\sin 2\phi &=& \sin^{(2)} 2\phi + \varepsilon\sin 2\phi^{(3)} + \varepsilon^2\sin 2\phi^{(4)}
\end{array}\right\}. \tag{C 9}
$$

In the above expressions $\varepsilon \sim \dot{\gamma}$ and the superscript "2" corresponds to the "Burnett-order" solution [i.e. the closed form expressions (4.10a) and (4.11) as given in §4.1] and the superscripts "3" and "4" correspond to the corrections at third and fourth orders, respectively.

Plugging (C 9) into corresponding third and fourth [(C 5-C 8)] order equations and after performing some cumbersome algebra we obtain a null-solution at third-order

$$\eta^{(3)} = 0 = \lambda^{(3)} = R^{(3)} = \sin 2\phi^{(3)}. \tag{C 10}$$

The solutions at fourth-order are given by :

$$\eta^{(4)} = -\frac{\begin{array}{l}\Big[\Big[\sqrt{\pi}(1+e)\nu g_0 \cos 2\phi^{(2)}\{5-2(1+e)(1-3e)\nu g_0\}\Big\{1024(5+3e){R^{(2)}}^4 \\ -192(1+3e){R^{(2)}}^2\Big({\eta^{(2)}}^2 - 4{\lambda^{(2)}}^2\Big) - 9(1-e)\Big({\eta^{(2)}}^4 - 8{\eta^{(2)}}^2{\lambda^{(2)}}^2 + 84{\lambda^{(2)}}^4\Big)\Big\}\Big] \\ -\Big[8\Big\{5\sqrt{\pi}\eta^{(2)}\cos 2\phi^{(2)} + 2(1+e)\nu g_0\Big(8(1+3e)R^{(2)} - (1-3e)\sqrt{\pi}\eta^{(2)}\cos 2\phi^{(2)}\Big)\Big\} \\ \times\Big\{210\sqrt{\pi}{\lambda^{(2)}}^2 R^{(2)}\cos 2\phi^{(2)} - 48(1+e)\sqrt{\pi}\nu g_0 R^{(2)}\cos 2\phi^{(2)}\Big((4-3e){\lambda^{(2)}}^2 - 8(1+e){R^{(2)}}^2\Big) \\ \qquad -3(1+e)\nu g_0\eta^{(2)}\Big(32(1-3e){R^{(2)}}^2 - (3-e)({\eta^{(2)}}^2 - {\lambda^{(2)}}^2)\Big)\Big\}\Big]\Big]\end{array}}{\begin{array}{l}168\Big[\sqrt{\pi}\cos 2\phi^{(2)}\Big\{2\sqrt{\pi}\cos 2\phi^{(2)}\{5-2(1+e)(1-3e)\nu g_0\}R^{(2)} - 3(1-e^2)\nu g_0\eta^{(2)}\Big\} \\ \times\{5-2(1+e)(1-3e)\nu g_0\} + 2\nu g_0\Big\{3(1+e)(3-e) + 10\sqrt{\pi}R_{St}^{(2)}\Big\}\Big\{5\sqrt{\pi}\eta^{(2)}\cos 2\phi^{(2)} \\ +2(1+e)\nu g_0\Big(8(1+3e)R^{(2)} - (1-3e)\sqrt{\pi}\eta^{(2)}\cos 2\phi^{(2)}\Big)\Big\}\Big]\end{array}} \tag{C 11}$$

$$\begin{array}{l}\lambda^{(4)} = \\ \dfrac{1}{77616\lambda^{(2)}}\Bigg(\Bigg[_a\dfrac{28(1+e)}{\Big\{3(1+e)(3-e) + 10\sqrt{\pi}R_{St}^{(2)}\Big\}}\Big\{1024(5+3e){R^{(2)}}^4 \\ +96{R^{(2)}}^2\Big(2(2-3e){\eta^{(2)}}^2 - 11(5-3e){\lambda^{(2)}}^2\Big) - 9(3-e)\Big({\eta^{(2)}}^4 - 11{\eta^{(2)}}^2{\lambda^{(2)}}^2 - 66{\lambda^{(2)}}^4\Big)\Big\}\Bigg]_a \\ -\Bigg[_b\dfrac{132}{\Big\{3(1+e)(3-e) + 10\sqrt{\pi}R_{St}^{(2)}\Big\}\sqrt{\pi}\{5-2(1+e)(1-3e)\nu g_0\}\nu^2 g_0^2\cos 2\phi^{(2)}} \\ \times\Big[35\sqrt{\pi}\eta^{(2)}\cos 2\phi^{(2)} + 8(1+e)\nu g_0\Big\{8(1+3e)R^{(2)} - (4-3e)\sqrt{\pi}\eta^{(2)}\cos 2\phi^{(2)}\Big\}\Big] \\ \times\Big[70\sqrt{\pi}{\lambda^{(2)}}^2 R^{(2)}\cos 2\phi^{(2)} + (1+e)\nu g_0\Big\{16\sqrt{\pi}R^{(2)}\cos 2\phi^{(2)}\Big(8(1+e){R^{(2)}}^2 - (4-3e){\lambda^{(2)}}^2\Big) \\ -32(1-3e){\eta^{(2)}}^2 R^{(2)} + (3-e)\eta^{(2)}\Big({\eta^{(2)}}^2 - 12{\lambda^{(2)}}^2\Big)\Big\}\Big]\Bigg]_b \\ +\Bigg[_c 1848\eta^{(4)}\Big[\dfrac{\sqrt{\pi}\{35-8(1+e)(4-3e)\nu g_0\}R^{(2)}\cos 2\phi^{(2)} - 6(3-e)(1+e)\nu g_0\eta^{(2)}}{\Big\{3(1+e)(3-e)\nu g_0 + 10\sqrt{\pi}\nu g_0 R_{St}^{(2)}\Big\}} \\ +\dfrac{\Big\{35\sqrt{\pi}\eta^{(2)}\cos 2\phi^{(2)} + 8(1+e)\nu g_0\Big(8(1+3e)R^{(2)} - (4-3e)\sqrt{\pi}\eta^{(2)}\cos 2\phi^{(2)}\Big)\Big\}}{\sqrt{\pi}\{5-2(1+e)(1-3e)\nu g_0\}\cos 2\phi^{(2)}}\Big]\Bigg]_c\Bigg)\end{array} \tag{C 12}$$

$$R^{(4)} = \frac{1}{42\sqrt{\pi}\{5-2(1+e)(1-3e)\nu g_0\}\cos 2\phi^{(2)}}\Big[-210\sqrt{\pi}{\lambda^{(2)}}^2 R^{(2)}\cos^{(2)} 2\phi$$
$$+48(1+e)\sqrt{\pi}\nu g_0 R^{(2)}\cos^{(2)} 2\phi\Big\{(4-3e){\lambda^{(2)}}^2 - 8(1+e){R^{(2)}}^2\Big\}$$
$$+3(1+e)\nu g_0 \eta^{(2)}\Big\{32(1-3e){R^{(2)}}^2 - (3-e)\Big({\eta^{(2)}}^2 - 12{\lambda^{(2)}}^2\Big)\Big\}$$
$$+42\eta^{(4)}\Big\{3(3-e)(1+e)+10\sqrt{\pi}R^{(2)}_{St}\Big\}\nu g_0\Big] \quad (C\,13)$$

$$\sin 2\phi^{(4)} = \frac{1}{21\sqrt{\pi}\{5-2(1+e)(1-3e)\nu g_0\}}\Big[105\sqrt{\pi}\Big(\eta^{(4)} - {\lambda^{(2)}}^2\sin^{(2)} 2\phi\Big)$$
$$-8(1+e)\nu g_0\sin^{(2)} 2\phi\Big\{4(5+3e)\eta^{(2)}R^{(2)}\cos^{(2)} 2\phi - 3\sqrt{\pi}\Big((4-3e){\lambda^{(2)}}^2 - 8(1+e){R^{(2)}}^2\Big)\Big\}\Big] \quad (C\,14)$$

where $R^{(2)}_{St} = R^{(2)}/\nu g_0 St_d$. In absence of interstitial gas, i.e., in the limit of $St_d \to \infty$, the St-dependent terms in (C 11-C 14) disappear and we obtain the corresponding super-super-Burnett order solutions for the dry-granular flow – the resulting expressions match exactly with the solution provided by Saha & Alam (2016) for the uniform shear flow of dry granular fluid.

In the dilute limit, the solutions at fourth-order are given by:

$$\eta^{(4)} = \frac{1}{56\{3(5-e)(1+e)\eta^{(2)} + 10\sqrt{\pi}R^{(2)}_{St}(St\cos 2\phi + 2\eta^{(2)})\}}\Big\{(1+e)(27-11e){\eta^{(2)}}^4$$
$$+252(1-e^2){\lambda^{(2)}}^4 + 560\sqrt{\pi}\cos 2\phi\eta^{(2)}{\lambda^{(2)}}^2 St R^{(2)}_{st} - 24(1+e)(13-5e){\eta^{(2)}}^2{\lambda^{(2)}}^2\Big\}, \quad (C\,15)$$

$$\lambda^{(4)} = \frac{7(1+e)}{17248\lambda^{(2)}\{3(3-e)(1+e)+10\sqrt{\pi}R^{(2)}_{St}\}}\Big\{132(19-11e){\lambda^{(2)}}^4$$
$$-(13+3e){\eta^{(2)}}^4 + 176{\eta^{(2)}}^2{\lambda^{(2)}}^2 - 88(5+3e)\eta^{(2)}\eta^{(4)}\Big\}, \quad (C\,16)$$

$$R^{(4)} = \frac{\nu}{560\sqrt{\pi}\eta^{(2)}\cos^{(2)} 2\phi}\Big\{3(1-e^2)\Big({\eta^{(2)}}^4 - 8{\eta^{(2)}}^2{\lambda^{(2)}}^2 + 84{\lambda^{(2)}}^4 + 56\eta^{(2)}\eta^{(4)}\Big)$$
$$-560\sqrt{\pi}\cos 2\phi St R^{(2)}_{St}\eta^{(4)}\Big\}, \quad (C\,17)$$

$$\sin 2\phi^{(4)} = \Big\{\eta^{(4)} - {\lambda^{(2)}}^2\sin^{(2)} 2\phi\Big\}. \quad (C\,18)$$

where $R^{(2)}_{St} = R^{(2)}/\nu$.

C.3. *Burnett-order equations for dilute gas-solid suspension: scaling arguments*

Considering the dry granular limit ($St \to \infty$) of a dilute gas-solid suspension, the Burnett-order second-moment balance equations are given by:

$$\left.\begin{array}{c} 20\sqrt{\pi}\eta R\cos 2\phi - 3(1-e^2)\nu(10+\eta^2) = 0 \\ 35\sqrt{\pi}\eta R\cos 2\phi - 3(1+e)(3-e)\nu(\eta^2+21\lambda^2) = 0 \\ 5\sqrt{\pi}R\cos 2\phi - 3(1+e)(3-e)\nu\eta = 0 \\ \eta - \sin 2\phi = 0 \end{array}\right\}. \quad (C\,19)$$

This admits an analytical solution of the form (Saha & Alam 2016)

$$\eta^2 = \frac{10(1-e)}{(11-3e)}, \quad \lambda^2 = \frac{20(1-e)}{7(11-3e)}, \quad \sin 2\phi = \eta, \quad R = \frac{3(3-e)(1+e)}{5\sqrt{\pi}\cos 2\phi}\frac{\sqrt{10}}{\sqrt{11-3e}}\nu\sqrt{1-e} \tag{C 20}$$

At leading order, we have

$$\eta = \frac{\sqrt{5(1-e)}}{2}, \ \lambda = \sqrt{\frac{5(1-e)}{14}}, \ \sin 2\phi = \frac{\sqrt{5(1-e)}}{4}, \ R = \frac{6}{\sqrt{5\pi}}\nu\sqrt{1-e}, \tag{C 21}$$

as was predicted also by Richman (1989). It is easy to verify that the following quantities,

$$\eta, \ \lambda, \ \sin 2\phi, \ R/\nu, \ \sim \ \sqrt{1-e}, \tag{C 22}$$

are of the same order which holds for a sheared dilute granular gas. Equation (C 22) helps to prove a scaling between the Stokes number and the inelasticity.

Proposition: The following scaling,

$$St \sim \sqrt{1-e}, \tag{C 23}$$

holds for a sheared dilute gas-solid suspension.

To clarify (C 23), let us consider the second-moment balance equations for a dilute suspension at the Burnett-order:

$$20\sqrt{\pi}\eta R\cos 2\phi - 3(1-e^2)\nu(10+\eta^2) - \frac{60\sqrt{\pi}R}{St} = 0 \tag{C 24a}$$

$$35\sqrt{\pi}\eta R\cos 2\phi + (1+e)\nu\big[-3(3-e)(\eta^2+21\lambda^2)\big] - \frac{210\sqrt{\pi}\lambda^2 R}{St} = 0, \tag{C 24b}$$

$$5\sqrt{\pi}R\cos 2\phi - 3(3-e)(1+e)\nu\eta - \frac{10\sqrt{\pi}\eta R}{St} = 0, \tag{C 24c}$$

$$5(\eta - \sin 2\phi) = 0. \tag{C 24d}$$

From (C 24c), we find

$$5\sqrt{\pi}R\cos 2\phi \ \sim R \sim \ \nu\sqrt{1-e} \quad \text{and} \quad 3(1+e)(3-e)\nu\eta \ \sim \ \nu\sqrt{1-e},$$

and hence

$$\frac{10\sqrt{\pi}\eta R}{St} \ \sim \ \frac{\sqrt{1-e}\nu\sqrt{1-e}}{\sqrt{1-e}} \ \sim \ \nu\sqrt{1-e},$$

which implies that all three terms in (C 24c) are of the same order. Similarly, from (C 24b) we have

$$35\sqrt{\pi}\eta R\cos 2\phi \ \sim \ \nu(1-e) \quad \text{and} \quad 3(1+e)(3-e)\nu(\eta^2+21\lambda^2) \ \sim \ \nu(1-e),$$

and hence

$$\frac{210\sqrt{\pi}\lambda^2 R}{St} \ \sim \ \frac{(1-e)\nu\sqrt{1-e}}{\sqrt{1-e}} \ \sim \ \nu(1-e),$$

again implying that three terms in (C 24b) are of the same order. From the above ordering arguments the proposition made in (C 23) is justified.

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REFERENCES

Alam, M. & Saha, S. 2017 Normal stress differences and beyond-Navier-Stokes hydrodynamics. *EPJ Conf. Proc.* **140**, 11014.

Alam, M., Saha, S. & Gupta, R. 2019 Unified theory for a sheared gas-solid suspension: from rapid-granular suspension to its small Stokes-number limit. *J. Fluid Mech.*, **870**, 1175-1193.

Richman, M. W. 1989 The source of second moment in dilute granular flows of highly inelastic spheres. *J. Rheol.* **33**, 1293-1306.

Saha, S. & Alam, M. 2014 Non-Newtonian stress, collisional dissipation and heat flux in the shear flow of inelastic disks: a reduction via Grad's moment method. *J. Fluid Mech.* **757**, 251-296.

Saha, S. & Alam, M. 2016 Normal stress differences, their origin and constitutive relations for a sheared granular fluid. *J. Fluid Mech.* **795**, 549-580.

Saha, S. & Alam, M. 2017 Revisiting ignited-quenched transition and the non-Newtonian rheology of a sheared dilute gas-solid suspension. *J. Fluid Mech.* **833**, 206-246.

Saha, S. & Alam, M. 2020 Burnett-order constitutive relations, second moment anisotropy and co-existing states in sheared dense gas-solid suspensions. *J. Fluid Mech.*, **887**, art. no. ?? (In press)