Supplementary Material (not for the paper publication)

Supplementary Material B. The transient boundary layer flow $[\breve{u}, \breve{v}]_{bl}$

We investigate the nature of $[\breve{u}, \breve{v}]_{bl}$ (5.6c) for $t \gg 1$. We anticipate that it is localised in a boundary layer of z-thickness $\Delta_{bl} = xt^{-1/2}$, near $z = 0$, for which convenient coordinates are (ζ, z):

$$\zeta \;=\; 2(1-\mathsf{x})t \;=\; O(1)\,, \qquad 2(1-\mathsf{x}) \;=\; \mathsf{z}^2 \,+\, O(\mathsf{z}^4)\,, \qquad \mathsf{z} \ll 1\,. \tag{1a–c}$$

The evaluation of (5.6c) for large t is helped by writing

$$[\,\breve{u}\,,\,\breve{v}\,]_{bl} \;=\; (\pi x)^{-1}[\,\mathsf{x}\mathsf{F}_i\,,\,\mathsf{F}_r + \mathsf{F}_0\,]\,, \qquad \mathsf{F}_0(t) \;=\; \int_t^\infty \frac{\mathrm{J}_1(2\tau)}{\tau}\,\mathrm{d}\tau \tag{2a,b}$$

with

$$\mathsf{F}_r + \mathrm{i}\mathsf{F}_i \;\equiv\; \mathsf{F}(\mathsf{x},t) \;=\; -\int_t^\infty \frac{\mathrm{J}_1(2\tau)}{\tau}\,\exp\bigl(\mathrm{i}2\mathsf{x}(t-\tau)\bigr)\,\mathrm{d}\tau\,. \tag{2c}$$

Since $\mathrm{J}_1(2\tau) = (\pi\tau)^{-1/2}\cos(2\tau - 3\pi/4) + O\bigl(\tau^{-3/2}\bigr)$, we have

$$\mathsf{F}_0 \;\approx\; \tfrac{1}{2}\pi^{-1/2}t^{-3/2}\cos(2t - \pi/4) \;=\; O\bigl(t^{-3/2}\bigr), \tag{3}$$

which is small compared to the resonant contribution to F:

$$\mathsf{F} \;=\; -\,\frac{\exp[\mathrm{i}(2t - 3\pi/4)]}{\pi^{1/2}}\int_t^\infty \frac{\exp\bigl[\mathrm{i}2(1-\mathsf{x})(\tau - t)\bigr]}{2\tau^{3/2}}\,\mathrm{d}\tau \;+\; O\bigl(t^{-3/2}\bigr),$$

obtained on the basis that x is close to unity. It may be expressed as

$$\mathsf{F} \;=\; \mathrm{i}\,\frac{\exp[\mathrm{i}(2t - \pi/4)]}{(\pi t)^{1/2}}\,\mathsf{G}(\zeta) \;+\; O\bigl(t^{-3/2}\bigr), \tag{4a}$$

where

$$\mathsf{G}(\zeta) \;=\; \frac{\zeta^{1/2}}{2}\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}\zeta'}}{(\zeta+\zeta')^{3/2}}\,\mathrm{d}\zeta' \;=\; 1 - (-\mathrm{i}\pi\zeta)^{1/2}\,\mathrm{e}^{-\mathrm{i}\zeta}\,\mathrm{erfc}\bigl((-\mathrm{i}\zeta)^{1/2}\bigr) \tag{4b}$$

$$=\; -\,\zeta^{1/2}\,\frac{\mathrm{d}}{\mathrm{d}\zeta}\biggl[\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}\zeta'}}{(\zeta+\zeta')^{1/2}}\biggr]\mathrm{d}\zeta' \;=\; -\,(\mathrm{i}\pi\zeta)^{1/2}\frac{\mathrm{d}}{\mathrm{d}\zeta}\bigl[\mathrm{e}^{-\mathrm{i}\zeta}\,\mathrm{erfc}\bigl((-\mathrm{i}\zeta)^{1/2}\bigr)\bigr], \tag{4c}$$

is a function of the similarity variable ζ (1a). Use of (4c) shows that

$$\int_0^\infty \frac{\mathsf{G}(\zeta)}{\zeta^{1/2}}\,\mathrm{d}\zeta \;=\; (\mathrm{i}\pi)^{1/2}\,. \tag{5}$$

With the help of (http://dlmf.nist.gov/7.6.E2), we may express (4b) in the form

$$\mathsf{G}(\zeta) \;=\; -(-\mathrm{i}\pi\zeta)^{1/2}\,\mathrm{e}^{-\mathrm{i}\zeta} \;+\; \mathsf{P}(\zeta)\,, \tag{6a}$$

where $\mathsf{P}(\zeta)$ is an entire function with the power series expansion

$$\begin{aligned}\mathsf{P}(\zeta) \;\equiv\; \mathsf{P}_r(\zeta) \,+\, \mathrm{i}\mathsf{P}_i(\zeta) \;&=\; 1 \,+\, (-\mathrm{i}\pi\zeta)^{1/2}\mathrm{e}^{-\mathrm{i}\zeta}\,\mathrm{erf}\bigl((-\mathrm{i}\zeta)^{1/2}\bigr)\\ &=\; 1 + \sum_{n=1}^{\infty}\frac{(-\mathrm{i}2\zeta)^n}{1\cdot 3\cdots(2n-1)}\,.\end{aligned} \tag{6b}$$

Explicitly the real and imaginary parts are

$$\mathsf{P}_r(\zeta) = 1 - (2\zeta)^2/3 + \cdots\,, \qquad \mathsf{P}_i(\zeta) = -2\zeta + (2\zeta)^3/15 + \cdots\,. \tag{6c}$$

The value of F determined by substitution of only the first term $-(-\mathrm{i}\pi\zeta)^{1/2}\,\mathrm{e}^{-\mathrm{i}\zeta}$ of G into (4a) is $-\sqrt{2(1-\mathsf{x})}\exp(2\mathrm{i}\mathsf{x}t) \approx -|\mathsf{z}|\exp(2\mathrm{i}\mathsf{x}t)$. It defines the contribution $-(\pi x)^{-1}|\mathsf{z}|[\mathsf{x}\sin(2\mathsf{x}t), \cos(2\mathsf{x}t)]$ to the flow $[\breve{u}, \breve{v}]_{bl}$ (2a), which exactly cancels the wave part of $[\breve{u}, \breve{v}]_{ms}$ (5.6b) so that their sum is simply $[0, \breve{v}_G]$. Hence the remaining second term $\mathsf{P}(\zeta)$ determines the complete boundary layer flow $[\breve{u}, \breve{v}]_{ms+bl}$:

$$\begin{bmatrix} (\pi x)\breve{u}_{ms+bl} \\ (\pi x)\breve{v}_{ms+bl} + 1 \end{bmatrix} = \frac{1}{(\pi t)^{1/2}} \begin{bmatrix} \mathsf{x}\mathsf{P}_r(\zeta) & -\mathsf{x}\mathsf{P}_i(\zeta) \\ -\mathsf{P}_i(\zeta) & -\mathsf{P}_r(\zeta) \end{bmatrix} \begin{bmatrix} \cos(2t - \pi/4) \\ \sin(2t - \pi/4) \end{bmatrix} + O(t^{-3/2})\,, \tag{7a}$$

in which (see (1))

$$\zeta = \mathsf{z}^2 t(1 + O(t^{-1})) \qquad \mathsf{x} = 1 + O(t^{-1}) \qquad \text{when} \qquad \zeta = O(1)\,. \tag{7b}$$

Note that the contribution from F_0 is $O(t^{-3/2})$ and contained in the error estimate. The $\zeta = 0$ values of (7a). At $z = 0$ the mainstream part $\big[\breve{u}_{ms},\, \breve{v}_{ms} + (\pi x)^{-1}\big]$ of (7a) vanishes. So what remains is $\big[\breve{u}_{bl},\, \breve{v}_{bl}\big]_{z=0}$, which with $\zeta = 0$, $\mathsf{x} = 1$ recovers (B1), valid for $t \gg 1$.

For large ζ, rather than (6), we use the asymptotic form

$$\mathsf{G}(\zeta) = \tfrac{1}{2}\mathrm{i}\zeta^{-1} + O\big(\zeta^{-2}\big) \qquad \text{for} \qquad |\zeta| \gg 1 \tag{8}$$

of (4b). To evaluate $[\breve{u}, \breve{v}]_{bl}$ from (2) in that limit, we find it tidier, though not essential, to reinstate the the asymptotic value (3) of F_0. Then substitution of (8) into (4a) determines

$$\begin{aligned} \mathsf{F} + \mathsf{F}_0 &= (\pi t)^{-1/2}\zeta^{-1}\big[-\exp\big(\mathrm{i}(2t - \pi/4)\big) + (1 - \mathsf{x})\cos(2t - \pi/4)\big] + O(t^{-3/2}) \\ &= -(\pi t)^{-1/2}\zeta^{-1}\big[\mathsf{x}\cos(2t - \pi/4) + \mathrm{i}\sin(2t - \pi/4)\big] + O(t^{-3/2})\,. \end{aligned} \tag{9}$$

In turn substitution into (2a) yields

$$[\,\breve{u}\,,\,\breve{v}\,]_{bl} = -(\pi\varpi\zeta)^{-1}(\pi t)^{-1/2}\big[\sin(2t - \pi/4)\,,\,\cos(2t - \pi/4)\,\big] + O(t^{-3/2})\,, \tag{10}$$

which tends to zero at fixed x as $z \to \infty$.

We make the approximation $\mathsf{x} \approx 1$ in (2a), continue to neglect F_0 and evaluate the mean value $(\pi x)^{-1}\langle \mathsf{F} \rangle$, using (4$a$), to obtain

$$\langle \breve{v}_{bl} \rangle + \mathrm{i}\langle \breve{u}_{bl} \rangle \approx \mathrm{i}\,(\pi x)^{-1}(\pi t)^{-1/2}\exp\big(\mathrm{i}(2t - \pi/4)\big)\int_0^1 \mathsf{G}(\zeta)\,\mathrm{d}z\,, \tag{11a}$$

which under the further approximation $\mathsf{z} \approx (\zeta/t)^{1/2}$ (see (7b)), implying $x^{-1}\mathrm{d}z \approx \frac{1}{2}(t\,\zeta)^{-1/2}\mathrm{d}\zeta$, yields

$$\langle \breve{v}_{bl} \rangle + \mathrm{i}\langle \breve{u}_{bl} \rangle \approx \mathrm{i}\,\frac{\exp\big(\mathrm{i}(2t - \pi/4)\big)}{2\pi t}\,\frac{1}{\pi^{1/2}}\int_0^{t/\varpi^2} \frac{\mathsf{G}(\zeta)}{\zeta^{1/2}}\,\mathrm{d}\zeta\,. \tag{11b}$$

Then in the limit $t/\varpi^2 \to \infty$, use of (5) determines

$$\big[\,\langle \breve{u}_{bl} \rangle\,,\,\langle \breve{v}_{bl} \rangle\,\big] = (2\pi t)^{-1}\big[\cos(2t)\,,\,-\sin(2t)\,\big] + O(t^{-3/2})\,, \tag{11c}$$

as given previously by (5.7c).