Supplementary Material

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1. Governing equations and boundary conditions

The linearised equations and boundary conditions are

$$\frac{\partial \omega}{\partial t} = \nu\left(\frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \omega}{\partial \theta^2}\right) \tag{1.1}$$

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = \omega, \tag{1.2}$$

$$\frac{\partial \eta}{\partial t} + \left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right)_{r=R_0} = 0, \tag{1.3}$$

$$\mu\left(\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} - \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}\right)_{r=R_0} = 0, \tag{1.4}$$

$$p(R_0,\theta,t) + 2\mu\left(\frac{1}{r}\frac{\partial^2 \psi}{\partial r\partial \theta} - \frac{1}{r^2}\frac{\partial \psi}{\partial \theta}\right)_{r=R_0} = \frac{T}{R_0^2}\left(\eta + \frac{\partial^2 \eta}{\partial \theta^2}\right) \tag{1.5}$$

$$\lim_{r\to\infty}\omega(r,\theta,t) = \texttt{finite},\ \lim_{r\to\infty}\psi(r,\theta,t) = \texttt{finite} \tag{1.6}$$

Note that in obtaining a linearised expression for $(\boldsymbol{\nabla}\cdot\mathbf{n})_{r=R_0+\eta}$, we have used the following expressions for the (surface) divergence of $\mathbf{n}$ in cylindrical coordinates.

$$\mathbf{n} = \frac{\boldsymbol{\nabla} F}{|\boldsymbol{\nabla} F|},\quad \boldsymbol{\nabla}\cdot\mathbf{n} \approx \boldsymbol{\nabla} F \equiv \left(1, -\frac{1}{r}\frac{\partial \eta}{\partial \theta}, 0\right),\quad (\boldsymbol{\nabla}\cdot\mathbf{n})_{r=R_0+\eta} \approx \frac{1}{R_0}\left(1 - \frac{\eta}{R_0}\right) - \frac{1}{R_0^2}\frac{\partial^2 \eta}{\partial \theta^2}. \tag{1.7}$$

In further analysis, we use governing equations 1.1 and 1.2 alongwith boundary conditions 1.3-1.6.

2. Linear stability analysis - Normal modes

2.1. *Discrete spectrum*

We seek normal mode solutions of standing wave form and set,

$$\eta(\theta,t) = a_0\cos(m\theta)\left[\frac{1}{2}\exp(\sigma t) + \texttt{c.c}\right], \tag{2.1}$$

$$\omega(r,\theta,t) = \sin(m\theta)\left[\frac{1}{2}\Omega(r)\exp(\sigma t) + \texttt{c.c.}\right], \tag{2.2}$$

$$\psi(r,\theta,t) = \sin(m\theta)\left[\frac{1}{2}\Psi(r)\exp(\sigma t) + \texttt{c.c}\right] \tag{2.3}$$

$$p(r,\theta,t) = \cos(m\theta)\left[\frac{1}{2}\mathcal{P}(r)\exp(\sigma t) + \texttt{c.c.}\right] \tag{2.4}$$

where `c.c.` stands for complex conjugate. Here $\Omega(r), \Psi(r)$ and $\mathcal{P}(r)$ are the eigenfunctions while σ is related to it's eigenvalue. We assume that a_0 and m are real (the latter restricted to only integer values for periodicity) while σ is allowed to be complex (temporal analysis). Due to σ being complex, $\Omega(r), \Psi(r)$ and $\mathcal{P}(r)$ are complex functions of a real argument as will be seen in subsequent algebra.

Substituting 2.1-2.4 into equations 1.1 and 1.2, we obtain

$$\left[\frac{\mathrm{d}^2\Omega}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\Omega}{\mathrm{d}r} - \left(\frac{m^2}{r^2} + \frac{\sigma}{\nu}\right)\Omega\right]\frac{\exp(\sigma t)}{2} + \texttt{c.c.} = 0 \tag{2.5}$$

and

$$\left[\frac{\mathrm{d}^2\Psi}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\Psi}{\mathrm{d}r} - \frac{m^2}{r^2}\Psi - \Omega\right]\frac{\exp(\sigma t)}{2} + \texttt{c.c.} = 0 \tag{2.6}$$

If equations 2.5 and 2.6 are to hold at all time t, the coefficient of $\exp(\sigma t)/2$ (or equivalently that of $\exp(\bar{\sigma} t)/2$), must be zero. The resultant equations are,

$$\frac{\mathrm{d}^2\Omega}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\Omega}{\mathrm{d}r} - \left(\frac{\sigma}{\nu} + \frac{m^2}{r^2}\right)\Omega = 0, \tag{2.7}$$

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\Psi}{\mathrm{d}r} - \frac{m^2}{r^2}\Psi = \Omega \tag{2.8}$$

Note that equation 2.7 has complex coefficients (since $\sigma \in \mathbb{C}$). Consequently both $\Omega(r)$ and $\Psi(r)$ are complex functions of a real argument (equation 2.8 has real coefficients but a complex inhomogenous term). The solution to the first of equations 2.14 is

$$\Omega(r) = \mathcal{C}\mathsf{K}_m(lr) + \mathcal{D}\mathrm{I}_m(lr) \tag{2.9}$$

where $\mathcal{C}, \mathcal{D}$ are complex constants of integration (coefficients of equation 2.7 are complex as $\sigma \in \mathbb{C}$), $l \equiv \sqrt{\frac{\sigma}{\nu}}$ and $\mathrm{I}_m, \mathsf{K}_m$ are mth order, modified Bessel functions of the first and second kind respectively. For preventing divergence of $\Omega(r)$ as $r \to \infty$, we set $\mathcal{D} = 0$. In doing this, we are inherently assuming that $\Re(l) > 0$†, since for fixed m (Abramowitz & Stegun 1965),

$$\lim_{z\to\infty} \mathrm{I}_m(z) \sim \frac{\exp(z)}{\sqrt{2\pi z}}, \tag{2.10}$$

† $\Re(z)$ and $\Im(z)$ denote the real and imaginary part of z

which diverges as $z \to \infty$ only if $\Re(z) > 0$. We thus have $\Omega(r) = \mathcal{C}\mathsf{K}_m(lr)$ and

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\Psi}{\mathrm{d}r} - \frac{m^2}{r^2}\Psi = \mathcal{C}\mathsf{K}_m(lr) \tag{2.11}$$

The general solution to equation 2.11 can be written as a linear combination of the two independent homogenous solutions $v_1(r) = r^m$ and $v_2(r) = r^{-m}$ and their Wronskian W (see equation 2.2.13 in Prosperetti (2011)),

$$\Psi(r) = \left[\alpha - \mathcal{C}\int^r \frac{v_2(q)\mathsf{K}_m(\hat{q})}{W(q)}dq\right]v_1(r) + \left[\beta + \mathcal{C}\int^r \frac{v_1(q)\mathsf{K}_m(\hat{q})}{W(q)}dq\right]v_2(r) \tag{2.12}$$

where α and β are real constants of integration (since coefficients of the left hand side of equation 2.8 are real) whose value depends on the choice of the lower limits of integration inside both the square brackets and $\hat{q} \equiv q\sqrt{\frac{\sigma}{\nu}} = ql$. We choose the lower limit as infinity and R_0 in the first and second integrals respectively. With $W(q) = -\frac{2m}{q}$, we obtain

$$\Psi(r) = \left[\alpha + \mathcal{C}\int_\infty^r \frac{q^{-m+1}\mathsf{K}_m(\hat{q})}{2m}dq\right]r^m + \left[\beta - \mathcal{C}\int_{R_0}^r \frac{q^{m+1}\mathsf{K}_m(\hat{q})}{2m}dq\right]r^{-m} \tag{2.13}$$

We set $\alpha = 0$ to prevent divergence as $r \to \infty$ † and obtain,

$$\Psi(r) = \mathcal{C}\left[\int_\infty^r \frac{q^{-m+1}\mathsf{K}_m(\hat{q})}{2m}dq\right]r^m + \left[\beta - \mathcal{C}\int_{R_0}^r \frac{q^{m+1}\mathsf{K}_m(\hat{q})}{2m}dq\right]r^{-m} \tag{2.14}$$

The first integral inside the square brackets in equation 2.14 may be simplified as follows.

$$\frac{\mathcal{C}}{2m}\int_\infty^r q^{-m+1}\mathsf{K}_m(\hat{q})\,dq = \frac{\mathcal{C}}{2m}\left(\frac{\nu}{\sigma}\right)^{1-\frac{m}{2}}\int_\infty^{\hat{r}} \hat{q}^{-m+1}\mathsf{K}_m(\hat{q})\,d\hat{q} \tag{2.15}$$

where $\hat{q} \equiv q\sqrt{\frac{\sigma}{\nu}} = ql$. Using the following identity (Weisstein 2001*a*),

$$\frac{d}{dz}\left(z^{-m}\mathsf{K}_m(z)\right) = -z^{-m}\mathsf{K}_{m+1}(z) \tag{2.16}$$

we obtain,

$$\frac{\mathcal{C}}{2m}\left(\frac{\nu}{\sigma}\right)^{1-\frac{m}{2}}\int_\infty^{\hat{r}} \hat{q}^{-m+1}\mathsf{K}_m(\hat{q})\,d\hat{q} = -\frac{\mathcal{C}}{2m}\left(\frac{\nu}{\sigma}\right)^{1/2} r^{-m+1}\mathsf{K}_{m-1}(\hat{r}) \tag{2.17}$$

Note that in writing the right side of equation 2.17, we have explicitly used the fact that $\lim_{\hat{q}\to\infty} \hat{q}^{-m+1}\mathsf{K}_{m-1}(\hat{q}) = 0$ for any positive integer m. Using similar arguments, we can also simplify the second integral in expression 2.14. Using the identity (Weisstein 2001*b*)

$$\frac{d}{dz}\left(z^m\mathsf{K}_m(z)\right) = -z^m\mathsf{K}_{m-1}(z)$$

† This is because $\mathsf{K}_m(z)$ decays as $\sqrt{\frac{\pi}{2z}}\exp(-z)$ as $z \to \infty$ and so the integral goes to zero exponentially fast.

we obtain

$$\frac{\mathcal{C}}{2m}\int_{R_0}^{r} q^{m+1}\mathsf{K}_m\left(\hat{q}\right)dq = \frac{\mathcal{C}}{2m}\left(\frac{\nu}{\sigma}\right)^{1+\frac{m}{2}}\int_{\hat{R}_0}^{\hat{r}} \hat{q}^{m+1}\mathsf{K}_m\left(\hat{q}\right)d\hat{q}$$
$$= -\frac{\mathcal{C}}{2m}\left(\frac{\nu}{\sigma}\right)^{1/2}\left[r^{m+1}\mathsf{K}_{m+1}\left(\hat{r}\right) - R_0^{m+1}\mathsf{K}_{m+1}\left(\hat{R}_0\right)\right] \quad (2.18)$$

where $\hat{R}_0 \equiv R_0\sqrt{\frac{\sigma}{\nu}}$. Combining expressions for the first and second integrals in equation 2.14, we obtain an expression for $\Psi(r)$

$$\Psi(r) = \left[-\frac{\mathcal{C}}{2ml}r^{-m+1}\mathsf{K}_{m-1}\left(\hat{r}\right)\right]r^{m}$$
$$+\left[\beta + \frac{\mathcal{C}}{2ml}\left\{r^{m+1}\mathsf{K}_{m+1}\left(\hat{r}\right) - R_0^{m+1}\mathsf{K}_{m+1}\left(\hat{R}_0\right)\right\}\right]r^{-m} \quad (2.19)$$

which can be re-written as

$$\Psi(r) = \beta r^{-m} + \frac{\mathcal{C}r}{2ml}\left\{\mathsf{K}_{m+1}\left(\hat{r}\right) - \mathsf{K}_{m-1}\left(\hat{r}\right)\right\} - \frac{\mathcal{C}R_0}{2ml}\left(\frac{r}{R_0}\right)^{-m}\mathsf{K}_{m+1}\left(\hat{R}_0\right) \quad (2.20)$$

Using the following identity (Abramowitz & Stegun 1965)

$$\mathsf{K}_{m+1}(z) - \mathsf{K}_{m-1}(z) = \frac{2m}{z}\mathsf{K}_m(z) \quad (2.21)$$

we simplify our expression for $\Psi(r)$ and obtain,

$$\Psi(r) = \beta r^{-m} + \left(\frac{\mathcal{C}}{l^2}\right)\mathsf{K}_m\left(\hat{r}\right) - \frac{\mathcal{C}R_0}{2ml}\left(\frac{r}{R_0}\right)^{-m}\mathsf{K}_{m+1}\left(\hat{R}_0\right) \quad (2.22)$$

For further algebra, we need expressions for the first and second derivative of $\Psi(r)$,

$$\frac{\mathrm{d}\Psi}{\mathrm{d}r} = -m\beta r^{-m-1} + \left(\frac{\mathcal{C}}{l}\right)\mathsf{K}'_m(\hat{r}) + \frac{\mathcal{C}}{2l}\left(\frac{r}{R_0}\right)^{-m-1}\mathsf{K}_{m+1}\left(\hat{R}_0\right) \quad (2.23)$$
$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}r^2} = m(m+1)\beta r^{-m-2} + \mathcal{C}\mathsf{K}''_m(\hat{r}) - \frac{\mathcal{C}}{2l}\frac{m+1}{R_0}\left(\frac{r}{R_0}\right)^{-m-2}\mathsf{K}_{m+1}\left(\hat{R}_0\right) \quad (2.24)$$

where $\mathsf{K}'_m(z) \equiv \dfrac{\mathrm{d}\mathsf{K}_m(z)}{\mathrm{d}z}$, $\mathsf{K}''_m(z) \equiv \dfrac{\mathrm{d}^2\mathsf{K}_m(z)}{\mathrm{d}z^2}$ and so on. In order to satisfy the boundary conditions, we need an expression for perturbation pressure. This is obtained from the (linearised) momentum equation for the radial component of velocity i.e.,

$$\frac{\partial u_r}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right) \quad (2.25)$$

which may be re-written in terms of streamfunction ψ

$$\begin{aligned}\frac{1}{\rho}\frac{\partial p}{\partial r} &= \frac{1}{r}\frac{\partial^2\psi}{\partial\theta\partial t} - \nu\left[\frac{1}{r^3}\frac{\partial^3\psi}{\partial\theta^3} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial r\partial\theta} + \frac{1}{r}\frac{\partial^3\psi}{\partial r^2\partial\theta}\right]\\ &= \frac{m\cos(m\theta)}{r}\left[\frac{1}{2}\left\{\sigma\Psi - \nu\left(\frac{\mathrm{d}^2\Psi}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\Psi}{\mathrm{d}r} - \frac{m^2}{r^2}\Psi\right)\right\}\exp(\sigma t) + \mathtt{c.c.}\right]\\ &= \frac{m\cos(m\theta)}{r}\left[\frac{1}{2}\left\{\sigma\Psi - \nu\mathcal{C}\mathsf{K}_m(\hat{r})\right\}\exp(\sigma t) + \mathtt{c.c.}\right]\\ &= m\cos(m\theta)\left[\frac{1}{2}\left\{\sigma\beta r^{-m-1} - \frac{\sigma\mathcal{C}}{2ml}\left(\frac{r}{R_0}\right)^{-m-1}\mathsf{K}_{m+1}\left(\hat{R}_0\right)\right\}\exp(\sigma t) + \mathtt{c.c.}\right]\end{aligned} \tag{2.26}$$

Equation 2.26 can be integrated from r to ∞ to obtain

$$\frac{1}{\rho}\left(p(\infty,\theta,t) - p(r,\theta,t)\right) = \cos(m\theta)\left[\frac{1}{2}\left\{\sigma\beta r^{-m} - \frac{\sigma\mathcal{C}R_0}{2ml}\left(\frac{r}{R_0}\right)^{-m}\mathsf{K}_{m+1}\left(\hat{R}_0\right)\right\}\exp(\sigma t) + \mathtt{c.c.}\right] \tag{2.27}$$

$$p(r,\theta,t) = \cos(m\theta)\left[\frac{1}{2}\mathcal{P}(r)\exp(\sigma t) + \mathtt{c.c.}\right] + p(\infty,\theta,t) \tag{2.28}$$

with $\mathcal{P}(r) \equiv \rho\left\{ -\sigma\beta r^{-m} + \frac{\sigma\mathcal{C}R_0}{2ml}\left(\frac{r}{R_0}\right)^{-m}\mathsf{K}_{m+1}\left(\hat{R}_0\right)\right\}$. Similar conclusions are obtained from the θ momentum equation which is

$$\frac{\partial u_\theta}{\partial t} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial\theta} + \nu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2} - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right) \tag{2.29}$$

Equation 2.29 may be re-written in terms of streamfunction as

$$\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial\theta} = -\frac{\partial^2\psi}{\partial t\partial r} + \nu\left(\frac{\partial^3\psi}{\partial r^3} + \frac{1}{r}\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^3\psi}{\partial\theta^2\partial r} - \frac{1}{r^2}\frac{\partial\psi}{\partial r} - \frac{2}{r^3}\frac{\partial^2\psi}{\partial\theta^2}\right) \tag{2.30}$$

Using 2.3 in 2.30, we obtain

$$\begin{aligned}\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial\theta} = \sin(m\theta)\Bigg[\Bigg\{ -\sigma\frac{d\Psi}{dr} + \nu\Bigg(\frac{d^3\Psi}{dr^3} + \frac{1}{r}\frac{d^2\Psi}{dr^2} - \frac{m^2}{r^2}\frac{d\Psi}{dr}\\ - \frac{1}{r^2}\frac{d\Psi}{dr} + \frac{2m^2}{r^3}\Psi\Bigg)\Bigg\}\frac{1}{2}\exp(\sigma t) + \mathtt{c.c.}\Bigg]\end{aligned}$$

$$\begin{aligned}\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial\theta} = \sin(m\theta)\Bigg[\Bigg\{ -\sigma\frac{d\Psi}{dr} + \nu\left(\frac{d^3\Psi}{dr^3} + \frac{1}{r}\frac{d^2\Psi}{dr^2} - \frac{m^2}{r^2}\frac{d\Psi}{dr}\right) - \nu\left(\frac{1}{r^2}\frac{d\Psi}{dr} - \frac{2m^2}{r^3}\Psi\right)\\ \Bigg\}\frac{1}{2}\exp(\sigma t) + \mathtt{c.c.}\Bigg]\end{aligned} \tag{2.31}$$

The derivative of 2.8 can be used to simplify equation 2.31. It may be re-written as

$$\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} = \sin(m\theta)\left[\left\{ -\sigma\frac{d\Psi}{dr} + \nu\left(\frac{d\Omega}{dr} + \frac{1}{r^2}\frac{d\Psi}{dr} - \frac{2m^2}{r^3}\Psi\right) - \nu\left(\frac{1}{r^2}\frac{d\Psi}{dr} - \frac{2m^2}{r^3}\Psi\right)\right.\right.$$
$$\left.\left.\right\}\frac{1}{2}\exp(\sigma t) + \texttt{c.c.}\right] \tag{2.32}$$

which further simplifies to,

$$\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} = \sin(m\theta)\left[\left\{ -\sigma\frac{d\Psi}{dr} + \nu\frac{d\Omega}{dr}\right\}\frac{1}{2}\exp(\sigma t) + \texttt{c.c.}\right] \tag{2.33}$$

Substituting the expressions for $\Psi(r)$ and $\Omega(r)$, we obtain

$$\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} = \sin(m\theta)\left[\left\{ -\sigma\left(-m\beta r^{-m-1} + \left(\frac{\mathcal{C}}{l}\right)\mathsf{K}'_m(\hat{r}) + \frac{\mathcal{C}}{2l}\left(\frac{r}{R_0}\right)^{-m-1}\mathsf{K}_{m+1}\left(\hat{R}_0\right)\right)\right.\right.$$
$$\left.\left.+\mathcal{C}l\nu\mathsf{K}'_m(\hat{r})\right\}\frac{1}{2}\exp(\sigma t) + \texttt{c.c.}\right] \tag{2.34}$$

which simplifies to,

$$\frac{\partial p}{\partial \theta} = \sin(m\theta)\left[\rho\left\{\sigma m\beta r^{-m} - \frac{\mathcal{C}\sigma}{2l}\left(\frac{r}{R_0}\right)^{-m} R_0\mathsf{K}_{m+1}\left(\hat{R}_0\right)\right\}\frac{1}{2}\exp(\sigma t) + \texttt{c.c.}\right] \tag{2.35}$$

Integrate equation 2.35 with respect to θ, we obtain

$$p(r,\theta,t) - p(r,0,t) = (\cos(m\theta) - 1)\left[\rho\left\{ -\sigma\beta r^{-m}\right.\right.$$
$$\left.\left.+\frac{\mathcal{C}\sigma}{2ml}\left(\frac{r}{R_0}\right)^{-m} R_0\mathsf{K}_{m+1}\left(\hat{R}_0\right)\right\}\frac{1}{2}\exp(\sigma t) + \texttt{c.c.}\right] \tag{2.36}$$

Comparing 2.36 with 2.27, we see that both lead to the same expression for $p(r,\theta,t)$ if we set $p(\infty,\theta,t) = 0$ with $p(r,0,t) = \left[\rho\left\{ -\sigma\beta r^{-m} + \frac{\mathcal{C}\sigma}{2ml}\left(\frac{r}{R_0}\right)^{-m} R_0\mathsf{K}_{m+1}\left(\hat{R}_0\right)\right\}\frac{1}{2}\exp(\sigma t) + \texttt{c.c.}\right]$. It is seen that $p(r,0,t) \to 0$ as $r \to \infty$. We thus have,

$$\Omega(r) = \mathcal{C}\mathsf{K}_m\left(\hat{r}\right) \tag{2.37}$$

$$\Psi(r) = \beta r^{-m} + \left(\frac{\mathcal{C}}{l^2}\right)\mathsf{K}_m\left(\hat{r}\right) - \frac{\mathcal{C}R_0}{2ml}\left(\frac{r}{R_0}\right)^{-m}\mathsf{K}_{m+1}\left(\hat{R}_0\right) \tag{2.38}$$

$$\mathcal{P}(r) = \rho\left\{ -\nu l^2\beta r^{-m} + \frac{\nu l\mathcal{C}R_0}{2m}\left(\frac{r}{R_0}\right)^{-m}\mathsf{K}_{m+1}\left(\hat{R}_0\right)\right\} \tag{2.39}$$

Equations 2.37-2.39 are to be used subsequently. Expressions 2.1-2.4 in boundary conditions 1.3-1.5 lead to

$$\left(\sigma a_0 + \frac{m}{R_0}\Psi(R_0)\right)\frac{1}{2}\exp(\sigma t) + \texttt{c.c.} = 0, \tag{2.40}$$

$$\left(\frac{\mathrm{d}^2\Psi}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}\Psi}{\mathrm{d}r} + \frac{m^2}{r^2}\Psi\right)_{r=R_0}\frac{1}{2}\exp(\sigma t) + \texttt{c.c.} = 0, \tag{2.41}$$

$$\left[\mathcal{P}(R_0) + 2\mu m\left(\frac{1}{r}\frac{\mathrm{d}\Psi}{\mathrm{d}r} - \frac{1}{r^2}\Psi\right)_{r=R_0} - \frac{Ta_0}{R_0^2}\left(1-m^2\right)\right]\frac{1}{2}\exp(\sigma t) + \texttt{c.c} = 0, \tag{2.42}$$

Since the above expressions must hold at all time t, the coefficients must be zero. This leads to the following homogenous equations.

$$\sigma a_0 + \frac{m}{R_0}\Psi(R_0) = 0 \tag{2.43}$$

$$\left(\frac{\mathrm{d}^2\Psi}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}\Psi}{\mathrm{d}r} + \frac{m^2}{r^2}\Psi\right)_{r=R_0} = 0 \tag{2.44}$$

$$\mathcal{P}(R_0) + 2\mu m\left(\frac{1}{r}\frac{\mathrm{d}\Psi}{\mathrm{d}r} - \frac{1}{r^2}\Psi\right)_{r=R_0} - \frac{Ta_0}{R_0^2}\left(1-m^2\right) = 0 \tag{2.45}$$

Substituting expressions for $\Psi(r)$ and $\mathcal{P}(r)$ from 2.37-2.39 into equations 2.43-2.45, we obtain (prime indicates differentiation with respect to the argument)

$$\frac{\nu l^2}{R_0}a_0 + \frac{1}{\hat{R}_0}\left(\frac{m}{\hat{R}_0}\mathsf{K}_m(\hat{R}_0) - \frac{1}{2}\mathsf{K}_{m+1}(\hat{R}_0)\right)\mathcal{C} + mR_0^{-m-2}\beta = 0 \tag{2.46}$$

$$2m(m+1)R_0^{-m-2}\beta + \left[\left(\mathsf{K}''_m - \frac{1}{\hat{r}}\mathsf{K}'_m + \frac{m^2}{\hat{r}^2}\mathsf{K}_m\right)_{\hat{r}=\hat{R}_0} - \frac{m+1}{\hat{R}_0}\mathsf{K}_{m+1}(\hat{R}_0)\right]\mathcal{C} = 0 \tag{2.47}$$

$$-\frac{T}{\rho R_0^2}(1-m^2)a_0 - \nu l^2 R_0^{-m}\,\beta + \frac{\nu\hat{R}_0}{2m}\mathsf{K}_{m+1}(\hat{R}_0)\,\mathcal{C} + 2\nu m\left[-(m+1)R_0^{-m-2}\beta\right.$$
$$\left. + \frac{1}{\hat{R}_0}\left\{\mathsf{K}'_m(\hat{R}_0) - \frac{\mathsf{K}_m(\hat{R}_0)}{\hat{R}_0} + \left(\frac{m+1}{2m}\right)\mathsf{K}_{m+1}(\hat{R}_0)\right\}\mathcal{C}\right] = 0 \tag{2.48}$$

The above set of equations are simplified further. We extensively use the following identities in subsequent algebra (Abramowitz & Stegun 1965)

$$\mathsf{K}'_m(z) = -\mathsf{K}_{m-1}(z) - \frac{m}{z}\mathsf{K}_m(z), \quad \mathsf{K}'_m(z) = -\mathsf{K}_{m+1}(z) + \frac{m}{z}\mathsf{K}_m(z), \tag{2.49}$$

$$-\mathsf{K}_{m-1}(z) + \mathsf{K}_{m+1}(z) = \frac{2m}{z}\mathsf{K}_m(z), \quad -\mathsf{K}_{m-1}(z) - \mathsf{K}_{m+1}(z) = 2\mathsf{K}'_m(z) \tag{2.50}$$

Consider equation 2.46. The coefficient of $\mathcal{C}$ using one of the identities in equation 2.49

can be re-written as

$$\begin{aligned}
\frac{1}{\hat{R}_0}\left(\frac{m}{\hat{R}_0}\mathsf{K}_m(\hat{R}_0)-\frac{1}{2}\mathsf{K}_{m+1}(\hat{R}_0)\right) &= \frac{1}{\hat{R}_0}\left(\frac{2m\mathsf{K}_m(\hat{R}_0)-\hat{R}_0\mathsf{K}_{m+1}(\hat{R}_0)}{2\hat{R}_0}\right)\\
&= \frac{1}{\hat{R}_0}\left(\frac{m\mathsf{K}_m(\hat{R}_0)+m\mathsf{K}_m(\hat{R}_0)-\hat{R}_0\mathsf{K}_{m+1}(\hat{R}_0)}{2\hat{R}_0}\right)\\
&= \frac{1}{\hat{R}_0}\left(\frac{m\mathsf{K}_m(\hat{R}_0)+\hat{R}_0\mathsf{K}'_m(\hat{R}_0)}{2\hat{R}_0}\right)\\
&= -\frac{1}{\hat{R}_0}\left(\frac{\hat{R}_0\mathsf{K}_{m-1}(\hat{R}_0)}{2\hat{R}_0}\right)\\
&= -\frac{\mathsf{K}_{m-1}(\hat{R}_0)}{2\hat{R}_0} \qquad (2.51)
\end{aligned}$$

Thus equation 2.46 reduces to

$$\frac{\nu l^2}{R_0}a_0 - \frac{\mathsf{K}_{m-1}(\hat{R}_0)}{2\hat{R}_0}\,\mathcal{C} + mR_0^{-m-2}\beta = 0 \qquad (2.52)$$

For simplifying equation 2.47, we differentiate the first identity in 2.49 leading to,

$$\mathsf{K}''_m(z) = -\mathsf{K}'_{m-1}(z) + \frac{m}{z^2}\mathsf{K}_m(z) - \frac{m}{z}\mathsf{K}'_m(z). \qquad (2.53)$$

Using 2.53 in the expression below, we obtain

$$\begin{aligned}
\mathsf{K}''_m(\hat{r}) - \frac{1}{\hat{r}}\mathsf{K}'_m(\hat{r}) + \frac{m^2}{\hat{r}^2}\mathsf{K}_m(\hat{r}) &= -\mathsf{K}'_{m-1}(\hat{r}) - \frac{m+1}{\hat{r}}\left(\mathsf{K}'_m(\hat{r}) - \frac{m}{\hat{r}}\mathsf{K}_m(\hat{r})\right)\\
&= -\mathsf{K}'_{m-1}(\hat{r}) + \frac{m+1}{\hat{r}}\mathsf{K}_{m+1}(\hat{r}) \qquad (2.54)
\end{aligned}$$

where we have used the second identity 2.49 in writing the second step. This can be used to simplify equation 2.47 to,

$$2m(m+1)R_0^{-m-2}\beta - \mathsf{K}'_{m-1}(\hat{R}_0)\mathcal{C} = 0 \qquad (2.55)$$

Similarly, equation 2.48 can be written as

$$\begin{aligned}
&-\frac{T}{\rho R_0^2}(1-m^2)a_0 - \nu l^2 R_0^{-m}\left(1+\frac{2m(m+1)}{\hat{R}_0^2}\right)\beta\\
&+\frac{\nu}{\hat{R}_0}\left[\frac{\hat{R}_0^2}{2m}\mathsf{K}_{m+1}(\hat{R}_0) + 2m\left(\mathsf{K}'_m(\hat{R}_0) - \frac{\mathsf{K}_m(\hat{R}_0)}{\hat{R}_0}\right) + (m+1)\,\mathsf{K}_{m+1}(\hat{R}_0)\right]\mathcal{C} = 0
\end{aligned} \qquad (2.56)$$

The coefficient of $\mathcal{C}$ equation in 2.56 can be simplified as follows. Using the identities written earlier in 2.49

$$
\begin{aligned}
&\frac{\nu}{\hat{R}_0}\left[\frac{\hat{R}_0^2}{2m}\mathsf{K}_{m+1}(\hat{R}_0)+2m\left(\mathsf{K}'_m(\hat{R}_0)-\frac{\mathsf{K}_m(\hat{R}_0)}{\hat{R}_0}\right)+(m+1)\,\mathsf{K}_{m+1}(\hat{R}_0)\right]\\
&=\frac{\nu}{\hat{R}_0}\left[\frac{\hat{R}_0^2}{2m}\mathsf{K}_{m+1}(\hat{R}_0)+2m\left(\mathsf{K}'_m(\hat{R}_0)-\frac{\mathsf{K}_m(\hat{R}_0)}{\hat{R}_0}\right)+(m+1)\left(-\mathsf{K}'_m(\hat{R}_0)+\frac{m}{\hat{R}_0}\mathsf{K}_m(\hat{R}_0)\right)\right]\\
&=\frac{\nu}{\hat{R}_0}\left[\frac{\hat{R}_0^2}{2m}\mathsf{K}_{m+1}(\hat{R}_0)-\frac{m\mathsf{K}_m(\hat{R}_0)}{\hat{R}_0}+m\left(\frac{m\mathsf{K}_m(\hat{R}_0)}{\hat{R}_0}+\mathsf{K}'_m(\hat{R}_0)\right)-\mathsf{K}'_m(\hat{R}_0)\right]\\
&=\frac{\nu}{\hat{R}_0}\left[\frac{\hat{R}_0^2}{2m}\mathsf{K}_{m+1}(\hat{R}_0)-\frac{m\mathsf{K}_m(\hat{R}_0)}{\hat{R}_0}-m\mathsf{K}_{m-1}(\hat{R}_0)-\mathsf{K}'_m(\hat{R}_0)\right]\\
&=\frac{\nu}{\hat{R}_0}\left[\frac{\hat{R}_0^2}{2m}\mathsf{K}_{m+1}(\hat{R}_0)-\frac{1}{2}\left(-\mathsf{K}_{m-1}(\hat{R}_0)+\mathsf{K}_{m+1}(\hat{R}_0)\right)-m\mathsf{K}_{m-1}(\hat{R}_0)\right.\\
&\quad\left.+\frac{1}{2}\left(\mathsf{K}_{m-1}(\hat{R}_0)+\mathsf{K}_{m+1}(\hat{R}_0)\right)\right]\\
&=\frac{\nu}{\hat{R}_0}\left[\frac{\hat{R}_0^2}{2m}\mathsf{K}_{m+1}(\hat{R}_0)-(m-1)\mathsf{K}_{m-1}(\hat{R}_0)\right] \qquad (2.57)
\end{aligned}
$$

Thus equation 2.56 simplifies to

$$
\begin{aligned}
&-\frac{T}{\rho R_0^2}(1-m^2)a_0-\nu l^2 R_0^{-m}\left(1+\frac{2m(m+1)}{\hat{R}_0^2}\right)\beta+\frac{\nu}{\hat{R}_0}\left[\frac{\hat{R}_0^2}{2m}\mathsf{K}_{m+1}(\hat{R}_0)\right.\\
&\left.-(m-1)\mathsf{K}_{m-1}(\hat{R}_0)\right]\mathcal{C}=0 \qquad (2.58)
\end{aligned}
$$

After these algebraic manipulations we obtain the final forms of equations 2.46-2.48.

$$
\frac{\nu l^2}{R_0}a_0-\frac{\mathsf{K}_{m-1}(\hat{R}_0)}{2\hat{R}_0}\,\mathcal{C}+mR_0^{-m-2}\beta=0 \qquad (2.59)
$$

$$
-\mathsf{K}'_{m-1}(\hat{R}_0)\,\mathcal{C}+2m(m+1)R_0^{-m-2}\beta=0 \qquad (2.60)
$$

$$
\begin{aligned}
&-\frac{T}{\rho R_0^2}(1-m^2)a_0+\frac{\nu}{\hat{R}_0}\left[\frac{\hat{R}_0^2}{2m}\mathsf{K}_{m+1}(\hat{R}_0)-(m-1)\mathsf{K}_{m-1}(\hat{R}_0)\right]\mathcal{C}\\
&-\nu l^2 R_0^{-m}\left(1+\frac{2m(m+1)}{\hat{R}_0^2}\right)\beta=0 \qquad (2.61)
\end{aligned}
$$

Equations 2.59-2.61 are linear equations in $a_0, \mathcal{C}$ and β. For a non-trivial solution, the determinant of the left hand side must be zero. Thus,

$$\begin{vmatrix} \nu l^2 & -\dfrac{\mathsf{K}_{m-1}(\hat{R}_0)}{2l} & \dfrac{m}{R_0} \\ 0 & -\mathsf{K}'_{m-1}(\hat{R}_0) & \dfrac{2m(m+1)}{R_0^2} \\ -\dfrac{T}{\rho R_0^2}(1-m^2) & \dfrac{\nu}{\hat{R}_0}\mathcal{G}(\hat{R}_0) & -\nu l^2\left(1+\dfrac{2m(m+1)}{\hat{R}_0^2}\right) \end{vmatrix} = 0$$

with $\mathcal{G}(\hat{R}_0) \equiv \dfrac{\hat{R}_0^2}{2m}\mathsf{K}_{m+1}(\hat{R}_0) - (m-1)\mathsf{K}_{m-1}(\hat{R}_0)$. Note that in the last column of the determinant, R_0^{-m} has been factored out as it appears in all the three terms. Similarly, $1/R_0$ has been factored out from the first row. Solving the determinant, we obtain

$$\begin{aligned} &\nu l^2\left[\nu l^2 \mathsf{K}'_{m-1}(\hat{R}_0)\left(1+\frac{2m(m+1)}{\hat{R}_0^2}\right) - \frac{\nu\mathcal{G}(\hat{R}_0)}{\hat{R}_0}\frac{2m(m+1)}{R_0^2}\right] \\ &+\frac{T\mathsf{K}_{m-1}(\hat{R}_0)}{l\rho R_0^4}m(m+1)(1-m^2) - \frac{T\mathsf{K}'_{m-1}(\hat{R}_0)}{\rho R_0^3}m(1-m^2) = 0 \end{aligned} \tag{2.62}$$

Re-arranging equation 2.62 we can write

$$\begin{aligned} &\nu l^2\left[\mathsf{K}'_{m-1}(\hat{R}_0)\nu l^2\left(1+\frac{2m(m+1)}{\hat{R}_0^2}\right) - \frac{\nu\mathcal{G}(\hat{R}_0)}{\hat{R}_0}\frac{2m(m+1)}{R_0^2}\right] \\ &+\frac{T}{\rho R_0^2}(1-m^2)\left(\frac{\mathsf{K}_{m-1}(\hat{R}_0)}{\hat{R}_0}\frac{m(m+1)}{R_0} - \mathsf{K}'_{m-1}(\hat{R}_0)\frac{m}{R_0}\right) = 0 \end{aligned} \tag{2.63}$$

Dividing equation 2.63 throughout by $\mathsf{K}'_{m-1}(\hat{R}_0)$ (it is checked that there are no zeroes of this on the complex plane whose $\Re\left(\hat{R}_0\right) > 0$), and re-arranging we obtain

$$\begin{aligned} &\left[\nu^2 l^4\left(1+\frac{2m(m+1)}{\hat{R}_0^2}\right) - \frac{2m(m+1)}{R_0^2\mathsf{K}'_{m-1}(\hat{R}_0)}\frac{\nu^2 l^2\mathcal{G}(\hat{R}_0)}{\hat{R}_0}\right] \\ &+\frac{T}{\rho R_0^3}m(m^2-1)\left(1-\frac{(m+1)\mathsf{K}_{m-1}(\hat{R}_0)}{\hat{R}_0\mathsf{K}'_{m-1}(\hat{R}_0)}\right) = 0 \end{aligned} \tag{2.64}$$

Equation 2.64 may be re-written as

$$\nu^2 l^4 + \frac{2m(m+1)}{\hat{R}_0^2}\left(1-\frac{\mathcal{G}(\hat{R}_0)}{\hat{R}_0\mathsf{K}'_{m-1}(\hat{R}_0)}\right)\nu^2 l^4 + \frac{Tm(m^2-1)}{\rho R_0^3}\left(1-\frac{(m+1)}{\hat{R}_0}\frac{\mathsf{K}_{m-1}(\hat{R}_0)}{\mathsf{K}'_{m-1}(\hat{R}_0)}\right) = 0 \tag{2.65}$$

Dividing and multiplying equation 2.65 by ν^2 and R_0^4 respectively, we obtain

$$\hat{R}_0^4 + \left\{2m(m+1)\left(1-\frac{\mathcal{G}(\hat{R}_0)}{\hat{R}_0\mathsf{K}'_{m-1}(\hat{R}_0)}\right)\right\}\hat{R}_0^2 + \mathrm{La}\, m(m^2-1)\left(1-\frac{(m+1)}{\hat{R}_0}\frac{\mathsf{K}_{m-1}(\hat{R}_0)}{\mathsf{K}'_{m-1}(\hat{R}_0)}\right) = 0 \tag{2.66}$$

2.2. *Continuous spectrum*

In our analysis in the previous section, we had assumed that $\Re(l) > 0$. We now allow for the possibility that $\Re(l) = 0$. Suppose $l \equiv \sqrt{\frac{\sigma}{\nu}} = \mathrm{I}\xi$ where $\xi \in \mathbb{R}^+$. This implies that σ is a real negative quantity ($\sigma = -\nu\xi^2$). In terms of ξ, equations 2.7 and 2.8 may be written as

$$\frac{\mathrm{d}^2\Omega}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\Omega}{\mathrm{d}r} + \left(\xi^2 - \frac{m^2}{r^2}\right)\Omega = 0, \tag{2.67}$$

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\Psi}{\mathrm{d}r} - \frac{m^2}{r^2}\Psi = \Omega \tag{2.68}$$

The solution to equation 2.67 is

$$\Omega(r) = \mathcal{C}\mathrm{J}_m(\xi r) + \mathcal{D}\mathrm{Y}_m(\xi r) \tag{2.69}$$

where $\mathcal{C}, \mathcal{D}$ are (real) constants of integration as coefficients of equation 2.67 are real. Equation 2.68 now becomes

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\Psi}{\mathrm{d}r} - \frac{m^2}{r^2}\Psi = \mathcal{C}\mathrm{J}_m(\xi r) + \mathcal{D}\mathrm{Y}_m(\xi r) \tag{2.70}$$

whose solution is (Prosperetti 2011)

$$\begin{aligned}\Psi(r) = &\left[\alpha - \int_{\infty}^{r} \frac{v_2(q)\left\{\mathcal{C}\mathrm{J}_m(\xi q) + \mathcal{D}\mathrm{Y}_m(\xi q)\right\}}{W(q)} dq\right] v_1(r) \\ &+ \left[\beta + \int_{R_0}^{r} \frac{v_1(q)\left\{\mathcal{C}\mathrm{J}_m(\xi q) + \mathcal{D}\mathrm{Y}_m(\xi q)\right\}}{W(q)} dq\right] v_2(r)\end{aligned} \tag{2.71}$$

With solution of the homogenous part of equation 2.70, $v_1(q) = q^m,\ v_2(q) = q^{-m},$ their Wronskian $W(q) = -\frac{2m}{q}$ and $\alpha = 0$ for boundedness at $r \to \infty$, we obtain

$$\begin{aligned}\Psi(r) = &\left[-\mathcal{C}\int_{r}^{\infty} \frac{q^{-m+1}\mathrm{J}_m(\xi q)}{2m} dq - \mathcal{D}\int_{r}^{\infty} \frac{q^{-m+1}\mathrm{Y}_m(\xi q)}{2m} dq\right] r^m \\ &+ \left[\beta - \mathcal{C}\int_{R_0}^{r} \frac{q^{m+1}\mathrm{J}_m(\xi q)}{2m} dq - \mathcal{D}\int_{R_0}^{r} \frac{q^{m+1}\mathrm{Y}_m(\xi q)}{2m} dq\right] r^{-m}\end{aligned} \tag{2.72}$$

Using the following identities (Abramowitz & Stegun 1965)

$$\frac{d}{dz}\left(z^{-m}\mathcal{Q}_m(z)\right) = -z^{-m}\mathcal{Q}_{m+1}(z), \quad \frac{d}{dz}\left(z^{m+1}\mathcal{Q}_{m+1}(z)\right) = z^{m+1}\mathcal{Q}_m(z) \tag{2.73}$$

where $\mathcal{Q}_m = \mathrm{J}_m$ or Y_m. We can simplify the integrals in equation 2.72 as follows.

$$\begin{aligned}
\frac{\mathcal{C}}{2m}\int_r^\infty q^{-m+1}\mathrm{J}_m\left(\xi q\right)dq &= \frac{\mathcal{C}\xi^{m-2}}{2m}\int_{\hat{r}}^\infty \hat{q}^{-m+1}\mathrm{J}_m\left(\hat{q}\right)d\hat{q}, \qquad (\hat{q}\equiv\xi q, \hat{r}\equiv\xi r)\\
&= \frac{\mathcal{C}\xi^{m-2}}{2m}\hat{r}^{-m+1}\mathrm{J}_{m-1}(\hat{r}), \qquad (m>1/2)\\
&= \frac{\mathcal{C}}{2m\xi}r^{-m+1}\mathrm{J}_{m-1}(\hat{r}) \qquad (2.74)
\end{aligned}$$

$$\begin{aligned}
\frac{\mathcal{C}}{2m}\int_{R_0}^r q^{m+1}\mathrm{J}_m\left(\xi q\right)dq &= \frac{\mathcal{C}\xi^{-m-2}}{2m}\int_{\hat{R}_0}^{\hat{r}} \hat{q}^{m+1}\mathrm{J}_m\left(\hat{q}\right)d\hat{q}, \qquad (\hat{R}_0\equiv\xi R_0)\\
&= \frac{\mathcal{C}\xi^{-m-2}}{2m}\left(\hat{r}^{m+1}\mathrm{J}_{m+1}(\hat{r}) - \hat{R}_0^{m+1}\mathrm{J}_{m+1}(\hat{R}_0)\right)\\
&= \frac{\mathcal{C}}{2m\xi}\left(r^{m+1}\mathrm{J}_{m+1}(\hat{r}) - R_0^{m+1}\mathrm{J}_{m+1}(\hat{R}_0)\right) \qquad (2.75)
\end{aligned}$$

As Y_m satisfies the same identities as J_m, the other two integrals can be handled in exactly the same way. We obtain now,

$$\begin{aligned}
\Psi(r) &= \left[-\frac{\mathcal{C}}{2m\xi}r^{-m+1}\mathrm{J}_{m-1}(\hat{r}) - \frac{\mathcal{D}}{2m\xi}r^{-m+1}\mathrm{Y}_{m-1}(\hat{r})\right]r^m\\
&+\left[\beta - \frac{\mathcal{C}}{2m\xi}\left\{r^{m+1}\mathrm{J}_{m+1}(\hat{r}) - R_0^{m+1}\mathrm{J}_{m+1}(\hat{R}_0)\right\}\right.\\
&\left.-\frac{\mathcal{D}}{2m\xi}\left\{r^{m+1}\mathrm{Y}_{m+1}(\hat{r}) - R_0^{m+1}\mathrm{Y}_{m+1}(\hat{R}_0)\right\}\right]r^{-m}\\
&= \beta r^{-m} - \frac{\mathcal{C}r}{2m\xi}\left\{\mathrm{J}_{m-1}(\hat{r}) + \mathrm{J}_{m+1}(\hat{r})\right\} - \frac{\mathcal{D}r}{2m\xi}\left\{\mathrm{Y}_{m-1}(\hat{r}) + \mathrm{Y}_{m+1}(\hat{r})\right\}\\
&+\frac{\mathcal{C}R_0}{2m\xi}\left(\frac{R_0}{r}\right)^m\mathrm{J}_{m+1}(\hat{R}_0) + \frac{\mathcal{D}R_0}{2m\xi}\left(\frac{R_0}{r}\right)^m\mathrm{Y}_{m+1}(\hat{R}_0) \qquad (2.76)
\end{aligned}$$

Using the identity (Abramowitz & Stegun 1965)

$$\mathcal{Q}_{m-1}(z) + \mathcal{Q}_{m+1}(z) = \frac{2m}{z}\mathcal{Q}_m(z) \qquad (2.77)$$

where $\mathcal{Q}_m(z) = \mathrm{J}_m(z)$ or $\mathrm{Y}_m(z)$, we can simplify equation 2.76 to

$$\Psi(r) = \beta r^{-m} - \frac{1}{\xi^2}\left\{\mathcal{C}\mathrm{J}_m(\hat{r}) + \mathcal{D}\mathrm{Y}_m(\hat{r})\right\} + \frac{R_0}{2m\xi}\left\{\mathcal{C}\mathrm{J}_{m+1}(\hat{R}_0) + \mathcal{D}\mathrm{Y}_{m+1}(\hat{R}_0)\right\}\left(\frac{R_0}{r}\right)^m \qquad (2.78)$$

For later algebra, we need the following expressions

$$
\begin{aligned}
\frac{d\Psi}{dr} &= -m\beta r^{-m-1} - \frac{1}{\xi}\left\{\mathcal{C}\frac{d\mathrm{J}_m(\hat{r})}{d\hat{r}} + \mathcal{D}\frac{d\mathrm{Y}_m(\hat{r})}{d\hat{r}}\right\} \\
&\quad - \frac{1}{2\xi}\left\{\mathcal{C}\mathrm{J}_{m+1}(\hat{R}_0) + \mathcal{D}\mathrm{Y}_{m+1}(\hat{R}_0)\right\}\left(\frac{R_0}{r}\right)^{m+1} \qquad (2.79) \\
\frac{d^2\Psi}{dr^2} &= m(m+1)\beta r^{-m-2} - \left\{\mathcal{C}\frac{d^2\mathrm{J}_m(\hat{r})}{d\hat{r}^2} + \mathcal{D}\frac{d^2\mathrm{Y}_m(\hat{r})}{d\hat{r}^2}\right\} \\
&\quad + \frac{m+1}{2\xi R_0}\left\{\mathcal{C}\mathrm{J}_{m+1}(\hat{R}_0) + \mathcal{D}\mathrm{Y}_{m+1}(\hat{R}_0)\right\}\left(\frac{R_0}{r}\right)^{m+2} \qquad (2.80)
\end{aligned}
$$

An expression for pressure is obtained from the radial momentum equation.

$$
\begin{aligned}
\frac{1}{\rho}\frac{\partial p}{\partial r} &= \frac{1}{r}\frac{\partial^2\psi}{\partial\theta\partial t} - \nu\left[\frac{1}{r^3}\frac{\partial^3\psi}{\partial\theta^3} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial r\partial\theta} + \frac{1}{r}\frac{\partial^3\psi}{\partial r^2\partial\theta}\right] \\
&= \frac{m}{r}\left(\sigma\Psi - \nu\Omega\right)\cos(m\theta)\exp(\sigma t). \qquad (2.81)
\end{aligned}
$$

The following expression is obtained

$$
\sigma\Psi - \nu\Omega = \sigma\beta r^{-m} + \frac{\sigma R_0}{2m\xi}\left\{\mathcal{C}\mathrm{J}_{m+1}(\hat{R}_0) + \mathcal{D}\mathrm{Y}_{m+1}(\hat{R}_0)\right\}\left(\frac{R_0}{r}\right)^{m} \qquad (2.82)
$$

and we have

$$
\frac{1}{\rho}\frac{\partial p}{\partial r} = \cos(m\theta)\exp(\sigma t)\left[m\sigma\beta + \frac{\sigma R_0^{m+1}}{2\xi}\left\{\mathcal{C}\mathrm{J}_{m+1}(\hat{R}_0) + \mathcal{D}\mathrm{Y}_{m+1}(\hat{R}_0)\right\}\right] r^{-m-1} \qquad (2.83)
$$

We thus obtain an expression for pressure by integrating from r to ∞. With $p(r,\theta,t) = \cos(m\theta)\exp(\sigma t)\mathcal{P}(r)$ and $P(\infty) = 0$, we obtain

$$
\mathcal{P}(r) = -\rho\left[\sigma\beta r^{-m} + \frac{\sigma R_0}{2\xi m}\left\{\mathcal{C}\mathrm{J}_{m+1}(\hat{R}_0) + \mathcal{D}\mathrm{Y}_{m+1}(\hat{R}_0)\right\}\left(\frac{R_0}{r}\right)^{m}\right] \qquad (2.84)
$$

We thus have

$$
\begin{aligned}
\eta(\theta,t) &= a_0\exp(\sigma t)\cos(m\theta) \qquad (2.85) \\
\omega(r,\theta,t) &= \exp(\sigma t)\sin(m\theta)\Omega(r), \qquad (2.86) \\
\psi(r,\theta,t) &= \exp(\sigma t)\sin(m\theta)\Psi(r) \qquad (2.87) \\
p(r,\theta,t) &= \exp(\sigma t)\cos(m\theta)\mathcal{P}(r) \qquad (2.88)
\end{aligned}
$$

Note that unlike the discrete spectrum case, σ in the continuous spectrum is real (as $\sigma = -\nu\xi^2$) and hence complex conjugation is not required in equations 2.85-2.88. Explicit boundary conditions have not been imposed so far (except boundedness at $r \to \infty$).

Substituting equations 2.85-2.88 in 1.3-1.5, we obtain

$$\sigma a_0 + \frac{m}{R_0}\Psi(R_0) = 0 \tag{2.89}$$

$$\left(\frac{d^2\Psi}{dr^2} - \frac{1}{r}\frac{d\Psi}{dr} + \frac{m^2}{r^2}\Psi\right)_{r=R_0} = 0 \tag{2.90}$$

$$\mathcal{P}(R_0) + 2\mu m\left(\frac{1}{r}\frac{\mathrm{d}\Psi}{\mathrm{d}r} - \frac{1}{r^2}\Psi\right)_{r=R_0} - \frac{T}{R_0^2}\left(1-m^2\right)a_0 = 0 \tag{2.91}$$

Equations 2.89-2.91 represent three linear homogenous equations in four unknowns viz. $a_0, \mathcal{C}, \mathcal{D}$ and β. In terms of these variables, 2.89 is

$$-\nu\xi^2 a_0 + \frac{1}{\xi}\left\{\frac{1}{2}\mathrm{J}_{m+1}(\hat{R}_0) - \frac{m}{\hat{R}_0}\mathrm{J}_m(\hat{R}_0)\right\}\mathcal{C} + \frac{1}{\xi}\left\{\frac{1}{2}\mathrm{Y}_{m+1}(\hat{R}_0) - \frac{m}{\hat{R}_0}\mathrm{Y}_m(\hat{R}_0)\right\}\mathcal{D} + mR_0^{-m-1}\beta = 0 \tag{2.92}$$

and equation 2.90 is

$$2m(m+1)R_0^{-m-2}\beta - \left\{\frac{\mathrm{d}^2\mathrm{J}_m}{\mathrm{d}\hat{r}^2} - \frac{1}{\hat{r}}\frac{\mathrm{d}\mathrm{J}_m}{\mathrm{d}\hat{r}} + \frac{m^2}{\hat{r}^2}\mathrm{J}_m(\hat{r})\right\}_{\hat{r}=\hat{R}_0}\mathcal{C}$$
$$-\left\{\frac{\mathrm{d}^2\mathrm{Y}_m}{\mathrm{d}\hat{r}^2} - \frac{1}{\hat{r}}\frac{\mathrm{d}\mathrm{Y}_m}{\mathrm{d}\hat{r}} + \frac{m^2}{\hat{r}^2}\mathrm{Y}_m(\hat{r})\right\}_{\hat{r}=\hat{R}_0}\mathcal{D} + \frac{m+1}{\xi R_0}\left\{\mathcal{C}\mathrm{J}_{m+1}(\hat{r}) + \mathcal{D}\mathrm{Y}_{m+1}(\hat{r})\right\}_{\hat{r}=\hat{R}_0} = 0 \tag{2.93}$$

Its can be shown that using identities for J_m and Y_m (Abramowitz & Stegun 1965) (see 2.53 and 2.54 for similar algebra) that

$$\frac{\mathrm{d}^2\mathcal{Q}_m}{\mathrm{d}\hat{r}^2} - \frac{1}{\hat{r}}\frac{\mathrm{d}\mathcal{Q}_m}{\mathrm{d}\hat{r}} + \frac{m^2}{\hat{r}^2}\mathcal{Q}_m(\hat{r}) = \frac{\mathrm{d}\mathcal{Q}_{m-1}}{\mathrm{d}\hat{r}} + \frac{m+1}{\hat{r}}\mathcal{Q}_{m+1}(\hat{r}) \tag{2.94}$$

with $\mathcal{Q}_m(z) = \mathrm{J}_m(z)$ or $\mathrm{Y}_m(z)$. Using this in equation 2.93 we may re-write it as,

$$2m(m+1)R_0^{-m-2}\beta - \left(\frac{\mathrm{d}\mathrm{J}_{m-1}}{\mathrm{d}\hat{r}}\right)_{\hat{r}=\hat{R}_0}\mathcal{C} - \left(\frac{\mathrm{d}\mathrm{Y}_{m-1}}{\mathrm{d}\hat{r}}\right)_{\hat{r}=\hat{R}_0}\mathcal{D} = 0. \tag{2.95}$$

Equation 2.95 may be re-written as

$$\left(\frac{\mathrm{d}\mathrm{J}_{m-1}}{\mathrm{d}\hat{r}}\right)_{\hat{r}=\hat{R}_0}\mathcal{C} + \left(\frac{\mathrm{d}\mathrm{Y}_{m-1}}{\mathrm{d}\hat{r}}\right)_{\hat{r}=\hat{R}_0}\mathcal{D} - 2m(m+1)R_0^{-m-2}\beta = 0 \tag{2.96}$$

Writing equation 2.91 as,

$$\rho\left[\nu\xi^2\beta R_0^{-m} + \frac{\nu\hat{R}_0}{2m}\left\{\mathcal{C}\mathrm{J}_{m+1}(\hat{R}_0) + \mathcal{D}\mathrm{Y}_{m+1}(\hat{R}_0)\right\}\right]$$
$$+ 2\mu m\left[-m\beta R_0^{-m-2} - \frac{\mathcal{C}}{\hat{R}_0}\left(\frac{\mathrm{d}\mathrm{J}_m(\hat{r})}{\mathrm{d}\hat{r}}\right)_{\hat{r}=\hat{R}_0} - \frac{\mathcal{D}}{\hat{R}_0}\left(\frac{\mathrm{d}\mathrm{Y}_m(\hat{r})}{\mathrm{d}\hat{r}}\right)_{\hat{r}=\hat{R}_0} - \frac{\mathcal{C}}{2\hat{R}_0}\mathrm{J}_{m+1}(\hat{R}_0)\right.$$
$$-\frac{\mathcal{D}}{2\hat{R}_0}\mathrm{Y}_{m+1}(\hat{R}_0) - \beta R_0^{-m-2} + \frac{1}{\hat{R}_0^2}\left\{\mathcal{C}\mathrm{J}_m(\hat{r}) + \mathcal{D}\mathrm{Y}_m(\hat{r})\right\}_{\hat{r}=\hat{R}_0}$$
$$\left.-\frac{1}{2m\hat{R}_0}\left\{\mathcal{C}\mathrm{J}_{m+1}(\hat{R}_0) + \mathcal{D}\mathrm{Y}_{m+1}(\hat{R}_0)\right\}\right] - \frac{T}{R_0^2}\left(1-m^2\right)a_0 = 0 \tag{2.97}$$

Equation 2.97 may be further simplified to

$$
\begin{aligned}
&-\frac{T\left(1-m^2\right)}{\rho R_0^2}a_0+\nu\xi^2 R_0^{-m}\left(1-\frac{2m\left(m+1\right)}{\hat{R}_0^2}\right)\beta\\
&+\frac{\nu}{\hat{R}_0}\left[\frac{\hat{R}_0^2}{2m}\mathrm{J}_{m+1}-2m\left(\mathrm{J}_m^{'}(\hat{R}_0)-\frac{\mathrm{J}_m(\hat{R}_0)}{\hat{R}_0}\right)-\left(m+1\right)\mathrm{J}_{m+1}(\hat{R}_0)\right]\mathcal{C}\\
&+\frac{\nu}{\hat{R}_0}\left[\frac{\hat{R}_0^2}{2m}\mathrm{Y}_{m+1}-2m\left(\mathrm{Y}_m^{'}(\hat{R}_0)-\frac{\mathrm{Y}_m(\hat{R}_0)}{\hat{R}_0}\right)-\left(m+1\right)\mathrm{Y}_{m+1}(\hat{R}_0)\right]\mathcal{D}=0
\end{aligned}
\tag{2.98}
$$

Using the same set of steps as equation 2.57 and identities for J_m and Y_m (Abramowitz & Stegun 1965), the coefficient of $\mathcal{C}$ and $\mathcal{D}$ may be simplified in 2.98 to obtain,

$$
\begin{aligned}
&-\frac{T\left(1-m^2\right)}{\rho R_0^2}a_0+\nu\xi^2 R_0^{-m}\left(1-\frac{2m\left(m+1\right)}{\hat{R}_0^2}\right)\beta\\
&+\frac{\nu}{\hat{R}_0}\left[\frac{\hat{R}_0^2}{2m}\mathrm{J}_{m+1}\left(\hat{R}_0\right)-\left(m-1\right)\mathrm{J}_{m-1}\left(\hat{R}_0\right)\right]\mathcal{C}+\frac{\nu}{\hat{R}_0}\left[\frac{\hat{R}_0^2}{2m}\mathrm{Y}_{m+1}\left(\hat{R}_0\right)-\left(m-1\right)\mathrm{Y}_{m-1}\left(\hat{R}_0\right)\right]\mathcal{D}\\
&=0
\end{aligned}
\tag{2.99}
$$

We write the final form for the three boundary conditions in equations 2.89,2.90 and 2.91 together below. As these are three equations in four unknowns, we can determine three ratios which we choose to be a_0/β, $\mathcal{C}/\beta$ and $\mathcal{D}/\beta$.

$$
-\nu\xi^2\frac{a_0}{\beta}+\frac{1}{\xi}\left\{\frac{1}{2}\mathrm{J}_{m+1}(\hat{R}_0)-\frac{m}{\hat{R}_0}\mathrm{J}_m(\hat{R}_0)\right\}\frac{\mathcal{C}}{\beta}+\frac{1}{\xi}\left\{\frac{1}{2}\mathrm{Y}_{m+1}(\hat{R}_0)-\frac{m}{\hat{R}_0}\mathrm{Y}_m(\hat{R}_0)\right\}\frac{\mathcal{D}}{\beta}=-mR_0^{-m-1}
\tag{2.100}
$$

$$
\left(\frac{\mathrm{dJ}_{m-1}}{\mathrm{d}\hat{r}}\right)_{\hat{r}=\hat{R}_0}\frac{\mathcal{C}}{\beta}+\left(\frac{\mathrm{dY}_{m-1}}{\mathrm{d}\hat{r}}\right)_{\hat{r}=\hat{R}_0}\frac{\mathcal{D}}{\beta}=2m(m+1)R_0^{-m-2}
\tag{2.101}
$$

$$
\begin{aligned}
&-\frac{T\left(1-m^2\right)}{\rho R_0^2}\frac{a_0}{\beta}+\frac{\nu}{\hat{R}_0}\left[\frac{\hat{R}_0^2}{2m}\mathrm{J}_{m+1}\left(\hat{R}_0\right)-\left(m-1\right)\mathrm{J}_{m-1}\left(\hat{R}_0\right)\right]\frac{\mathcal{C}}{\beta}\\
&+\frac{\nu}{\hat{R}_0}\left[\frac{\hat{R}_0^2}{2m}\mathrm{Y}_{m+1}\left(\hat{R}_0\right)-\left(m-1\right)\mathrm{Y}_{m-1}\left(\hat{R}_0\right)\right]\frac{\mathcal{D}}{\beta}=-\nu\xi^2R_0^{-m}\left(1-\frac{2m\left(m+1\right)}{\hat{R}_0^2}\right)
\end{aligned}
\tag{2.102}
$$

Equations 2.100, 2.101 and 2.102 are three *inhomogenous* equations. For obtaining a non-trivial solution, the determinant of the coefficients of the left hand side should not be zero. It is clear that there is no dispersion relation in the present case and a non-trivial value of a_0/β, $\mathcal{C}/\beta$, $\mathcal{D}/\beta$ is obtained for every value of $0<\xi<\infty$ by solving equations 2.100-2.102. Note that since $\sigma=-\nu\xi^2$, this implies that $-\infty<\sigma<0$ for the continuous spectrum.

3. Initial value problem (IVP)

For studying the IVP, we set

$$\eta(\theta, t) = a(t)\cos(m\theta), \tag{3.1}$$
$$\omega(r, \theta, t) = \sin(m\theta)\Omega(r, t), \tag{3.2}$$
$$\psi(r, \theta, t) = \sin(m\theta)\Psi(r, t) \tag{3.3}$$
$$p(r, \theta, t) = \cos(m\theta)\mathcal{P}(r, t) \tag{3.4}$$

In order to avoid profusion of symbols in the algebra, we have retained the same symbols for the radial part of the ω, ψ and p as in earlier sections. Substituting expressions 3.1-3.4 in equation 1.1 and 1.2, we obtain

$$\frac{\partial \Omega}{\partial t} = \nu\left[\frac{\partial^2 \Omega}{\partial r^2} + \frac{1}{r}\frac{\partial \Omega}{\partial r} - \frac{m^2}{r^2}\Omega\right] \tag{3.5}$$
$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} - \frac{m^2}{r^2}\Psi = \Omega \tag{3.6}$$

Define the Laplace transform and inverse transform pair $f(t)$ and $\tilde{f}(s)$ as,

$$\tilde{f}(s) \equiv \int_0^\infty f(t)\exp(-st)dt \tag{3.7}$$

With initial conditions $a(0) = a_0, \dot{a}(0) = 0,\ \Omega(r, 0) = 0$, the Laplace transform of equations 3.5 and 3.6 is (Laplace transformed variables are indicated with a tilde on top)

$$\frac{\partial^2 \tilde{\Omega}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{\Omega}}{\partial r} - \left(\frac{s}{\nu} + \frac{m^2}{r^2}\right)\tilde{\Omega} = 0, \quad \frac{\partial^2 \tilde{\Psi}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{\Psi}}{\partial r} - \frac{m^2}{r^2}\tilde{\Psi} = \tilde{\Omega}(r, s) \tag{3.8}$$

The algebra henceforth is entirely similar to the discrete spectrum algebra except that now equations 3.8 are partial differential equations. The solution to the first of equations 3.8 is

$$\tilde{\Omega}(r, s) = \tilde{\mathcal{C}}(s)\mathsf{K}_m(hr) + \tilde{\mathcal{D}}(s)\mathrm{I}_m(hr) \tag{3.9}$$

where $h \equiv \sqrt{\frac{s}{\nu}}$ and $\tilde{\mathcal{C}}(s), \tilde{\mathcal{D}}(s)$ are complex functions of s, $h \equiv \sqrt{\frac{s}{\nu}}$ and $\mathrm{I}_m, \mathsf{K}_m$ are mth order modified Bessel functions of the first and second kind respectively. For preventing divergence of $\tilde{\Omega}(r, s)$ as $r \to \infty$, we set $\tilde{\mathcal{D}}(s) = 0$. We thus have $\tilde{\Omega}(r, s) = \tilde{\mathcal{C}}(s)\mathsf{K}_m(hr)$.

$$\frac{\partial^2 \tilde{\Psi}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{\Psi}}{\partial r} - \frac{m^2}{r^2}\tilde{\Psi} = \tilde{\mathcal{C}}(s)\mathsf{K}_m(hr), \quad \Re(h) > 0 \tag{3.10}$$

Similar to the discrete spectrum case, we obtain

$$\tilde{\Omega}(r,s) = \tilde{\mathcal{C}}(s)\mathsf{K}_m\left(hr\right) \tag{3.11}$$

$$\tilde{\Psi}(r,s) = \tilde{\beta}(s)r^{-m} + \left(\frac{\tilde{\mathcal{C}}(s)}{h^2}\right)\mathsf{K}_m\left(hr\right) - \frac{\tilde{\mathcal{C}}(s)R_0}{2mh}\left(\frac{r}{R_0}\right)^{-m}\mathsf{K}_{m+1}\left(hR_0\right) \tag{3.12}$$

$$\frac{\mathsf{d}\tilde{\Psi}}{\mathsf{d}r} = -m\tilde{\beta}(s)r^{-m-1} + \left(\frac{\tilde{\mathcal{C}}(s)}{h}\right)\mathsf{K}'_m(hr) + \frac{\tilde{\mathcal{C}}(s)}{2h}\left(\frac{r}{R_0}\right)^{-m-1}\mathsf{K}_{m+1}\left(hR_0\right) \tag{3.13}$$

$$\frac{\mathsf{d}^2\tilde{\Psi}}{\mathsf{d}r^2} = m(m+1)\tilde{\beta}(s)r^{-m-2} + \tilde{\mathcal{C}}(s)\mathsf{K}''_m(hr) - \frac{\tilde{\mathcal{C}}(s)}{2h}\frac{m+1}{R_0}\left(\frac{r}{R_0}\right)^{-m-2}\mathsf{K}_{m+1}\left(hR_0\right) \tag{3.14}$$

$$\tilde{\mathcal{P}}(r,s) = \rho\left\{-s\tilde{\beta}(s)r^{-m} + \frac{s\tilde{\mathcal{C}}(s)R_0}{2mh}\left(\frac{r}{R_0}\right)^{-m}\mathsf{K}_{m+1}\left(hR_0\right)\right\} \tag{3.15}$$

Laplace transforming the boundary conditions 1.3-1.5 and using 3.1-3.4, we obtain

$$s\tilde{a}(s) + \frac{m}{R_0}\tilde{\Psi}(R_0,s) = a_0 \tag{3.16}$$

$$\left(\frac{\partial^2\tilde{\Psi}}{\partial r^2} - \frac{1}{r}\frac{\partial\tilde{\Psi}}{\partial r} + \frac{m^2}{r^2}\tilde{\Psi}\right)_{r=R_0} = 0 \tag{3.17}$$

$$\tilde{\mathcal{P}}(R_0,s) + 2\mu m\left(\frac{1}{r}\frac{\partial\tilde{\Psi}}{\partial r} - \frac{1}{r^2}\tilde{\Psi}\right)_{r=R_0} - \frac{T}{R_0^2}\left(1-m^2\right)\tilde{a}(s) = 0 \tag{3.18}$$

Substituting expressions for $\tilde{\Psi}(r,s)$ and its derivatives alongwith $\tilde{\mathcal{P}}(r,s)$ from 3.12-3.15 into equations 3.16-3.18 and following the same algebra as in the discrete spectrum, we obtain

$$\frac{s}{R_0}\tilde{a}(s) - \frac{\mathsf{K}_{m-1}(hR_0)}{2hR_0}\,\tilde{\mathcal{C}}(s) + mR_0^{-m-2}\tilde{\beta}(s) = \frac{a_0}{R_0} \tag{3.19}$$

$$-\mathsf{K}'_{m-1}(hR_0)\,\tilde{\mathcal{C}}(s) + 2m(m+1)R_0^{-m-2}\tilde{\beta}(s) = 0 \tag{3.20}$$

$$-\frac{T}{\rho R_0^2}(1-m^2)\tilde{a}(s) + \frac{\nu}{hR_0}\left[\frac{h^2R_0^2}{2m}\mathsf{K}_{m+1}(hR_0) - (m-1)\mathsf{K}_{m-1}(hR_0)\right]\tilde{\mathcal{C}}(s)$$
$$-sR_0^{-m}\left(1 + \frac{2m(m+1)}{h^2R_0^2}\right)\tilde{\beta}(s) = 0 \tag{3.21}$$

Equations 3.19-3.21 are linear equations in $\tilde{a}(s)$, $\tilde{\beta}(s)$ and $\tilde{\mathcal{C}}(s)$. To solve for $\tilde{a}(s)$, using Cramer's rule we obtain,

$$\tilde{a}(s) = \frac{\begin{vmatrix} a_0 & -\dfrac{\mathsf{K}_{m-1}(hR_0)}{2h} & \dfrac{m}{R_0} \\ 0 & -\mathsf{K}'_{m-1}(hR_0) & \dfrac{2m(m+1)}{R_0^2} \\ 0 & \dfrac{\nu}{hR_0}\mathcal{G}(hR_0) & -s\left(1+\dfrac{2m(m+1)}{h^2R_0^2}\right) \end{vmatrix}}{\begin{vmatrix} s & -\dfrac{\mathsf{K}_{m-1}(hR_0)}{2h} & \dfrac{m}{R_0} \\ 0 & -\mathsf{K}'_{m-1}(hR_0) & \dfrac{2m(m+1)}{R_0^2} \\ -\dfrac{T}{\rho R_0^2}(1-m^2) & \dfrac{\nu}{hR_0}\mathcal{G}(hR_0) & -s\left(1+\dfrac{2m(m+1)}{h^2R_0^2}\right) \end{vmatrix}}$$

where $\mathcal{G}(hR_0) \equiv \dfrac{h^2R_0^2K_{m+1}(hR_0)}{2m} - (m-1)\mathsf{K}_{m-1}(hR_0)$. Note that the homogenous part of equations 3.19-3.21 is identical to what was obtained earlier (see determinant above equation 2.62), if one replaces s with σ and h with l. The expression for $\tilde{a}(s)$ may be simplified as follows,

$$\tilde{a}(s) = \frac{\left[s\mathsf{K}'_{m-1}(hR_0)\left(1+\dfrac{2m(m+1)}{h^2R_0^2}\right) - \dfrac{2\nu m(m+1)}{hR_0^3}\mathcal{G}(hR_0)\right]\dfrac{1}{\mathsf{K}'_{m-1}(hR_0)}}{s^2+\dfrac{2\nu m(m+1)}{R_0^2}\left(1-\dfrac{\mathcal{G}(hR_0)}{hR_0\mathsf{K}'_{m-1}(hR_0)}\right)s+\dfrac{Tm(m^2-1)}{\rho R_0^3}\left(1-\dfrac{(m+1)}{hR_0}\dfrac{\mathsf{K}_{m-1}(hR_0)}{\mathsf{K}'_{m-1}(hR_0)}\right)}a_0$$

$$= \frac{\left[s\left(1+\dfrac{2m(m+1)}{h^2R_0^2}\right) - \dfrac{2\nu m(m+1)}{hR_0^3}\dfrac{\mathcal{G}(hR_0)}{\mathsf{K}'_{m-1}(hR_0)}\right]}{s^2+\dfrac{2\nu m(m+1)}{R_0^2}\left(1-\dfrac{\mathcal{G}(hR_0)}{hR_0\mathsf{K}'_{m-1}(hR_0)}\right)s+\dfrac{Tm(m^2-1)}{\rho R_0^3}\left(1-\dfrac{(m+1)}{hR_0}\dfrac{\mathsf{K}_{m-1}(hR_0)}{\mathsf{K}'_{m-1}(hR_0)}\right)}a_0$$

$$= \frac{\left[s+\dfrac{2m\nu(m+1)}{R_0^2} - \dfrac{2\nu m(m+1)}{hR_0^3}\dfrac{\mathcal{G}(hR_0)}{\mathsf{K}'_{m-1}(hR_0)}\right]}{s^2+\dfrac{2\nu m(m+1)}{R_0^2}\left(1-\dfrac{\mathcal{G}(hR_0)}{hR_0\mathsf{K}'_{m-1}(hR_0)}\right)s+\dfrac{Tm(m^2-1)}{\rho R_0^3}\left(1-\dfrac{(m+1)}{hR_0}\dfrac{\mathsf{K}_{m-1}(hR_0)}{\mathsf{K}'_{m-1}(hR_0)}\right)}a_0$$

$$= \frac{\left[s+\dfrac{2m\nu(m+1)}{R_0^2}\left(1-\dfrac{\mathcal{G}(hR_0)}{hR_0\mathsf{K}'_{m-1}(hR_0)}\right)\right]}{s^2+\dfrac{2\nu m(m+1)}{R_0^2}\left(1-\dfrac{\mathcal{G}(hR_0)}{hR_0\mathsf{K}'_{m-1}(hR_0)}\right)s+\dfrac{Tm(m^2-1)}{\rho R_0^3}\left(1-\dfrac{(m+1)}{hR_0}\dfrac{\mathsf{K}_{m-1}(hR_0)}{\mathsf{K}'_{m-1}(hR_0)}\right)}a_0 \tag{3.22}$$

The final expression for $\tilde{a}(s)$ may be written in compact form as,

$$\tilde{a}(s) = \left[\frac{s+\mathcal{M}(s)}{s^2+s\mathcal{M}(s)+w_0^2\chi(s)}\right]a_0 \tag{3.23}$$

with

$$\mathcal{M}(s) \equiv \frac{2\nu m(m+1)}{R_0^2}\left(1 - \frac{\mathcal{G}(hR_0)}{hR_0\mathsf{K}'_{m-1}(hR_0)}\right),\ \chi(s) \equiv 1 - \frac{(m+1)}{hR_0}\frac{\mathsf{K}_{m-1}(hR_0)}{\mathsf{K}'_{m-1}(hR_0)}$$

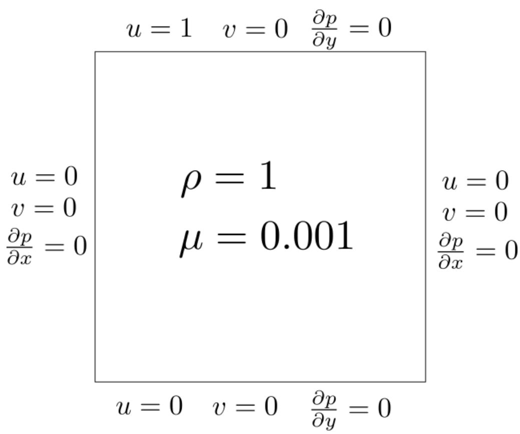
and $w_0^2 \equiv \dfrac{Tm(m^2-1)}{\rho R_0^3}$.

Note that ω_0 is the inviscid frequency of oscillation of a Fourier mode of index m.

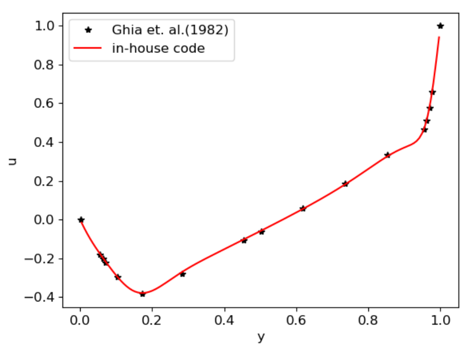
4. Test cases

4.1. *Lid driven cavity*

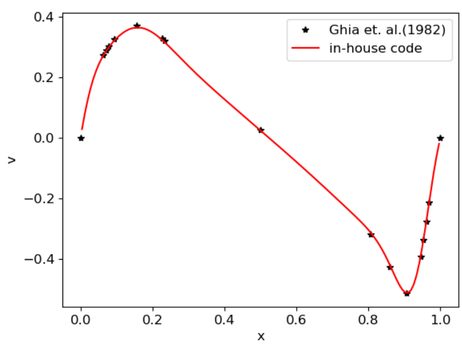
Lid driven cavity is the classic test case for one phase Navier-Stokes equations in CFD. Here we have a square cavity of side length unity with no slip and no-penetration conditions on three boundaries. except the lid which moves with constant velocity. We benchmark our solver against the data of Ghia *et al.* (1982)



(a) Simulation geometry for square cavity



(b) u velocity along $y = 0.5$



(c) v velocity along $x = 0.5$

Figure 1: Reynolds no. Re=1000

4.2. *Dam break test (Cartesian), Experimental Verification*

In this section, we validate our in-house developed solver against experimental data for the dam-break test. A rectangular two dimensional column of fluid is initialised with zero velocity and spreads under the influence of gravity.

(i) Domain: $[0, 1.61] \times [0, 0.805]$ m
(ii) Initial Interface: Dam of height 0.3 m and length 1.0 m
(iii) Grid size: 128×256
(iv) $\rho_g = 1.226$ kg/m^3, $\rho_l = 997.0$ m^3,
(v) $\mu_g = 1.78 \times 10^{-5}$, $\mu_l = 8.8733 \times 10^{-4}$
(vi) $\sigma_{lg} = 0.0728$ N/m
(vii) $g = 9.81$ m/s^2
(viii) Boundary Condition : no slip at all boundaries.

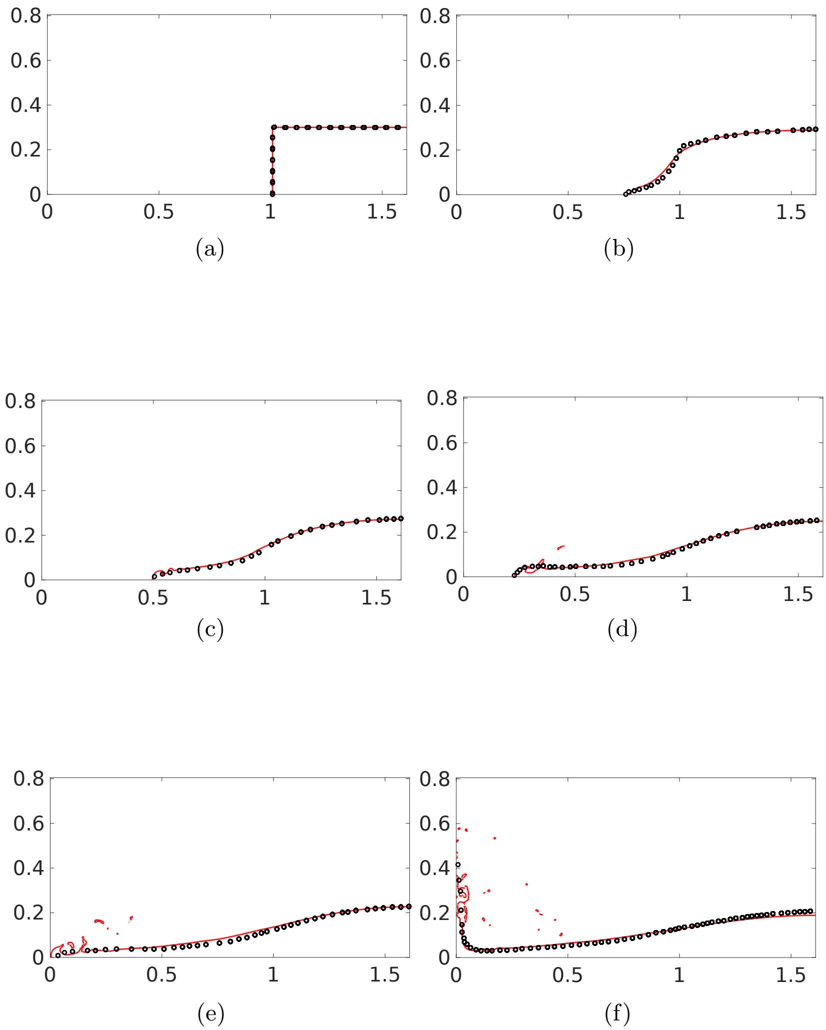


Figure 2: Verification of in-house code (solid line) and with experimental data (symbols) obtained from Lobovskỳ *et al.* (2014)

4.3. *Free oscillations at the interface between two fluids*

In order to benchmark the inhouse viscous solver, we simulate free capillary oscillations of two immiscible, incompressible viscous fluids of density and viscosity $\rho_u, \mu_u,\ \rho_l, \mu_l$ (subscript u for upper and l for lower) on horizontally and vertically unbounded domain. The interface is initiated in the form of a single Fourier mode with zero velocity everywhere and the subsequent motion of the interface is tracked in time at a particular x location. We use symmetry boundary conditions on all four sides of the computational domain taken to be $[0, 1] \times [0, 3]$. In the linearised limit, the analytical expression for amplitude of the standing wave (in the Laplace domain) is given by Prosperetti (1981).

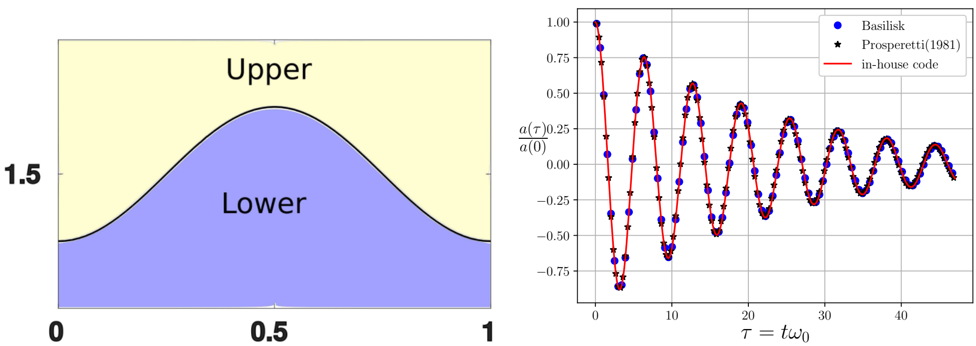
$$\tilde{a}(s) = \left(\frac{s + \Lambda(s)}{s^2 + \Lambda(s)s + \omega_0^2} \right) a_0 \tag{4.1}$$

where,

$$\Lambda(s) = \frac{4k(-\rho_l\rho_u s + k(\mu_u - \mu_l)(\rho_u(k - \lambda_l) - \rho_l(k - \lambda_u)) + k^2(\mu_l - \mu_u)^2(k - \lambda_l)(k - \lambda_u)s^{-1})}{(\rho_l + \rho_u)(\rho_l(k - \lambda_u) + \rho_u(k - \lambda_l))},$$

$$\lambda_{l,u} \equiv \sqrt{k^2 + \frac{s}{\nu_{l,u}}} \qquad \text{and} \qquad \omega_0^2 = \frac{Tk^3}{\rho_l + \rho_u}.$$

For these simulations, we have chosen $\rho_l = 1,\ \rho_u = 0.01,\ \mu_l = 0.01,\ \mu_u = 0.0001,\ T = 1,\ k = 2\pi$ with zero gravity. The simulation geometry is shown in figure 3a. Results from DNS conducted using our inhouse solver, the numerical Laplace inversion of expression 4.1 and the open-source solver Basilisk (Popinet 2014) is shown in figure 3b.



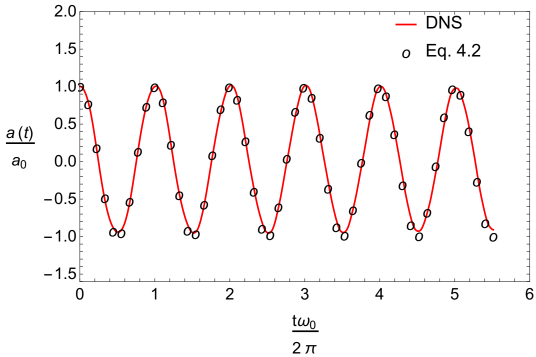
(a) Simulation geometry: Grid - 64 × 192

(b) Amplitude vs time

Figure 3: Planar wave viscous oscillations

4.4. *Inviscid capillary oscillations*

The simulation geometry is the same as that studied in the main manuscript with



(a) Case 0 in table 2 of the main manuscript

Figure 4: Benchmarking of the inhouse developed DNS code with frequency predicted by equation 4.2

$$\omega_0^2 = \frac{T}{R_0^3}\left[\frac{m(m^2-1)}{\rho^{\mathcal{I}}+\rho^{\mathcal{O}}}\right]. \tag{4.2}$$

5. Extraction of vorticity data from DNS

Our in-house solver is written in two dimensional Cartesian coordinates (x-y) while the analytical expressions for $\hat{\Omega}(r,t)$ have been obtained using plane-polar (r-θ) coordinates. We show here that the out-of-plane z component of the vorticity $\omega_z^{\texttt{Cartesian}} \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ as obtained from DNS, is the same as the (axial) z component of vorticity in cylindrical coordinates (or polar coordinates) ($\omega \equiv \frac{1}{r}\{\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial \theta}\}$). We have the following relations between plane polar coordinates (r,θ) and Cartesian coordinates $((x,y))$

$$x = r\cos(\theta),\ y = r\sin(\theta) \tag{5.1}$$
$$r = \left(x^2+y^2\right)^{1/2},\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \tag{5.2}$$

And the relations between velocity components in Cartesian (u,v) and polar components (u_r,u_θ)

$$u = u_r\cos(\theta) - u_\theta\sin(\theta) \tag{5.3}$$
$$v = u_\theta\cos(\theta) + u_r\sin(\theta) \tag{5.4}$$

The partial derivatives have the following relation

$$\frac{\partial}{\partial x} = \cos(\theta)\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta} \tag{5.5}$$
$$\frac{\partial}{\partial y} = \sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta} \tag{5.6}$$

Substituting expressions 5.5 and 5.6 into,

$$\begin{aligned}\omega_z^{\texttt{Cartesian}} &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \\ &= \left\{\cos(\theta)\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}\right\}\left[u_\theta \cos(\theta) + u_r \sin(\theta)\right] \\ &\quad -\left\{\sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}\right\}\left[u_r \cos(\theta) - u_\theta \sin(\theta)\right]\end{aligned} \tag{5.7}$$

After some algebra on 5.7, we can show that

$$\omega_z^{\texttt{Cartesian}} = \frac{1}{r}\left\{\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right\} = \omega \tag{5.8}$$

6. List of symbols

For the benefit of the reader, we provide here a list of symbols which are used in the main manuscript in the nomnclature below.

7. Appendix

The series representation of $\mathsf{K}_m(z)$ for $m = 0, 1, 2$ and 3 are obtained below from Mathematica (Wolfram Research, Inc. 2017) or eqn. 4.16 in the main manuscript. For $m > 0$, it is seen that $z = 0$ is a pole and a logarithmic branch point.

$$\mathsf{K}_0(z) = -\left[\gamma + \ln\left(\frac{z}{2}\right)\right] + \ldots$$

$$\mathsf{K}_1(z) = \frac{1}{z} + \frac{1}{4}\left[2\ln\left(\frac{z}{2}\right) - 1 + 2\gamma\right] z + \ldots \tag{7.1}$$

$$\mathsf{K}_2(z) = \frac{2}{z^2} - \frac{1}{2} + \frac{1}{32}\left[3 - 4\gamma - 4\ln\left(\frac{z}{2}\right)\right] z^2 + \ldots \tag{7.2}$$

$$\mathsf{K}_3(z) = \frac{8}{z^3} - \frac{1}{z} + \frac{z}{8} + \frac{1}{576}\left[-11 + 12\gamma + 12\ln\left(\frac{z}{2}\right)\right] z^3 + \ldots \tag{7.3}$$

Nomenclature

- η Location of perturbed interface
- $\hat{z}$ Axial coordinate
- I_m m$^{\text{th}}$ order modified Bessel function of the first kind
- J_m m$^{\text{th}}$ order Bessel function of the first kind
- K_m m$^{\text{th}}$ order modified Bessel function of the second kind
- $\mu^{\mathcal{I}}$ The viscosity of inner fluid
- $\mu^{\mathcal{O}}$ The viscosity of outer fluid
- Y_m m$^{\text{th}}$ order Bessel function of the second kind
- Ω Radial part of perturbation vorticity ω
- ω Axial component of perturbation vorticity
- ω_0 Inviscid frequency
- $\mathcal{P}$ Radial part of perturbation pressure p
- Ψ Radial part of perturbation streamfunction ψ
- ψ Axial component of perturbation streamfunction
- $\rho^{\mathcal{I}}$ The density of inner fluid
- $\rho^{\mathcal{O}}$ The density of outer fluid
- σ Complex frequency
- θ Azimuthal coordinate
- $\underline{\underline{\boldsymbol{\sigma}}}^{\texttt{tot}}$ Stress tensor
- k Axial wavenumber
- m Azimuthal wavenumber
- p Perturbation pressure
- r Radial coordinate
- R_0 Radius of cylindrical filament
- s Laplace variable
- T Surface tension coefficient
- La Laplace number La

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