1. Supplementary material: other electrical conditions on top and bottom caps

The top and bottom boundary conditions (2.6c) and (2.6d) for the electrical potential and currents are well adapted to model the metal pad roll instability in Hall-Héroult cells, because they mimic the fact that the top Carbon anodes are better conductors than the cryolite, and the bottom cathode is not as good a conductor as the aluminium. Nevertheless, this is an approximation and we have also studied the impact of using different top and bottom boundary conditions. More precisely we consider the following two sets of boundary conditions:

$$\text{(zero } \varphi) \qquad \varphi_1|_{z=H_1} = 0, \qquad \varphi_2|_{z=-H_2} = 0, \tag{1.1a}$$

$$\text{(zero } j_z) \qquad j_{1,z}|_{z=H_1} = 0, \qquad j_{2,z}|_{z=-H_2} = 0, \tag{1.1b}$$

respectively, for very well conducting top and bottom plates or very badly conducting top and bottom plates. For these boundary conditions, we find the potentials in the magneto-static inviscid limit to be

$$\text{(zero } \varphi) \qquad \begin{bmatrix} \widehat{\varphi}_1 \\ \widehat{\varphi}_2 \end{bmatrix} = \frac{JkA}{\omega} \frac{\sigma_1^{-1} - \sigma_2^{-1}}{\sigma_1^{-1}\tanh(kH_1) + \sigma_2^{-1}\tanh(kH_2)} \times \begin{bmatrix} -\mathrm{i}\sinh(k(z-H_1))/(\sigma_1\cosh(kH_1)) \\ -\mathrm{i}\sinh(k(z+H_2))/(\sigma_2\cosh(kH_2)) \end{bmatrix} J_m(kr)e^{\mathrm{i}m\theta}, \tag{1.2a}$$

$$\text{(zero } j_z) \qquad \begin{bmatrix} \widehat{\varphi}_1 \\ \widehat{\varphi}_2 \end{bmatrix} = \frac{JkA}{\omega} \frac{\sigma_1^{-1} - \sigma_2^{-1}}{(\sigma_1\tanh(kH_1))^{-1} + (\sigma_2\tanh(kH_2))^{-1}} \times \begin{bmatrix} \mathrm{i}\cosh(k(z-H_1))/(\sigma_1\sinh(kH_1)) \\ -\mathrm{i}\cosh(k(z+H_2))/(\sigma_2\sinh(kH_2)) \end{bmatrix} J_m(kr)e^{\mathrm{i}m\theta}. \tag{1.2b}$$

We then recalculate $\mathcal{P}_{\mathrm{v}}$ and find the inviscid magnetostatic growth rate

$$\lambda_{\mathrm{v}} = \frac{\omega}{2} \frac{JB_z}{(\rho_2-\rho_1)g + \gamma_{1|2}k^2} \, \frac{m}{(kR)^2 - m^2} \, \Xi, \tag{1.3}$$

with

$$\text{(zero } \varphi) \qquad \Xi = \frac{(\sigma_1^{-1} - \sigma_2^{-1})\left[\sum_{i=1,2}\tanh(kH_i)\right]}{\sum_{i=1,2}\sigma_i^{-1}\tanh(kH_i)}, \tag{1.4a}$$

$$\text{(zero } j_z) \qquad \Xi = \frac{(\sigma_1^{-1} - \sigma_2^{-1})\sum_{i=1,2}\left[\dfrac{kH_i}{\sinh^2(kH_i)} + \dfrac{1}{\tanh(kH_i)}\right]}{\sum_{i=1,2}(\sigma_i\tanh(kH_i))^{-1}}. \tag{1.4b}$$

In the deep cell limit, $kH_i \gg 1$, and for large conductivity jumps, $\sigma_1 \ll \sigma_2$, we find that

$$\lambda_{\mathrm{v,deep}} \approx \frac{JB_z}{(\rho_2-\rho_1)g + \gamma_{1|2}k^2} \frac{m\omega_{\mathrm{deep}}}{(kR)^2 - m^2}, \tag{1.5}$$

for both sets of boundary conditions. This expression is exactly the same formula as (2.41a), which is not a real surprise. All the motion occurs near the interface, so far away from the top and bottom boundaries that precise boundary conditions there have little importance. In the shallow cell limit, $kH_i \ll 1$, and for large conductivity jumps,

$\sigma_1 \ll \sigma_2$, we have

$$\text{(zero } \varphi) \qquad \lambda_{\text{v,shallow}} \approx \frac{J B_z}{(\rho_2-\rho_1)g + \gamma_{1|2}k^2} \left[\frac{1}{2}\left(1+\frac{H_2}{H_1}\right)\right] \frac{m\omega_{\text{shallow}}}{\kappa_{mn}^2 - m^2}, \tag{1.6a}$$

$$\text{(zero } j_z) \qquad \lambda_{\text{v,shallow}} \approx \frac{J B_z}{(\rho_2-\rho_1)g + \gamma_{1|2}k^2} \left(1+\frac{H_1}{H_2}\right) \frac{m\omega_{\text{shallow}}}{\kappa_{mn}^2 - m^2}. \tag{1.6b}$$

These formulas are very different from the shallow cell formula we have computed with mixed boundary conditions. Comparing all three cases of top and bottom boundary conditions, we conclude that the mixed boundary conditions used in the body of the article, always yield the most unstable configuration in the shallow cell limit.