Supplementary Material for Fast Stokesian Dynamics

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Particle ID	x-coordinate	y-coordinate	z-coordinate
1	0.529285890092826	3.523022483164158	5.484806932245434
2	1.905248504411454	2.107000803902882	1.574380806104360
3	2.955617519755345	4.473835710288266	4.844550431283280
4	5.801000061455573	4.930024624007738	2.299588452754040
5	0.798962336335867	1.440675393585724	5.483621569662730
6	4.544374905027849	1.227142494676998	5.473367596622032
7	2.841597610452971	2.600989553918899	5.599282851024916
8	1.406000704309130	2.256978689809362	3.583544479413773
9	3.342587523865602	4.325905161264344	1.084384992549901
10	4.855121516019277	0.929962613703946	3.037851721042064
11	3.377932960037302	2.359447798613839	2.970065584591988
12	2.239095471877138	0.334672721541239	0.382811014387224
13	4.114271938344111	2.376400786030453	1.017253092301853
14	1.185542341656091	4.255984236236199	3.459244962579186
15	4.445356106771235	0.328213714656732	1.076946378311737
16	1.175081018317673	4.835484235611831	1.066966641996011
17	5.788503713015421	3.083781368922106	3.288857324994003
18	0.296278319430643	0.727758099169412	1.237198342780773
19	0.729117057541273	0.378881438216108	3.430999872949822
20	1.030932696441608	5.482041093433065	5.281179387787007
21	2.462809095100618	5.933084463400179	2.272668410058109
22	5.559050487117455	4.874846862249195	0.211221623395827
23	4.734101342091497	3.268704200734260	5.477307153882863
24	4.022679285101558	4.300499688898289	2.989917777083694
25	5.164161056490291	5.378106401177105	4.490755326462062
26	2.864433727327856	0.487551123260315	4.558213863484444
27	6.196445423923576	2.986711186416819	1.318689297115926

Table 1. Cartesian coordinates of the hard sphere fluid used in the eigenspectrum calculations.

1. FSD Plugin for HOOMD-Blue

The source code for a GPU implementation designed to be used as a plug-in for the open source molecular dynamics package HOOMD-blue is included in the file `PSEv3-master.zip`. Installation instructions for the software as well as documentation can be found in the self-contained sub-directory `docs/_build/html/`. Open the file `index.html` in any web browser to access the documentation.

2. Hard Sphere Fluid Configuration

The hard sphere fluid configuration used to compute the eigenspectra is comprised of 27 particles with radius $a = 1$ in a periodically replicated isotropic square unit cell with cell dimension $L = 6.214453999582854\,a$. The origin of the coordinate system is located in the lower left corner of the simulation box, so the particle coordinates in each of the three cartesian directions vary between 0 and L. The set of coordinates are provided in Table 1.

3. Polydisperse Real Space Scalar Mobility Functions

This section contains the results of the integrals for the real space scalar mobility function presented in Appendix A of the main text. The mobility functions are made dimensionless on $6\pi\eta$, i.e. $6\pi\eta a\, M_{\mathrm{UF}}$, $6\pi\eta a^2\, M_{\mathrm{UC}}$, and $6\pi\eta a^3\, M_{\mathrm{DC}}$.

3.1. *Real Space Mobility Functions, Translation-Force Coupling*

The integrals presented in Appendix A are easier to compute under simplifying expressions for the relative magnitude of r, a, and b, giving four simplified sub-cases.

3.1.1. ***Case 1***, $r > a + b$

First scalar mobility function

$$f_0^{(1)} = 0 \tag{3.1}$$

$$f_1^{(1)} = \frac{-2\xi^2\left(7r^2(a+b) - r(a+b)^2 + (a+b)^3 + 9r^3\right) + a + b - 3r}{128\sqrt{\pi}abr^3\xi^3} \tag{3.2}$$

$$f_2^{(1)} = \frac{2\xi^2\left(7r^2(a+b) + r(a+b)^2 + (a+b)^3 - 9r^3\right) - a - b - 3r}{128\sqrt{\pi}abr^3\xi^3} \tag{3.3}$$

$$f_3^{(1)} = \frac{-2\xi^2\left(7r^2(a-b) + r(a-b)^2 + (a-b)^3 - 9r^3\right) + a - b + 3r}{128\sqrt{\pi}abr^3\xi^3} \tag{3.4}$$

$$f_4^{(1)} = \frac{2\xi^2(a-b+r)\left(a^2 - 2a(b+r) + b^2 + 2br + 9r^2\right) - a + b + 3r}{128\sqrt{\pi}abr^3\xi^3} \tag{3.5}$$

$$f_5^{(1)} = \frac{4\xi^4(a+b-r)^2\left(2r(a+b) + (a+b)^2 + 9r^2\right) - 3}{256abr^3\xi^4} \tag{3.6}$$

$$f_6^{(1)} = \frac{3 - 4\xi^4(a-b+r)^2\left(a^2 - 2a(b+r) + b^2 + 2br + 9r^2\right)}{256abr^3\xi^4} \tag{3.7}$$

$$f_7^{(1)} = \frac{3 - 4\xi^4(-a+b+r)^2\left(2r(a-b) + (a-b)^2 + 9r^2\right)}{256abr^3\xi^4} \tag{3.8}$$

$$f_8^{(1)} = \frac{4\xi^4(a+b+r)^2\left(-2r(a+b) + (a+b)^2 + 9r^2\right) - 3}{256abr^3\xi^4} \tag{3.9}$$

Second scalar mobility function

$$f_0^{(2)} = 0 \tag{3.10}$$

$$f_1^{(2)} = \frac{2\xi^2(a+b+r)^2(a+b-3r) - a - b + 3r}{64\sqrt{\pi}abr^3\xi^3} \tag{3.11}$$

$$f_2^{(2)} = \frac{-2\xi^2(a+b-r)^2(a+b+3r) + a + b + 3r}{64\sqrt{\pi}abr^3\xi^3} \tag{3.12}$$

$$f_3^{(2)} = \frac{2\xi^2(-a+b+r)^2(a-b+3r) - a + b - 3r}{64\sqrt{\pi}abr^3\xi^3} \tag{3.13}$$

$$f_4^{(2)} = \frac{-2\xi^2(a-b-3r)(a-b+r)^2 + a - b - 3r}{64\sqrt{\pi}abr^3\xi^3} \tag{3.14}$$

$$f_5^{(2)} = \frac{3 - 4\xi^4(a+b-r)^3(a+b+3r)}{128abr^3\xi^4} \tag{3.15}$$

$$f_6^{(2)} = \frac{4\xi^4(a-b-3r)(a-b+r)^3 - 3}{128abr^3\xi^4} \tag{3.16}$$

$$f_7^{(2)} = \frac{4\xi^4(a-b-r)^3(a-b+3r) - 3}{128abr^3\xi^4} \tag{3.17}$$

$$f_8^{(2)} = \frac{3 - 4\xi^4(a+b-3r)(a+b+r)^3}{128abr^3\xi^4} \tag{3.18}$$

3.1.2. ***Case 2***, $r < a+b, r > b-a, r > a-b$

First scalar mobility function

$$f_0^{(1)} = -\frac{(a+b-r)^2\left(a^2 + 2a(b+r) + b^2 + 2br + 9r^2\right)}{32abr^3} \tag{3.19}$$

$$f_1^{(1)} = \frac{-2\xi^2\left(7r^2(a+b) - r(a+b)^2 + (a+b)^3 + 9r^3\right) + a + b - 3r}{128\sqrt{\pi}abr^3\xi^3} \tag{3.20}$$

$$f_2^{(1)} = \frac{2\xi^2\left(7r^2(a+b) + r(a+b)^2 + (a+b)^3 - 9r^3\right) - a - b - 3r}{128\sqrt{\pi}abr^3\xi^3} \tag{3.21}$$

$$f_3^{(1)} = \frac{-2\xi^2\left(7r^2(a-b) + r(a-b)^2 + (a-b)^3 - 9r^3\right) + a - b + 3r}{128\sqrt{\pi}abr^3\xi^3} \tag{3.22}$$

$$f_4^{(1)} = \frac{2\xi^2(a-b+r)\left(a^2 - 2a(b+r) + b^2 + 2br + 9r^2\right) - a + b + 3r}{128\sqrt{\pi}abr^3\xi^3} \tag{3.23}$$

$$f_5^{(1)} = \frac{4\xi^4(a+b-r)^2\left(2r(a+b) + (a+b)^2 + 9r^2\right) - 3}{256abr^3\xi^4} \tag{3.24}$$

$$f_6^{(1)} = \frac{3 - 4\xi^4(a-b+r)^2\left(a^2 - 2a(b+r) + b^2 + 2br + 9r^2\right)}{256abr^3\xi^4} \tag{3.25}$$

$$f_7^{(1)} = \frac{3 - 4\xi^4(-a+b+r)^2\left(2r(a-b) + (a-b)^2 + 9r^2\right)}{256abr^3\xi^4} \tag{3.26}$$

$$f_8^{(1)} = \frac{4\xi^4(a+b+r)^2\left(-2r(a+b) + (a+b)^2 + 9r^2\right) - 3}{256abr^3\xi^4} \tag{3.27}$$

Second scalar mobility function

$$f_0^{(2)} = \frac{(a+b-r)^3(a+b+3r)}{16abr^3} \tag{3.28}$$

$$f_1^{(2)} = \frac{2\xi^2(a+b+r)^2(a+b-3r)-a-b+3r}{64\sqrt{\pi}abr^3\xi^3} \tag{3.29}$$

$$f_2^{(2)} = \frac{-2\xi^2(a+b-r)^2(a+b+3r)+a+b+3r}{64\sqrt{\pi}abr^3\xi^3} \tag{3.30}$$

$$f_3^{(2)} = \frac{2\xi^2(-a+b+r)^2(a-b+3r)-a+b-3r}{64\sqrt{\pi}abr^3\xi^3} \tag{3.31}$$

$$f_4^{(2)} = \frac{-2\xi^2(a-b-3r)(a-b+r)^2+a-b-3r}{64\sqrt{\pi}abr^3\xi^3} \tag{3.32}$$

$$f_5^{(2)} = \frac{3-4\xi^4(a+b-r)^3(a+b+3r)}{128abr^3\xi^4} \tag{3.33}$$

$$f_6^{(2)} = \frac{4\xi^4(a-b-3r)(a-b+r)^3-3}{128abr^3\xi^4} \tag{3.34}$$

$$f_7^{(2)} = \frac{4\xi^4(a-b-r)^3(a-b+3r)-3}{128abr^3\xi^4} \tag{3.35}$$

$$f_8^{(2)} = \frac{3-4\xi^4(a+b-3r)(a+b+r)^3}{128abr^3\xi^4} \tag{3.36}$$

3.1.3. ***Case 3***, *particle α inside particle β*, $r < a+b, r < b-a, r > a-b$

First scalar mobility function

$$f_0^{(1)} = -\frac{a^2b+b^3+3br^2-4r^3}{4br^3} \tag{3.37}$$

$$f_1^{(1)} = \frac{-2\xi^2\left(7r^2(a+b)-r(a+b)^2+(a+b)^3+9r^3\right)+a+b-3r}{128\sqrt{\pi}abr^3\xi^3} \tag{3.38}$$

$$f_2^{(1)} = \frac{2\xi^2\left(7r^2(a+b)+r(a+b)^2+(a+b)^3-9r^3\right)-a-b-3r}{128\sqrt{\pi}abr^3\xi^3} \tag{3.39}$$

$$f_3^{(1)} = \frac{-2\xi^2\left(7r^2(a-b)+r(a-b)^2+(a-b)^3-9r^3\right)+a-b+3r}{128\sqrt{\pi}abr^3\xi^3} \tag{3.40}$$

$$f_4^{(1)} = \frac{2\xi^2(a-b+r)\left(a^2-2a(b+r)+b^2+2br+9r^2\right)-a+b+3r}{128\sqrt{\pi}abr^3\xi^3} \tag{3.41}$$

$$f_5^{(1)} = \frac{4\xi^4(a+b-r)^2\left(2r(a+b)+(a+b)^2+9r^2\right)-3}{256abr^3\xi^4} \tag{3.42}$$

$$f_6^{(1)} = \frac{3-4\xi^4(a-b+r)^2\left(a^2-2a(b+r)+b^2+2br+9r^2\right)}{256abr^3\xi^4} \tag{3.43}$$

$$f_7^{(1)} = \frac{3-4\xi^4(-a+b+r)^2\left(2r(a-b)+(a-b)^2+9r^2\right)}{256abr^3\xi^4} \tag{3.44}$$

$$f_8^{(1)} = \frac{4\xi^4(a+b+r)^2\left(-2r(a+b)+(a+b)^2+9r^2\right)-3}{256abr^3\xi^4} \tag{3.45}$$

Second scalar mobility function

$$f_0^{(2)} = \frac{a^2b + (b-r)^2(b+2r)}{2br^3} \tag{3.46}$$

$$f_1^{(2)} = \frac{2\xi^2(a+b+r)^2(a+b-3r) - a - b + 3r}{64\sqrt{\pi}abr^3\xi^3} \tag{3.47}$$

$$f_2^{(2)} = \frac{-2\xi^2(a+b-r)^2(a+b+3r) + a + b + 3r}{64\sqrt{\pi}abr^3\xi^3} \tag{3.48}$$

$$f_3^{(2)} = \frac{2\xi^2(-a+b+r)^2(a-b+3r) - a + b - 3r}{64\sqrt{\pi}abr^3\xi^3} \tag{3.49}$$

$$f_4^{(2)} = \frac{-2\xi^2(a-b-3r)(a-b+r)^2 + a - b - 3r}{64\sqrt{\pi}abr^3\xi^3} \tag{3.50}$$

$$f_5^{(2)} = \frac{3 - 4\xi^4(a+b-r)^3(a+b+3r)}{128abr^3\xi^4} \tag{3.51}$$

$$f_6^{(2)} = \frac{4\xi^4(a-b-3r)(a-b+r)^3 - 3}{128abr^3\xi^4} \tag{3.52}$$

$$f_7^{(2)} = \frac{4\xi^4(a-b-r)^3(a-b+3r) - 3}{128abr^3\xi^4} \tag{3.53}$$

$$f_8^{(2)} = \frac{3 - 4\xi^4(a+b-3r)(a+b+r)^3}{128abr^3\xi^4} \tag{3.54}$$

3.1.4. ***Case 4***, *particle β inside particle α, $r < a+b, r > b-a, r < a-b$*

First scalar mobility function

$$f_0^{(1)} = -\frac{a^3 + ab^2 + 3ar^2 - 4r^3}{4ar^3} \tag{3.55}$$

$$f_1^{(1)} = \frac{-2\xi^2\left(7r^2(a+b) - r(a+b)^2 + (a+b)^3 + 9r^3\right) + a + b - 3r}{128\sqrt{\pi}abr^3\xi^3} \tag{3.56}$$

$$f_2^{(1)} = \frac{2\xi^2\left(7r^2(a+b) + r(a+b)^2 + (a+b)^3 - 9r^3\right) - a - b - 3r}{128\sqrt{\pi}abr^3\xi^3} \tag{3.57}$$

$$f_3^{(1)} = \frac{-2\xi^2\left(7r^2(a-b) + r(a-b)^2 + (a-b)^3 - 9r^3\right) + a - b + 3r}{128\sqrt{\pi}abr^3\xi^3} \tag{3.58}$$

$$f_4^{(1)} = \frac{2\xi^2(a-b+r)\left(a^2 - 2a(b+r) + b^2 + 2br + 9r^2\right) - a + b + 3r}{128\sqrt{\pi}abr^3\xi^3} \tag{3.59}$$

$$f_5^{(1)} = \frac{4\xi^4(a+b-r)^2\left(2r(a+b) + (a+b)^2 + 9r^2\right) - 3}{256abr^3\xi^4} \tag{3.60}$$

$$f_6^{(1)} = \frac{3 - 4\xi^4(a-b+r)^2\left(a^2 - 2a(b+r) + b^2 + 2br + 9r^2\right)}{256abr^3\xi^4} \tag{3.61}$$

$$f_7^{(1)} = \frac{3 - 4\xi^4(-a+b+r)^2\left(2r(a-b) + (a-b)^2 + 9r^2\right)}{256abr^3\xi^4} \tag{3.62}$$

$$f_8^{(1)} = \frac{4\xi^4(a+b+r)^2\left(-2r(a+b) + (a+b)^2 + 9r^2\right) - 3}{256abr^3\xi^4} \tag{3.63}$$

Second scalar mobility function

$$f_0^{(2)} = \frac{a^3 + ab^2 - 3ar^2 + 2r^3}{2ar^3} \tag{3.64}$$

$$f_1^{(2)} = \frac{2\xi^2(a+b+r)^2(a+b-3r) - a - b + 3r}{64\sqrt{\pi}abr^3\xi^3} \tag{3.65}$$

$$f_2^{(2)} = \frac{-2\xi^2(a+b-r)^2(a+b+3r) + a + b + 3r}{64\sqrt{\pi}abr^3\xi^3} \tag{3.66}$$

$$f_3^{(2)} = \frac{2\xi^2(-a+b+r)^2(a-b+3r) - a + b - 3r}{64\sqrt{\pi}abr^3\xi^3} \tag{3.67}$$

$$f_4^{(2)} = \frac{-2\xi^2(a-b-3r)(a-b+r)^2 + a - b - 3r}{64\sqrt{\pi}abr^3\xi^3} \tag{3.68}$$

$$f_5^{(2)} = \frac{3 - 4\xi^4(a+b-r)^3(a+b+3r)}{128abr^3\xi^4} \tag{3.69}$$

$$f_6^{(2)} = \frac{4\xi^4(a-b-3r)(a-b+r)^3 - 3}{128abr^3\xi^4} \tag{3.70}$$

$$f_7^{(2)} = \frac{4\xi^4(a-b-r)^3(a-b+3r) - 3}{128abr^3\xi^4} \tag{3.71}$$

$$f_8^{(2)} = \frac{3 - 4\xi^4(a+b-3r)(a+b+r)^3}{128abr^3\xi^4} \tag{3.72}$$

3.1.5. ***Case 5***, *self contribution* ($\alpha = \beta$ *or* $r = 0$)

$$6\pi\eta a\, M_{UF,im}^{\alpha\alpha,(r)} = \frac{1 - e^{-4a^2\xi^2} + 4\pi^{1/2}a\xi\,\mathrm{erfc}\,(2a\xi)}{4\pi^{1/2}\xi a}\,\delta_{im}. \tag{3.73}$$

3.2. *Real Space Mobility Functions, Translation-Dipole Coupling*

The integrals presented in Appendix A of the main tex provide the tensor coupling

$$6\pi\eta a^2 \left(M_{\mathrm{UC}}^{\alpha\beta,\mathrm{real}}\right)_{ijk} = g^{(1)}(r)\ (\delta_{ij}\hat{r}_k - \hat{r}_i\hat{r}_j\hat{r}_k) + g^{(2)}(r)\ (\delta_{ik}\hat{r}_j + \delta_{jk}\hat{r}_i - 4\,\hat{r}_i\hat{r}_j\hat{r}_k) \tag{3.74}$$

where all the scalar functions are of the form

$$\begin{aligned} g^{(i)}(r) = g_0^{(i)} &+ g_1^{(i)}\, e^{-(a+b+r)^2\xi^2} + g_2^{(i)}\, e^{-(a+b-r)^2\xi^2} + g_3^{(i)}\, e^{-(-a+b+r)^2\xi^2} \\ &+ g_4^{(i)}\, e^{-(a-b+r)^2\xi^2} + g_5^{(i)}\,\mathrm{erfc}\,((-a-b+r)\xi) + g_6^{(i)}\,\mathrm{erfc}\,((a-b+r)\xi) \\ &+ g_7^{(i)}\,\mathrm{erfc}\,((-a+b+r)\xi) + g_8^{(i)}\,\mathrm{erfc}\,((a+b+r)\xi) \end{aligned} \tag{3.75}$$

The gradient-force coupling for particle pair α, β can be obtained from the velocity-dipole coupling through symmetry relations, i.e. $M_{imn}^{UC}(\alpha,\beta) = M_{nmi}^{DF}(\beta,\alpha)$.

3.2.1. ***Case 1***, $r > a + b$

First scalar mobility function

$$g_0^{(1)} = 0 \tag{3.76}$$

$$g_1^{(1)} = \frac{3(39a-25b)}{320\sqrt{\pi}ab^3\xi} + \frac{-6\xi^4(a-5b)(a+b)^4+3\xi^2(a+b)^2(a-5b)+63a-45b}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{3\left(-4\xi^2(3a-10b)(a+b)^2+11a-45b\right)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{12\xi^2(3a-10b)(a+b)-75}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a-5b)(a+b)\left(2\xi^2(a+b)^2-3\right)-45}{640\sqrt{\pi}ab^3\xi^5r^3} + \frac{15r}{64\sqrt{\pi}ab^3\xi} \tag{3.77}$$

$$g_2^{(1)} = \frac{3(25b-39a)}{320\sqrt{\pi}ab^3\xi} + \frac{6\xi^4(a+b)^4(a-5b)-3\xi^2(a-5b)(a+b)^2-63a+45b}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{3\left(4\xi^2(a+b)^2(3a-10b)-11a+45b\right)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{12\xi^2(3a-10b)(a+b)-75}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a-5b)(a+b)\left(2\xi^2(a+b)^2-3\right)-45}{640\sqrt{\pi}ab^3\xi^5r^3} + \frac{15r}{64\sqrt{\pi}ab^3\xi} \tag{3.78}$$

$$g_3^{(1)} = \frac{3(39a+25b)}{320\sqrt{\pi}ab^3\xi} + \frac{-6\xi^4(a-b)^4(a+5b)+3\xi^2(a-b)^2(a+5b)+63a+45b}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{3\left(-4\xi^2(a-b)^2(3a+10b)+11a+45b\right)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{75-12\xi^2(a-b)(3a+10b)}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a-b)(a+5b)\left(3-2\xi^2(a-b)^2\right)+45}{640\sqrt{\pi}ab^3\xi^5r^3} - \frac{15r}{64\sqrt{\pi}ab^3\xi} \tag{3.79}$$

$$g_4^{(1)} = -\frac{3(39a+25b)}{320\sqrt{\pi}ab^3\xi} + \frac{6\xi^4(a-b)^4(a+5b)-3\xi^2(a-b)^2(a+5b)-9(7a+5b)}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{3\left(4\xi^2(a-b)^2(3a+10b)-11a-45b\right)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{75-12\xi^2(a-b)(3a+10b)}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a-b)(a+5b)\left(3-2\xi^2(a-b)^2\right)+45}{640\sqrt{\pi}ab^3\xi^5r^3} - \frac{15r}{64\sqrt{\pi}ab^3\xi} \tag{3.80}$$

$$g_5^{(1)} = \frac{27\left(b^2-a^2\right)}{64ab^3} + \frac{12\xi^6(a+b)^5(a-5b)-135\xi^2(a-b)(a+b)-45}{1280ab^3\xi^6r^4} - \frac{15r^2}{64ab^3} + \frac{12\xi^4(a-3b)(a+b)^3-9}{256ab^3\xi^4r^2} + \frac{3r}{5b^3} \tag{3.81}$$

$$g_6^{(1)} = \frac{-12\xi^6(a-b)^5(a+5b)+135\xi^2(a-b)(a+b)+45}{1280ab^3\xi^6r^4} + \frac{15r^2}{64ab^3} + \frac{27(a-b)(a+b)}{64ab^3} + \frac{3r}{5b^3} + \frac{9-12\xi^4(a-b)^3(a+3b)}{256ab^3\xi^4r^2} \tag{3.82}$$

$$g_7^{(1)} = \frac{-12\xi^6(a-b)^5(a+5b)+135\xi^2(a-b)(a+b)+45}{1280ab^3\xi^6r^4} + \frac{15r^2}{64ab^3} + \frac{27(a-b)(a+b)}{64ab^3} - \frac{3r}{5b^3} + \frac{9-12\xi^4(a-b)^3(a+3b)}{256ab^3\xi^4r^2} \tag{3.83}$$

$$g_8^{(1)} = \frac{27\left(b^2-a^2\right)}{64ab^3} + \frac{12\xi^6(a+b)^5(a-5b)-135\xi^2(a-b)(a+b)-45}{1280ab^3\xi^6r^4} + \frac{12\xi^4(a-3b)(a+b)^3-9}{256ab^3\xi^4r^2} - \frac{15r^2}{64ab^3} - \frac{3r}{5b^3} \tag{3.84}$$

Second scalar mobility function

$$g_0^{(2)} = 0 \tag{3.85}$$

$$g_1^{(2)} = \frac{3(11a-5b)}{320\sqrt{\pi}ab^3\xi} + \frac{6\xi^4(a+b)^4(a-5b)-3\xi^2(a-5b)(a+b)^2-63a+45b}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{-12\xi^2(2a-5b)(a+b)^2-3a+45b}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{12\xi^2(2a-5b)(a+b)-15}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a+b)(a-5b)\left(3-2\xi^2(a+b)^2\right)+45}{640\sqrt{\pi}ab^3\xi^5r^3} + \frac{3r}{64\sqrt{\pi}ab^3\xi} \tag{3.86}$$

$$g_2^{(2)} = \frac{3(5b-11a)}{320\sqrt{\pi}ab^3\xi} + \frac{-6\xi^4(a-5b)(a+b)^4+3\xi^2(a+b)^2(a-5b)+63a-45b}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{3\left(4\xi^2(a+b)^2(2a-5b)+a-15b\right)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{12\xi^2(2a-5b)(a+b)-15}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a+b)(a-5b)\left(3-2\xi^2(a+b)^2\right)+45}{640\sqrt{\pi}ab^3\xi^5r^3} + \frac{3r}{64\sqrt{\pi}ab^3\xi} \tag{3.87}$$

$$g_3^{(2)} = \frac{3(11a+5b)}{320\sqrt{\pi}ab^3\xi} + \frac{6\xi^4(a-b)^4(a+5b)-3\xi^2(a-b)^2(a+5b)-9(7a+5b)}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{-12\xi^2(a-b)^2(2a+5b)-3(a+15b)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{15-12\xi^2(a-b)(2a+5b)}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a-b)(a+5b)\left(2\xi^2(a-b)^2-3\right)-45}{640\sqrt{\pi}ab^3\xi^5r^3} - \frac{3r}{64\sqrt{\pi}ab^3\xi} \tag{3.88}$$

$$g_4^{(2)} = -\frac{3(11a+5b)}{320\sqrt{\pi}ab^3\xi} + \frac{-6\xi^4(a-b)^4(a+5b)+3\xi^2(a-b)^2(a+5b)+63a+45b}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{3\xi^2(a-b)(a+5b)\left(2\xi^2(a-b)^2-3\right)-45}{640\sqrt{\pi}ab^3\xi^5r^3} + \frac{3\left(4\xi^2(a-b)^2(2a+5b)+a+15b\right)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{15-12\xi^2(a-b)(2a+5b)}{640\sqrt{\pi}ab^3\xi^3r} - \frac{3r}{64\sqrt{\pi}ab^3\xi} \tag{3.89}$$

$$g_5^{(2)} = \frac{-12\xi^6(a-5b)(a+b)^5+135\xi^2(a-b)(a+b)+45}{1280ab^3\xi^6r^4} - \frac{3r^2}{64ab^3} - \frac{9(a-b)(a+b)}{64ab^3} + \frac{3r}{20b^3} + \frac{12\xi^4(a-3b)(a+b)^3-9}{256ab^3\xi^4r^2} \tag{3.90}$$

$$g_6^{(2)} = \frac{12\xi^6(a-b)^5(a+5b)-135\xi^2(a-b)(a+b)-45}{1280ab^3\xi^6r^4} + \frac{3r^2}{64ab^3} + \frac{9(a-b)(a+b)}{64ab^3} + \frac{3r}{20b^3} + \frac{9-12\xi^4(a-b)^3(a+3b)}{256ab^3\xi^4r^2} \tag{3.91}$$

$$g_7^{(2)} = \frac{12\xi^6(a-b)^5(a+5b) - 135\xi^2(a-b)(a+b) - 45}{1280ab^3\xi^6r^4} + \frac{3r^2}{64ab^3} + \frac{9(a-b)(a+b)}{64ab^3} - \frac{3r}{20b^3} + \frac{9 - 12\xi^4(a-b)^3(a+3b)}{256ab^3\xi^4r^2} \tag{3.92}$$

$$g_8^{(2)} = \frac{-12\xi^6(a-5b)(a+b)^5 + 135\xi^2(a-b)(a+b) + 45}{1280ab^3\xi^6r^4} - \frac{3r^2}{64ab^3} - \frac{9(a-b)(a+b)}{64ab^3} - \frac{3r}{20b^3} + \frac{12\xi^4(a-3b)(a+b)^3 - 9}{256ab^3\xi^4r^2} \tag{3.93}$$

3.2.2. ***Case 2***, $r < a+b, r > b-a, r > a-b$

First scalar mobility function

$$g_0^{(1)} = -\frac{3(a-5b)(a+b)^5}{160ab^3r^4} - \frac{3(a-3b)(a+b)^3}{32ab^3r^2} + \frac{15r^2}{32ab^3} + \frac{27(a-b)(a+b)}{32ab^3} - \frac{6r}{5b^3} \tag{3.94}$$

$$g_1^{(1)} = \frac{3(39a-25b)}{320\sqrt{\pi}ab^3\xi} + \frac{-6\xi^4(a-5b)(a+b)^4 + 3\xi^2(a+b)^2(a-5b) + 63a - 45b}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{3\left(-4\xi^2(3a-10b)(a+b)^2 + 11a - 45b\right)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{12\xi^2(3a-10b)(a+b) - 75}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a-5b)(a+b)\left(2\xi^2(a+b)^2 - 3\right) - 45}{640\sqrt{\pi}ab^3\xi^5r^3} + \frac{15r}{64\sqrt{\pi}ab^3\xi} \tag{3.95}$$

$$g_2^{(1)} = \frac{3(25b-39a)}{320\sqrt{\pi}ab^3\xi} + \frac{6\xi^4(a+b)^4(a-5b) - 3\xi^2(a-5b)(a+b)^2 - 63a + 45b}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{3\left(4\xi^2(a+b)^2(3a-10b) - 11a + 45b\right)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{12\xi^2(3a-10b)(a+b) - 75}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a-5b)(a+b)\left(2\xi^2(a+b)^2 - 3\right) - 45}{640\sqrt{\pi}ab^3\xi^5r^3} + \frac{15r}{64\sqrt{\pi}ab^3\xi} \tag{3.96}$$

$$g_3^{(1)} = \frac{3(39a+25b)}{320\sqrt{\pi}ab^3\xi} + \frac{-6\xi^4(a-b)^4(a+5b) + 3\xi^2(a-b)^2(a+5b) + 63a + 45b}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{3\left(-4\xi^2(a-b)^2(3a+10b) + 11a + 45b\right)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{75 - 12\xi^2(a-b)(3a+10b)}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a-b)(a+5b)\left(3 - 2\xi^2(a-b)^2\right) + 45}{640\sqrt{\pi}ab^3\xi^5r^3} - \frac{15r}{64\sqrt{\pi}ab^3\xi} \tag{3.97}$$

$$g_4^{(1)} = -\frac{3(39a+25b)}{320\sqrt{\pi}ab^3\xi} + \frac{6\xi^4(a-b)^4(a+5b) - 3\xi^2(a-b)^2(a+5b) - 9(7a+5b)}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{3\left(4\xi^2(a-b)^2(3a+10b) - 11a - 45b\right)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{75 - 12\xi^2(a-b)(3a+10b)}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a-b)(a+5b)\left(3 - 2\xi^2(a-b)^2\right) + 45}{640\sqrt{\pi}ab^3\xi^5r^3} - \frac{15r}{64\sqrt{\pi}ab^3\xi} \tag{3.98}$$

$$g_5^{(1)} = \frac{27\left(b^2-a^2\right)}{64ab^3} + \frac{12\xi^6(a+b)^5(a-5b) - 135\xi^2(a-b)(a+b) - 45}{1280ab^3\xi^6r^4} + \frac{12\xi^4(a-3b)(a+b)^3 - 9}{256ab^3\xi^4r^2} - \frac{15r^2}{64ab^3} + \frac{3r}{5b^3} \tag{3.99}$$

$$g_6^{(1)} = \frac{9 - 12\xi^4(a-b)^3(a+3b)}{256ab^3\xi^4r^2} + \frac{15r^2}{64ab^3} + \frac{27(a-b)(a+b)}{64ab^3} + \frac{3r}{5b^3} + \frac{-12\xi^6(a-b)^5(a+5b) + 135\xi^2(a-b)(a+b) + 45}{1280ab^3\xi^6r^4} \tag{3.100}$$

$$g_7^{(1)} = \frac{9 - 12\xi^4(a-b)^3(a+3b)}{256ab^3\xi^4r^2} + \frac{15r^2}{64ab^3} + \frac{27(a-b)(a+b)}{64ab^3} - \frac{3r}{5b^3} + \frac{-12\xi^6(a-b)^5(a+5b) + 135\xi^2(a-b)(a+b) + 45}{1280ab^3\xi^6r^4} \tag{3.101}$$

$$g_8^{(1)} = \frac{27\left(b^2-a^2\right)}{64ab^3} + \frac{12\xi^6(a+b)^5(a-5b) - 135\xi^2(a-b)(a+b) - 45}{1280ab^3\xi^6r^4} + \frac{12\xi^4(a-3b)(a+b)^3 - 9}{256ab^3\xi^4r^2} - \frac{15r^2}{64ab^3} - \frac{3r}{5b^3} \tag{3.102}$$

Second scalar mobility function

$$g_0^{(2)} = \frac{3(a-5b)(a+b)^5}{160ab^3r^4} - \frac{3(a-3b)(a+b)^3}{32ab^3r^2} + \frac{3r^2}{32ab^3} + \frac{9(a-b)(a+b)}{32ab^3} - \frac{3r}{10b^3} \tag{3.103}$$

$$g_1^{(2)} = \frac{3(11a-5b)}{320\sqrt{\pi}ab^3\xi} + \frac{6\xi^4(a+b)^4(a-5b) - 3\xi^2(a-5b)(a+b)^2 - 63a + 45b}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{-12\xi^2(2a-5b)(a+b)^2 - 3a + 45b}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{12\xi^2(2a-5b)(a+b) - 15}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a+b)(a-5b)\left(3 - 2\xi^2(a+b)^2\right) + 45}{640\sqrt{\pi}ab^3\xi^5r^3} + \frac{3r}{64\sqrt{\pi}ab^3\xi} \tag{3.104}$$

$$g_2^{(2)} = \frac{3(5b-11a)}{320\sqrt{\pi}ab^3\xi} + \frac{-6\xi^4(a-5b)(a+b)^4 + 3\xi^2(a+b)^2(a-5b) + 63a - 45b}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{3\left(4\xi^2(a+b)^2(2a-5b) + a - 15b\right)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{12\xi^2(2a-5b)(a+b) - 15}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a+b)(a-5b)\left(3 - 2\xi^2(a+b)^2\right) + 45}{640\sqrt{\pi}ab^3\xi^5r^3} + \frac{3r}{64\sqrt{\pi}ab^3\xi} \tag{3.105}$$

$$g_3^{(2)} = \frac{3(11a+5b)}{320\sqrt{\pi}ab^3\xi} + \frac{6\xi^4(a-b)^4(a+5b) - 3\xi^2(a-b)^2(a+5b) - 9(7a+5b)}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{-12\xi^2(a-b)^2(2a+5b) - 3(a+15b)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{15 - 12\xi^2(a-b)(2a+5b)}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a-b)(a+5b)\left(2\xi^2(a-b)^2 - 3\right) - 45}{640\sqrt{\pi}ab^3\xi^5r^3} - \frac{3r}{64\sqrt{\pi}ab^3\xi} \tag{3.106}$$

$$g_4^{(2)} = \frac{-6\xi^4(a-b)^4(a+5b)+3\xi^2(a-b)^2(a+5b)+63a+45b}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{3\left(4\xi^2(a-b)^2(2a+5b)+a+15b\right)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{15-12\xi^2(a-b)(2a+5b)}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a-b)(a+5b)\left(2\xi^2(a-b)^2-3\right)-45}{640\sqrt{\pi}ab^3\xi^5r^3} - \frac{3(11a+5b)}{320\sqrt{\pi}ab^3\xi} - \frac{3r}{64\sqrt{\pi}ab^3\xi} \tag{3.107}$$

$$g_5^{(2)} = \frac{12\xi^4(a-3b)(a+b)^3-9}{256ab^3\xi^4r^2} - \frac{3r^2}{64ab^3} - \frac{9(a-b)(a+b)}{64ab^3} + \frac{3r}{20b^3} + \frac{-12\xi^6(a-5b)(a+b)^5+135\xi^2(a-b)(a+b)+45}{1280ab^3\xi^6r^4} \tag{3.108}$$

$$g_6^{(2)} = \frac{9-12\xi^4(a-b)^3(a+3b)}{256ab^3\xi^4r^2} + \frac{3r^2}{64ab^3} + \frac{9(a-b)(a+b)}{64ab^3} + \frac{3r}{20b^3} + \frac{12\xi^6(a-b)^5(a+5b)-135\xi^2(a-b)(a+b)-45}{1280ab^3\xi^6r^4} \tag{3.109}$$

$$g_7^{(2)} = \frac{9-12\xi^4(a-b)^3(a+3b)}{256ab^3\xi^4r^2} + \frac{3r^2}{64ab^3} + \frac{9(a-b)(a+b)}{64ab^3} - \frac{3r}{20b^3} + \frac{12\xi^6(a-b)^5(a+5b)-135\xi^2(a-b)(a+b)-45}{1280ab^3\xi^6r^4} \tag{3.110}$$

$$g_8^{(2)} = \frac{12\xi^4(a-3b)(a+b)^3-9}{256ab^3\xi^4r^2} - \frac{3r^2}{64ab^3} - \frac{9(a-b)(a+b)}{64ab^3} - \frac{3r}{20b^3} + \frac{-12\xi^6(a-5b)(a+b)^5+135\xi^2(a-b)(a+b)+45}{1280ab^3\xi^6r^4} \tag{3.111}$$

3.2.3. ***Case 3***, *particle α inside particle β, $r < a+b, r < b-a, r > a-b$*

First scalar mobility function

$$g_0^{(1)} = \frac{3\left(5a^2+3b^2\right)}{10r^4} - \frac{12r}{5b^3} + \frac{3}{2r^2} \tag{3.112}$$

$$g_1^{(1)} = \frac{3(39a-25b)}{320\sqrt{\pi}ab^3\xi} + \frac{-6\xi^4(a-5b)(a+b)^4+3\xi^2(a+b)^2(a-5b)+63a-45b}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{3\left(-4\xi^2(3a-10b)(a+b)^2+11a-45b\right)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{12\xi^2(3a-10b)(a+b)-75}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a-5b)(a+b)\left(2\xi^2(a+b)^2-3\right)-45}{640\sqrt{\pi}ab^3\xi^5r^3} + \frac{15r}{64\sqrt{\pi}ab^3\xi} \tag{3.113}$$

$$g_2^{(1)} = \frac{3(25b-39a)}{320\sqrt{\pi}ab^3\xi} + \frac{6\xi^4(a+b)^4(a-5b)-3\xi^2(a-5b)(a+b)^2-63a+45b}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{3\left(4\xi^2(a+b)^2(3a-10b)-11a+45b\right)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{12\xi^2(3a-10b)(a+b)-75}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a-5b)(a+b)\left(2\xi^2(a+b)^2-3\right)-45}{640\sqrt{\pi}ab^3\xi^5r^3} + \frac{15r}{64\sqrt{\pi}ab^3\xi} \tag{3.114}$$

$$g_3^{(1)} = \frac{3(39a+25b)}{320\sqrt{\pi}ab^3\xi} + \frac{-6\xi^4(a-b)^4(a+5b)+3\xi^2(a-b)^2(a+5b)+63a+45b}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{3\left(-4\xi^2(a-b)^2(3a+10b)+11a+45b\right)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{75-12\xi^2(a-b)(3a+10b)}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a-b)(a+5b)\left(3-2\xi^2(a-b)^2\right)+45}{640\sqrt{\pi}ab^3\xi^5r^3} - \frac{15r}{64\sqrt{\pi}ab^3\xi} \tag{3.115}$$

$$g_4^{(1)} = -\frac{3(39a+25b)}{320\sqrt{\pi}ab^3\xi} + \frac{6\xi^4(a-b)^4(a+5b)-3\xi^2(a-b)^2(a+5b)-9(7a+5b)}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{3\left(4\xi^2(a-b)^2(3a+10b)-11a-45b\right)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{75-12\xi^2(a-b)(3a+10b)}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a-b)(a+5b)\left(3-2\xi^2(a-b)^2\right)+45}{640\sqrt{\pi}ab^3\xi^5r^3} - \frac{15r}{64\sqrt{\pi}ab^3\xi} \tag{3.116}$$

$$g_5^{(1)} = \frac{27\left(b^2-a^2\right)}{64ab^3} + \frac{12\xi^6(a+b)^5(a-5b)-135\xi^2(a-b)(a+b)-45}{1280ab^3\xi^6r^4} + \frac{12\xi^4(a-3b)(a+b)^3-9}{256ab^3\xi^4r^2} - \frac{15r^2}{64ab^3} + \frac{3r}{5b^3} \tag{3.117}$$

$$g_6^{(1)} = \frac{9-12\xi^4(a-b)^3(a+3b)}{256ab^3\xi^4r^2} + \frac{15r^2}{64ab^3} + \frac{27(a-b)(a+b)}{64ab^3} + \frac{3r}{5b^3} + \frac{-12\xi^6(a-b)^5(a+5b)+135\xi^2(a-b)(a+b)+45}{1280ab^3\xi^6r^4} \tag{3.118}$$

$$g_7^{(1)} = \frac{9-12\xi^4(a-b)^3(a+3b)}{256ab^3\xi^4r^2} + \frac{15r^2}{64ab^3} + \frac{27(a-b)(a+b)}{64ab^3} - \frac{3r}{5b^3} + \frac{-12\xi^6(a-b)^5(a+5b)+135\xi^2(a-b)(a+b)+45}{1280ab^3\xi^6r^4} \tag{3.119}$$

$$g_8^{(1)} = \frac{27\left(b^2-a^2\right)}{64ab^3} + \frac{12\xi^6(a+b)^5(a-5b)-135\xi^2(a-b)(a+b)-45}{1280ab^3\xi^6r^4} + \frac{12\xi^4(a-3b)(a+b)^3-9}{256ab^3\xi^4r^2} - \frac{15r^2}{64ab^3} - \frac{3r}{5b^3} \tag{3.120}$$

Second scalar mobility function

$$g_0^{(2)} = -\frac{3\left(5a^2+3b^2\right)}{10r^4} - \frac{3r}{5b^3} + \frac{3}{2r^2} \tag{3.121}$$

$$g_1^{(2)} = \frac{3(11a-5b)}{320\sqrt{\pi}ab^3\xi} + \frac{6\xi^4(a+b)^4(a-5b)-3\xi^2(a-5b)(a+b)^2-63a+45b}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{-12\xi^2(2a-5b)(a+b)^2-3a+45b}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{12\xi^2(2a-5b)(a+b)-15}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a+b)(a-5b)\left(3-2\xi^2(a+b)^2\right)+45}{640\sqrt{\pi}ab^3\xi^5r^3} + \frac{3r}{64\sqrt{\pi}ab^3\xi} \tag{3.122}$$

$$g_2^{(2)} = \frac{3(5b-11a)}{320\sqrt{\pi}ab^3\xi} + \frac{-6\xi^4(a-5b)(a+b)^4 + 3\xi^2(a+b)^2(a-5b) + 63a - 45b}{640\sqrt{\pi}ab^3\xi^5r^4} \tag{3.123}$$
$$+ \frac{3\left(4\xi^2(a+b)^2(2a-5b) + a - 15b\right)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{12\xi^2(2a-5b)(a+b) - 15}{640\sqrt{\pi}ab^3\xi^3r}$$
$$+ \frac{3\xi^2(a+b)(a-5b)\left(3 - 2\xi^2(a+b)^2\right) + 45}{640\sqrt{\pi}ab^3\xi^5r^3} + \frac{3r}{64\sqrt{\pi}ab^3\xi}$$

$$g_3^{(2)} = \frac{3(11a+5b)}{320\sqrt{\pi}ab^3\xi} + \frac{6\xi^4(a-b)^4(a+5b) - 3\xi^2(a-b)^2(a+5b) - 9(7a+5b)}{640\sqrt{\pi}ab^3\xi^5r^4} \tag{3.124}$$
$$+ \frac{-12\xi^2(a-b)^2(2a+5b) - 3(a+15b)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{15 - 12\xi^2(a-b)(2a+5b)}{640\sqrt{\pi}ab^3\xi^3r}$$
$$+ \frac{3\xi^2(a-b)(a+5b)\left(2\xi^2(a-b)^2 - 3\right) - 45}{640\sqrt{\pi}ab^3\xi^5r^3} - \frac{3r}{64\sqrt{\pi}ab^3\xi}$$

$$g_4^{(2)} = -\frac{3(11a+5b)}{320\sqrt{\pi}ab^3\xi} + \frac{-6\xi^4(a-b)^4(a+5b) + 3\xi^2(a-b)^2(a+5b) + 63a + 45b}{640\sqrt{\pi}ab^3\xi^5r^4} \tag{3.125}$$
$$+ \frac{3\left(4\xi^2(a-b)^2(2a+5b) + a + 15b\right)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{15 - 12\xi^2(a-b)(2a+5b)}{640\sqrt{\pi}ab^3\xi^3r}$$
$$+ \frac{3\xi^2(a-b)(a+5b)\left(2\xi^2(a-b)^2 - 3\right) - 45}{640\sqrt{\pi}ab^3\xi^5r^3} - \frac{3r}{64\sqrt{\pi}ab^3\xi}$$

$$g_5^{(2)} = \frac{12\xi^4(a-3b)(a+b)^3 - 9}{256ab^3\xi^4r^2} - \frac{3r^2}{64ab^3} - \frac{9(a-b)(a+b)}{64ab^3} + \frac{3r}{20b^3} \tag{3.126}$$
$$+ \frac{-12\xi^6(a-5b)(a+b)^5 + 135\xi^2(a-b)(a+b) + 45}{1280ab^3\xi^6r^4}$$

$$g_6^{(2)} = \frac{9 - 12\xi^4(a-b)^3(a+3b)}{256ab^3\xi^4r^2} + \frac{3r^2}{64ab^3} + \frac{9(a-b)(a+b)}{64ab^3} + \frac{3r}{20b^3} \tag{3.127}$$
$$+ \frac{12\xi^6(a-b)^5(a+5b) - 135\xi^2(a-b)(a+b) - 45}{1280ab^3\xi^6r^4}$$

$$g_7^{(2)} = \frac{9 - 12\xi^4(a-b)^3(a+3b)}{256ab^3\xi^4r^2} + \frac{3r^2}{64ab^3} + \frac{9(a-b)(a+b)}{64ab^3} - \frac{3r}{20b^3} \tag{3.128}$$
$$+ \frac{12\xi^6(a-b)^5(a+5b) - 135\xi^2(a-b)(a+b) - 45}{1280ab^3\xi^6r^4}$$

$$g_8^{(2)} = \frac{12\xi^4(a-3b)(a+b)^3 - 9}{256ab^3\xi^4r^2} - \frac{3r^2}{64ab^3} - \frac{9(a-b)(a+b)}{64ab^3} - \frac{3r}{20b^3} \tag{3.129}$$
$$+ \frac{-12\xi^6(a-5b)(a+b)^5 + 135\xi^2(a-b)(a+b) + 45}{1280ab^3\xi^6r^4}$$

3.2.4. ***Case 4***, *particle β inside particle α, $r < a+b, r > b-a, r < a-b$*

First scalar mobility function

$$g_0^{(1)} = \frac{3\left(5a^2 + 3b^2\right)}{10r^4} + \frac{3}{2r^2} \tag{3.130}$$

$$g_1^{(1)} = \frac{3(39a-25b)}{320\sqrt{\pi}ab^3\xi} + \frac{-6\xi^4(a-5b)(a+b)^4+3\xi^2(a+b)^2(a-5b)+63a-45b}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{3\left(-4\xi^2(3a-10b)(a+b)^2+11a-45b\right)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{12\xi^2(3a-10b)(a+b)-75}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a-5b)(a+b)\left(2\xi^2(a+b)^2-3\right)-45}{640\sqrt{\pi}ab^3\xi^5r^3} + \frac{15r}{64\sqrt{\pi}ab^3\xi} \tag{3.131}$$

$$g_2^{(1)} = \frac{3(25b-39a)}{320\sqrt{\pi}ab^3\xi} + \frac{6\xi^4(a+b)^4(a-5b)-3\xi^2(a-5b)(a+b)^2-63a+45b}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{3\left(4\xi^2(a+b)^2(3a-10b)-11a+45b\right)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{12\xi^2(3a-10b)(a+b)-75}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a-5b)(a+b)\left(2\xi^2(a+b)^2-3\right)-45}{640\sqrt{\pi}ab^3\xi^5r^3} + \frac{15r}{64\sqrt{\pi}ab^3\xi} \tag{3.132}$$

$$g_3^{(1)} = \frac{3(39a+25b)}{320\sqrt{\pi}ab^3\xi} + \frac{-6\xi^4(a-b)^4(a+5b)+3\xi^2(a-b)^2(a+5b)+63a+45b}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{3\left(-4\xi^2(a-b)^2(3a+10b)+11a+45b\right)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{75-12\xi^2(a-b)(3a+10b)}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a-b)(a+5b)\left(3-2\xi^2(a-b)^2\right)+45}{640\sqrt{\pi}ab^3\xi^5r^3} - \frac{15r}{64\sqrt{\pi}ab^3\xi} \tag{3.133}$$

$$g_4^{(1)} = \frac{6\xi^4(a-b)^4(a+5b)-3\xi^2(a-b)^2(a+5b)-9(7a+5b)}{640\sqrt{\pi}ab^3\xi^5r^4} - \frac{3(39a+25b)}{320\sqrt{\pi}ab^3\xi} + \frac{3\left(4\xi^2(a-b)^2(3a+10b)-11a-45b\right)}{640\sqrt{\pi}ab^3\xi^3r^2} - \frac{15r}{64\sqrt{\pi}ab^3\xi} + \frac{75-12\xi^2(a-b)(3a+10b)}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a-b)(a+5b)\left(3-2\xi^2(a-b)^2\right)+45}{640\sqrt{\pi}ab^3\xi^5r^3} \tag{3.134}$$

$$g_5^{(1)} = \frac{27\left(b^2-a^2\right)}{64ab^3} + \frac{12\xi^6(a+b)^5(a-5b)-135\xi^2(a-b)(a+b)-45}{1280ab^3\xi^6r^4} + \frac{12\xi^4(a-3b)(a+b)^3-9}{256ab^3\xi^4r^2} - \frac{15r^2}{64ab^3} + \frac{3r}{5b^3} \tag{3.135}$$

$$g_6^{(1)} = \frac{9-12\xi^4(a-b)^3(a+3b)}{256ab^3\xi^4r^2} + \frac{15r^2}{64ab^3} + \frac{27(a-b)(a+b)}{64ab^3} + \frac{3r}{5b^3} + \frac{-12\xi^6(a-b)^5(a+5b)+135\xi^2(a-b)(a+b)+45}{1280ab^3\xi^6r^4} \tag{3.136}$$

$$g_7^{(1)} = \frac{9-12\xi^4(a-b)^3(a+3b)}{256ab^3\xi^4r^2} + \frac{15r^2}{64ab^3} + \frac{27(a-b)(a+b)}{64ab^3} - \frac{3r}{5b^3} + \frac{-12\xi^6(a-b)^5(a+5b)+135\xi^2(a-b)(a+b)+45}{1280ab^3\xi^6r^4} \tag{3.137}$$

$$g_8^{(1)} = \frac{27\left(b^2-a^2\right)}{64ab^3} + \frac{12\xi^6(a+b)^5(a-5b)-135\xi^2(a-b)(a+b)-45}{1280ab^3\xi^6r^4} + \frac{12\xi^4(a-3b)(a+b)^3-9}{256ab^3\xi^4r^2} - \frac{15r^2}{64ab^3} - \frac{3r}{5b^3} \tag{3.138}$$

Second scalar mobility function

$$g_0^{(2)} = \frac{3}{2r^2} - \frac{3\left(5a^2+3b^2\right)}{10r^4} \tag{3.139}$$

$$g_1^{(2)} = \frac{3(11a-5b)}{320\sqrt{\pi}ab^3\xi} + \frac{6\xi^4(a+b)^4(a-5b)-3\xi^2(a-5b)(a+b)^2-63a+45b}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{-12\xi^2(2a-5b)(a+b)^2-3a+45b}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{12\xi^2(2a-5b)(a+b)-15}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a+b)(a-5b)\left(3-2\xi^2(a+b)^2\right)+45}{640\sqrt{\pi}ab^3\xi^5r^3} + \frac{3r}{64\sqrt{\pi}ab^3\xi} \tag{3.140}$$

$$g_2^{(2)} = \frac{3(5b-11a)}{320\sqrt{\pi}ab^3\xi} + \frac{-6\xi^4(a-5b)(a+b)^4+3\xi^2(a+b)^2(a-5b)+63a-45b}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{3\left(4\xi^2(a+b)^2(2a-5b)+a-15b\right)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{12\xi^2(2a-5b)(a+b)-15}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a+b)(a-5b)\left(3-2\xi^2(a+b)^2\right)+45}{640\sqrt{\pi}ab^3\xi^5r^3} + \frac{3r}{64\sqrt{\pi}ab^3\xi} \tag{3.141}$$

$$g_3^{(2)} = \frac{3(11a+5b)}{320\sqrt{\pi}ab^3\xi} + \frac{6\xi^4(a-b)^4(a+5b)-3\xi^2(a-b)^2(a+5b)-9(7a+5b)}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{-12\xi^2(a-b)^2(2a+5b)-3(a+15b)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{15-12\xi^2(a-b)(2a+5b)}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a-b)(a+5b)\left(2\xi^2(a-b)^2-3\right)-45}{640\sqrt{\pi}ab^3\xi^5r^3} - \frac{3r}{64\sqrt{\pi}ab^3\xi} \tag{3.142}$$

$$g_4^{(2)} = -\frac{3(11a+5b)}{320\sqrt{\pi}ab^3\xi} + \frac{-6\xi^4(a-b)^4(a+5b)+3\xi^2(a-b)^2(a+5b)+63a+45b}{640\sqrt{\pi}ab^3\xi^5r^4} + \frac{3\left(4\xi^2(a-b)^2(2a+5b)+a+15b\right)}{640\sqrt{\pi}ab^3\xi^3r^2} + \frac{15-12\xi^2(a-b)(2a+5b)}{640\sqrt{\pi}ab^3\xi^3r} + \frac{3\xi^2(a-b)(a+5b)\left(2\xi^2(a-b)^2-3\right)-45}{640\sqrt{\pi}ab^3\xi^5r^3} - \frac{3r}{64\sqrt{\pi}ab^3\xi} \tag{3.143}$$

$$g_5^{(2)} = \frac{12\xi^4(a-3b)(a+b)^3-9}{256ab^3\xi^4r^2} - \frac{3r^2}{64ab^3} - \frac{9(a-b)(a+b)}{64ab^3} + \frac{3r}{20b^3} + \frac{-12\xi^6(a-5b)(a+b)^5+135\xi^2(a-b)(a+b)+45}{1280ab^3\xi^6r^4} \tag{3.144}$$

$$g_6^{(2)} = \frac{9-12\xi^4(a-b)^3(a+3b)}{256ab^3\xi^4r^2} + \frac{3r^2}{64ab^3} + \frac{9(a-b)(a+b)}{64ab^3} + \frac{3r}{20b^3} + \frac{12\xi^6(a-b)^5(a+5b)-135\xi^2(a-b)(a+b)-45}{1280ab^3\xi^6r^4} \tag{3.145}$$

$$g_7^{(2)} = \frac{9-12\xi^4(a-b)^3(a+3b)}{256ab^3\xi^4r^2} + \frac{3r^2}{64ab^3} + \frac{9(a-b)(a+b)}{64ab^3} - \frac{3r}{20b^3} + \frac{12\xi^6(a-b)^5(a+5b)-135\xi^2(a-b)(a+b)-45}{1280ab^3\xi^6r^4} \tag{3.146}$$

$$g_8^{(2)} = \frac{12\xi^4(a-3b)(a+b)^3-9}{256ab^3\xi^4r^2} - \frac{3r^2}{64ab^3} - \frac{9(a-b)(a+b)}{64ab^3} - \frac{3r}{20b^3} + \frac{-12\xi^6(a-5b)(a+b)^5+135\xi^2(a-b)(a+b)+45}{1280ab^3\xi^6r^4} \tag{3.147}$$

3.2.5. ***Case 5**, self contribution ($\alpha = \beta$ or $r = 0$)*

The self contribution to the translation-dipole coupling is zero for all scalar coefficients.

3.3. *Real Space Mobility Functions, Gradient-Dipole Coupling*

The integrals presented in Appendix A of the main text give the tensor coupling

$$\begin{aligned} 6\pi\eta a^3 \left(M_{\mathrm{DC}}^{\alpha\beta,\mathrm{real}}\right)_{ijkl} &= h_1(r)\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} - 4\delta_{il}\delta_{jk}\right) + h_2(r)\left(\delta_{jk}\hat{r}_i\hat{r}_l - \hat{r}_i\hat{r}_j\hat{r}_k\hat{r}_l\right) \\ &\quad + h_3(r)\left(\delta_{ij}\hat{r}_k\hat{r}_l + \delta_{ik}\hat{r}_j\hat{r}_l + \delta_{jl}\hat{r}_i\hat{r}_k + \delta_{kl}\hat{r}_i\hat{r}_j + \delta_{il}\hat{r}_j\hat{r}_k \right. \\ &\quad \left. -6\hat{r}_i\hat{r}_j\hat{r}_k\hat{r}_l - \delta_{il}\delta_{jk}\right). \end{aligned}$$

where all the scalar functions are of the form

$$\begin{aligned} h^{(i)}(r) &= h_0^{(i)} + h_1^{(i)}\, e^{-(a+b+r)^2\xi^2} + h_2^{(i)}\, e^{-(a+b-r)^2\xi^2} + h_3^{(i)}\, e^{-(-a+b+r)^2\xi^2} \\ &\quad + h_4^{(i)}\, e^{-(a-b+r)^2\xi^2} + h_5^{(i)}\,\mathrm{erfc}\left((-a-b+r)\xi\right) + h_6^{(i)}\,\mathrm{erfc}\left((a-b+r)\xi\right) \\ &\quad + h_7^{(i)}\,\mathrm{erfc}\left((-a+b+r)\xi\right) + h_8^{(i)}\,\mathrm{erfc}\left((a+b+r)\xi\right) \end{aligned} \tag{3.148}$$

3.3.1. ***Case 1**, $r > a + b$*

First scalar mobility function

$$h_0^{(1)} = 0 \tag{3.149}$$

$$
\begin{aligned}
h_1^{(1)} =& -\frac{3r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{3r(a+b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{6\xi^2\left(11a^2-2ab+11b^2\right)+3}{2048\sqrt{\pi}a^3b^3\xi^3} \\
&+ \frac{9(a+b)}{40960\sqrt{\pi}a^3b^3\xi^7r^5}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)-4\xi^4(a+b)^2\left(a^2-6ab+b^2\right)\right. \\
&\left.-6\xi^2\left(29a^2-54ab+29b^2\right)-45\right] + \frac{9}{40960\sqrt{\pi}a^3b^3\xi^7r^4}\left[-8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)\right. \\
&\left.+12\xi^4(a+b)^2\left(a^2-6ab+b^2\right)+30\xi^2\left(5a^2-6ab+5b^2\right)-45\right] \\
&+ \frac{3(a+b)\left(-4\xi^4(a+b)^2\left(17a^2-62ab+17b^2\right)+4\xi^2\left(a^2+14ab+b^2\right)-15\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^3} \\
&+ \frac{3\left(4\xi^4(a+b)^2\left(17a^2-62ab+17b^2\right)-60\xi^2(a-b)^2+75\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^2} \\
&+ \frac{3(a+b)\left(2\xi^2\left(73a^2-118ab+73b^2\right)-15\right)}{10240\sqrt{\pi}a^3b^3\xi^3r}
\end{aligned} \tag{3.150}
$$

$$
\begin{aligned}
h_2^{(1)} =& -\frac{3r^2}{1024\sqrt{\pi}a^3b^3\xi} - \frac{3r(a+b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{3\left(22a^2\xi^2-4ab\xi^2+22b^2\xi^2+1\right)}{2048\sqrt{\pi}a^3b^3\xi^3} \\
&- \frac{3(a+b)\left(146a^2\xi^2-236ab\xi^2+146b^2\xi^2-15\right)}{10240\sqrt{\pi}a^3b^3\xi^3r} + \frac{3(a+b)}{20480\sqrt{\pi}a^3b^3\xi^5r^3}\left[68a^4\xi^4\right. \\
&\left.-112a^3b\xi^4-4a^2\left(90b^2\xi^4+\xi^2\right)-56a\left(2b^3\xi^4+b\xi^2\right)+68b^4\xi^4-4b^2\xi^2+15\right] \\
&+ \frac{3}{20480\sqrt{\pi}a^3b^3\xi^5r^2}\left[68a^4\xi^4-112a^3b\xi^4-60a^2\left(6b^2\xi^4+\xi^2\right)-8ab\xi^2\left(14b^2\xi^2-15\right)\right. \\
&\left.+68b^4\xi^4-60b^2\xi^2+75\right] - \frac{9(a+b)}{40960\sqrt{\pi}a^3b^3\xi^7r^5}\left[8a^6\xi^6-16a^5b\xi^6-4a^4\left(34b^2\xi^6+\xi^4\right)\right. \\
&+a^3\left(16b\xi^4-224b^3\xi^6\right)-2a^2\xi^2\left(68b^4\xi^4-20b^2\xi^2+87\right)-4ab\xi^2\left(4b^4\xi^4-4b^2\xi^2-81\right) \\
&\left.+8b^6\xi^6-4b^4\xi^4-174b^2\xi^2-45\right] - \frac{9}{40960\sqrt{\pi}a^3b^3\xi^7r^4}\left[8a^6\xi^6-16a^5b\xi^6\right. \\
&-4a^4\xi^4\left(34b^2\xi^2+3\right)+a^3\left(48b\xi^4-224b^3\xi^6\right)-2a^2\xi^2\left(68b^4\xi^4-60b^2\xi^2+75\right) \\
&\left.+4ab\xi^2\left(-4b^4\xi^4+12b^2\xi^2+45\right)+8b^6\xi^6-12b^4\xi^4-150b^2\xi^2+45\right]
\end{aligned} \tag{3.151}
$$

$$
\begin{aligned}
h_3^{(1)} =& \frac{3r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{3r(a-b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{-6\xi^2\left(11a^2+2ab+11b^2\right)-3}{2048\sqrt{\pi}a^3b^3\xi^3} \\
&+ \frac{9(a-b)}{40960\sqrt{\pi}a^3b^3\xi^7r^5}\left[8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)-4\xi^4(a-b)^2\left(a^2+6ab+b^2\right)\right. \\
&\left.-6\xi^2\left(29a^2+54ab+29b^2\right)-45\right] + \frac{9}{40960\sqrt{\pi}a^3b^3\xi^7r^4}\left[8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)\right. \\
&\left.-12\xi^4(a-b)^2\left(a^2+6ab+b^2\right)-30\xi^2\left(5a^2+6ab+5b^2\right)+45\right] \\
&+ \frac{3(a-b)}{20480\sqrt{\pi}a^3b^3\xi^5r^3}\left(-4\xi^4(a-b)^2\left(17a^2+62ab+17b^2\right)+4\xi^2\left(a^2-14ab+b^2\right)-15\right) \\
&+ \frac{3\left(-4\xi^4(a-b)^2\left(17a^2+62ab+17b^2\right)+60\xi^2(a+b)^2-75\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^2} \\
&+ \frac{3(a-b)\left(2\xi^2\left(73a^2+118ab+73b^2\right)-15\right)}{10240\sqrt{\pi}a^3b^3\xi^3r}
\end{aligned} \tag{3.152}
$$

$$h_4^{(1)} = \frac{3\xi^6 r^2}{1024\sqrt{\pi}a^3b^3\xi^7} + \frac{3\xi^6 r(b-a)}{1024\sqrt{\pi}a^3b^3\xi^7} - \frac{3\left(\xi^6\left(22a^2+4ab+22b^2\right)+\xi^4\right)}{2048\sqrt{\pi}a^3b^3\xi^7} \tag{3.153}$$
$$- \frac{3\xi^4(a-b)\left(2\xi^2\left(73a^2+118ab+73b^2\right)-15\right)}{10240\sqrt{\pi}a^3b^3r\xi^7}$$
$$+ \frac{9(a-b)}{40960\sqrt{\pi}a^3b^3r^5\xi^7}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)+4\xi^4(a-b)^2\left(a^2+6ab+b^2\right)\right.$$
$$\left.+6\xi^2\left(29a^2+54ab+29b^2\right)+45\right] + \frac{9}{40960\sqrt{\pi}a^3b^3r^4\xi^7}\left[8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)\right.$$
$$\left.-12\xi^4(a-b)^2\left(a^2+6ab+b^2\right)-30\xi^2\left(5a^2+6ab+5b^2\right)+45\right]$$
$$+ \frac{3\xi^2(a-b)\left(4\xi^4(a-b)^2\left(17a^2+62ab+17b^2\right)-4\xi^2\left(a^2-14ab+b^2\right)+15\right)}{20480\sqrt{\pi}a^3b^3r^3\xi^7}$$
$$+ \frac{3\xi^2\left(-4\xi^4(a-b)^2\left(17a^2+62ab+17b^2\right)+60\xi^2(a+b)^2-75\right)}{20480\sqrt{\pi}a^3b^3r^2\xi^7}$$

$$h_5^{(1)} = \frac{3r^3}{1024a^3b^3} + \frac{3}{40}\frac{a^3+b^3}{a^3b^3} - \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4 r} - \frac{9r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.154}$$
$$+ \frac{9\left(-16\xi^8(a+b)^6\left(a^2-6ab+b^2\right)+360\xi^4\left(a^2-b^2\right)^2+240\xi^2\left(a^2+b^2\right)-45\right)}{81920a^3b^3\xi^8r^5}$$
$$+ \frac{12\xi^6(a+b)^4\left(a^2-4ab+b^2\right)-27\xi^2\left(a^2+b^2\right)+9}{1024a^3b^3\xi^6r^3}$$

$$h_6^{(1)} = -\frac{3r^3}{1024a^3b^3} + \frac{3}{40}\frac{a^3-b^3}{a^3b^3} + \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4 r} + \frac{9r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.155}$$
$$+ \frac{9\left(16\xi^8(a-b)^6\left(a^2+6ab+b^2\right)-360\xi^4\left(a^2-b^2\right)^2-240\xi^2\left(a^2+b^2\right)+45\right)}{81920a^3b^3\xi^8r^5}$$
$$+ \frac{-12\xi^6(a-b)^4\left(a^2+4ab+b^2\right)+27\xi^2\left(a^2+b^2\right)-9}{1024a^3b^3\xi^6r^3}$$

$$h_7^{(1)} = -\frac{3r^3}{1024a^3b^3} + \frac{3}{40}\frac{b^3-a^3}{a^3b^3} + \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4 r} + \frac{9r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.156}$$
$$+ \frac{9\left(16\xi^8(a-b)^6\left(a^2+6ab+b^2\right)-360\xi^4\left(a^2-b^2\right)^2-240\xi^2\left(a^2+b^2\right)+45\right)}{81920a^3b^3\xi^8r^5}$$
$$+ \frac{-12\xi^6(a-b)^4\left(a^2+4ab+b^2\right)+27\xi^2\left(a^2+b^2\right)-9}{1024a^3b^3\xi^6r^3}$$

$$h_8^{(1)} = \frac{3r^3}{1024a^3b^3} + \frac{3}{40}\frac{-a^3-b^3}{a^3b^3} - \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4 r} - \frac{9r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.157}$$
$$+ \frac{9\left(-16\xi^8(a+b)^6\left(a^2-6ab+b^2\right)+360\xi^4\left(a^2-b^2\right)^2+240\xi^2\left(a^2+b^2\right)-45\right)}{81920a^3b^3\xi^8r^5}$$
$$+ \frac{12\xi^6(a+b)^4\left(a^2-4ab+b^2\right)-27\xi^2\left(a^2+b^2\right)+9}{1024a^3b^3\xi^6r^3}$$

Second scalar mobility function

$$h_0^{(2)} = 0 \tag{3.158}$$

$$\begin{aligned}
h_1^{(2)} = {} & \frac{63r^2}{1024\sqrt{\pi}a^3b^3\xi} - \frac{63r(a+b)}{1024\sqrt{\pi}a^3b^3\xi} - \frac{9\left(2\xi^2\left(13a^2-14ab+13b^2\right)+7\right)}{2048\sqrt{\pi}a^3b^3\xi^3} \\
& + \frac{9(a+b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[-8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)+4\xi^4(a+b)^2\left(a^2-6ab+b^2\right)\right. \\
& \left.+6\xi^2\left(29a^2-54ab+29b^2\right)+45\right] + \frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)\right. \\
& \left.-12\xi^4(a+b)^2\left(a^2-6ab+b^2\right)-30\xi^2\left(5a^2-6ab+5b^2\right)+45\right] \\
& + \frac{9(a+b)\left(-4\xi^4(a+b)^2\left(5a^2-22ab+5b^2\right)+4\xi^2(5a-b)(a-5b)+69\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3} \\
& + \frac{9\left(4\xi^2\left(\xi^2(a+b)^2\left(5a^2-22ab+5b^2\right)-11a^2+54ab-11b^2\right)+39\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2} \\
& + \frac{9(a+b)\left(2\xi^2\left(13a^2-14ab+13b^2\right)+21\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}
\end{aligned} \tag{3.159}$$

$$\begin{aligned}
h_2^{(2)} = {} & \frac{63r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{63r(a+b)}{1024\sqrt{\pi}a^3b^3\xi} - \frac{9\left(2\xi^2\left(13a^2-14ab+13b^2\right)+7\right)}{2048\sqrt{\pi}a^3b^3\xi^3} \\
& + \frac{9(a+b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)-4\xi^4(a+b)^2\left(a^2-6ab+b^2\right)\right. \\
& \left.-6\xi^2\left(29a^2-54ab+29b^2\right)-45\right] + \frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)\right. \\
& \left.-12\xi^4(a+b)^2\left(a^2-6ab+b^2\right)-30\xi^2\left(5a^2-6ab+5b^2\right)+45\right] \\
& + \frac{9(a+b)\left(4\xi^4(a+b)^2\left(5a^2-22ab+5b^2\right)-4\xi^2(a-5b)(5a-b)-69\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3} \\
& + \frac{9\left(4\xi^2\left(\xi^2(a+b)^2\left(5a^2-22ab+5b^2\right)-11a^2+54ab-11b^2\right)+39\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2} \\
& - \frac{9(a+b)\left(2\xi^2\left(13a^2-14ab+13b^2\right)+21\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}
\end{aligned}$$

$$
\begin{aligned}
h_3^{(2)} = & -\frac{63r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{63r(b-a)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{18\xi^2\left(13a^2+14ab+13b^2\right)+63}{2048\sqrt{\pi}a^3b^3\xi^3} \\
& + \frac{9(a-b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)+4\xi^4(a-b)^2\left(a^2+6ab+b^2\right)\right. \\
& \left.+6\xi^2\left(29a^2+54ab+29b^2\right)+45\right] + \frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)\right. \\
& \left.+12\xi^4(a-b)^2\left(a^2+6ab+b^2\right)+30\xi^2\left(5a^2+6ab+5b^2\right)-45\right] \\
& + \frac{9(a-b)\left(-4\xi^4(a-b)^2\left(5a^2+22ab+5b^2\right)+4\xi^2(5a+b)(a+5b)+69\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3} \\
& + \frac{9\left(-4\xi^4(a-b)^2\left(5a^2+22ab+5b^2\right)+4\xi^2\left(11a^2+54ab+11b^2\right)-39\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2} \\
& + \frac{9(a-b)\left(2\xi^2\left(13a^2+14ab+13b^2\right)+21\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}
\end{aligned} \tag{3.160}
$$

$$
\begin{aligned}
h_4^{(2)} = & -\frac{63r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{63r(a-b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{18\xi^2\left(13a^2+14ab+13b^2\right)+63}{2048\sqrt{\pi}a^3b^3\xi^3} \\
& + \frac{9(a-b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)-4\xi^4(a-b)^2\left(a^2+6ab+b^2\right)\right. \\
& \left.-6\xi^2\left(29a^2+54ab+29b^2\right)-45\right] + \frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)\right. \\
& \left.+12\xi^4(a-b)^2\left(a^2+6ab+b^2\right)+30\xi^2\left(5a^2+6ab+5b^2\right)-45\right] \\
& + \frac{9(a-b)\left(4\xi^4(a-b)^2\left(5a^2+22ab+5b^2\right)-4\xi^2(5a+b)(a+5b)-69\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3} \\
& + \frac{9\left(-4\xi^4(a-b)^2\left(5a^2+22ab+5b^2\right)+4\xi^2\left(11a^2+54ab+11b^2\right)-39\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2} \\
& - \frac{9(a-b)\left(2\xi^2\left(13a^2+14ab+13b^2\right)+21\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}
\end{aligned} \tag{3.161}
$$

$$
\begin{aligned}
h_5^{(2)} = & -\frac{63r^3}{1024a^3b^3} - \frac{324\xi^4\left(a^2-b^2\right)^2+81}{2048a^3b^3\xi^4r} + \frac{45r\left(a^2+b^2\right)}{256a^3b^3} \\
& + \frac{9\left(16\xi^8(a+b)^6\left(a^2-6ab+b^2\right)-360\xi^4\left(a^2-b^2\right)^2-240\xi^2\left(a^2+b^2\right)+45\right)}{16384a^3b^3\xi^8r^5} \\
& + \frac{9\left(4\xi^6(a+b)^4\left(a^2-4ab+b^2\right)-9\xi^2\left(a^2+b^2\right)+3\right)}{1024a^3b^3\xi^6r^3}
\end{aligned} \tag{3.162}
$$

$$
\begin{aligned}
h_6^{(2)} = & \frac{63r^3}{1024a^3b^3} + \frac{324\xi^4\left(a^2-b^2\right)^2+81}{2048a^3b^3\xi^4r} - \frac{45r\left(a^2+b^2\right)}{256a^3b^3} \\
& + \frac{9\left(-16\xi^8(a-b)^6\left(a^2+6ab+b^2\right)+360\xi^4\left(a^2-b^2\right)^2+240\xi^2\left(a^2+b^2\right)-45\right)}{16384a^3b^3\xi^8r^5} \\
& + \frac{9\left(-4\xi^6(a-b)^4\left(a^2+4ab+b^2\right)+9\xi^2\left(a^2+b^2\right)-3\right)}{1024a^3b^3\xi^6r^3}
\end{aligned} \tag{3.163}
$$

$$h_7^{(2)} = \frac{63r^3}{1024a^3b^3} + \frac{324\xi^4\left(a^2-b^2\right)^2+81}{2048a^3b^3\xi^4 r} - \frac{45r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.164}$$
$$+ \frac{9\left(-16\xi^8(a-b)^6\left(a^2+6ab+b^2\right)+360\xi^4\left(a^2-b^2\right)^2+240\xi^2\left(a^2+b^2\right)-45\right)}{16384a^3b^3\xi^8r^5}$$
$$+ \frac{9\left(-4\xi^6(a-b)^4\left(a^2+4ab+b^2\right)+9\xi^2\left(a^2+b^2\right)-3\right)}{1024a^3b^3\xi^6r^3}$$

$$h_8^{(2)} = -\frac{63r^3}{1024a^3b^3} - \frac{324\xi^4\left(a^2-b^2\right)^2+81}{2048a^3b^3\xi^4 r} + \frac{45r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.165}$$
$$+ \frac{9\left(16\xi^8(a+b)^6\left(a^2-6ab+b^2\right)-360\xi^4\left(a^2-b^2\right)^2-240\xi^2\left(a^2+b^2\right)+45\right)}{16384a^3b^3\xi^8r^5}$$
$$+ \frac{9\left(4\xi^6(a+b)^4\left(a^2-4ab+b^2\right)-9\xi^2\left(a^2+b^2\right)+3\right)}{1024a^3b^3\xi^6r^3}$$

Third scalar mobility function

$$h_0^{(3)} = 0 \tag{3.166}$$

$$h_1^{(3)} = -\frac{9r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{9r(a+b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{18\xi^2\left(3a^2-2ab+3b^2\right)+9}{2048\sqrt{\pi}a^3b^3\xi^3} \tag{3.167}$$
$$+ \frac{9(a+b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[-8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)+4\xi^4(a+b)^2\left(a^2-6ab+b^2\right)\right.$$
$$\left.+6\xi^2\left(29a^2-54ab+29b^2\right)+45\right] + \frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)\right.$$
$$\left.-12\xi^4(a+b)^2\left(a^2-6ab+b^2\right)-30\xi^2\left(5a^2-6ab+5b^2\right)+45\right]$$
$$+ \frac{9(a+b)\left(4\xi^2\left(a^2+\xi^2(3a-b)(a+b)^2(a-3b)-10ab+b^2\right)+21\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3}$$
$$+ \frac{9\left(4\xi^2\left(a^2+6ab+b^2\right)-4\xi^4(a-3b)(3a-b)(a+b)^2-9\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2}$$
$$- \frac{9(a+b)\left(2\xi^2\left(3a^2-2ab+3b^2\right)+3\right)}{2048\sqrt{\pi}a^3b^3\xi^3 r}$$

$$h_2^{(3)} = -\frac{9r^2}{1024\sqrt{\pi}a^3b^3\xi} - \frac{9r(a+b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{18\xi^2\left(3a^2-2ab+3b^2\right)+9}{2048\sqrt{\pi}a^3b^3\xi^3} \tag{3.168}$$
$$+\frac{9(a+b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)-4\xi^4(a+b)^2\left(a^2-6ab+b^2\right)\right.$$
$$\left.-6\xi^2\left(29a^2-54ab+29b^2\right)-45\right]+\frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)\right.$$
$$\left.-12\xi^4(a+b)^2\left(a^2-6ab+b^2\right)-30\xi^2\left(5a^2-6ab+5b^2\right)+45\right]$$
$$+\frac{9(a+b)\left(-4\xi^2\left(a^2-10ab+b^2\right)-4\xi^4(a-3b)(3a-b)(a+b)^2-21\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3}$$
$$+\frac{9\left(4\xi^2\left(a^2+6ab+b^2\right)-4\xi^4(a-3b)(3a-b)(a+b)^2-9\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2}$$
$$+\frac{9(a+b)\left(2\xi^2\left(3a^2-2ab+3b^2\right)+3\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}$$

$$h_3^{(3)} = \frac{9r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{9r(a-b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{-18\xi^2\left(3a^2+2ab+3b^2\right)-9}{2048\sqrt{\pi}a^3b^3\xi^3} \tag{3.169}$$
$$+\frac{9(a-b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)+4\xi^4(a-b)^2\left(a^2+6ab+b^2\right)\right.$$
$$\left.+6\xi^2\left(29a^2+54ab+29b^2\right)+45\right]+\frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)\right.$$
$$\left.+12\xi^4(a-b)^2\left(a^2+6ab+b^2\right)+30\xi^2\left(5a^2+6ab+5b^2\right)-45\right]$$
$$+\frac{9(a-b)\left(4\xi^2\left(a^2+\xi^2(a-b)^2(3a+b)(a+3b)+10ab+b^2\right)+21\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3}$$
$$+\frac{9\left(-4\xi^2\left(a^2-6ab+b^2\right)+4\xi^4(a-b)^2(3a+b)(a+3b)+9\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2}$$
$$-\frac{9(a-b)\left(2\xi^2\left(3a^2+2ab+3b^2\right)+3\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}$$

$$h_4^{(3)} = \frac{9r^2}{1024\sqrt{\pi}a^3b^3\xi} - \frac{9r(a-b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{-18\xi^2\left(3a^2+2ab+3b^2\right)-9}{2048\sqrt{\pi}a^3b^3\xi^3} \tag{3.170}$$
$$+\frac{9(a-b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)-4\xi^4(a-b)^2\left(a^2+6ab+b^2\right)\right.$$
$$\left.-6\xi^2\left(29a^2+54ab+29b^2\right)-45\right]+\frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)\right.$$
$$\left.+12\xi^4(a-b)^2\left(a^2+6ab+b^2\right)+30\xi^2\left(5a^2+6ab+5b^2\right)-45\right]$$
$$+\frac{9(a-b)\left(-4\xi^2\left(a^2+10ab+b^2\right)-4\xi^4(a-b)^2(3a+b)(a+3b)-21\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3}$$
$$+\frac{9\left(-4\xi^2\left(a^2-6ab+b^2\right)+4\xi^4(a-b)^2(3a+b)(a+3b)+9\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2}$$
$$+\frac{9(a-b)\left(2\xi^2\left(3a^2+2ab+3b^2\right)+3\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}$$

$$h_5^{(3)} = \frac{9r^3}{1024a^3b^3} + \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4 r} - \frac{9r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.171}$$
$$+ \frac{9\left(16\xi^8(a+b)^6\left(a^2-6ab+b^2\right)-360\xi^4\left(a^2-b^2\right)^2-240\xi^2\left(a^2+b^2\right)+45\right)}{16384a^3b^3\xi^8r^5}$$
$$+ \frac{9\left(-4\xi^6(a+b)^4\left(a^2-4ab+b^2\right)+9\xi^2\left(a^2+b^2\right)-3\right)}{1024a^3b^3\xi^6r^3}$$

$$h_6^{(3)} = -\frac{9r^3}{1024a^3b^3} - \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4 r} + \frac{9r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.172}$$
$$+ \frac{9\left(-16\xi^8(a-b)^6\left(a^2+6ab+b^2\right)+360\xi^4\left(a^2-b^2\right)^2+240\xi^2\left(a^2+b^2\right)-45\right)}{16384a^3b^3\xi^8r^5}$$
$$+ \frac{9\left(4\xi^6(a-b)^4\left(a^2+4ab+b^2\right)-9\xi^2\left(a^2+b^2\right)+3\right)}{1024a^3b^3\xi^6r^3}$$

$$h_7^{(3)} = -\frac{9r^3}{1024a^3b^3} - \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4 r} + \frac{9r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.173}$$
$$+ \frac{9\left(-16\xi^8(a-b)^6\left(a^2+6ab+b^2\right)+360\xi^4\left(a^2-b^2\right)^2+240\xi^2\left(a^2+b^2\right)-45\right)}{16384a^3b^3\xi^8r^5}$$
$$+ \frac{9\left(4\xi^6(a-b)^4\left(a^2+4ab+b^2\right)-9\xi^2\left(a^2+b^2\right)+3\right)}{1024a^3b^3\xi^6r^3}$$

$$h_8^{(3)} = \frac{9r^3}{1024a^3b^3} + \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4 r} - \frac{9r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.174}$$
$$+ \frac{9\left(16\xi^8(a+b)^6\left(a^2-6ab+b^2\right)-360\xi^4\left(a^2-b^2\right)^2-240\xi^2\left(a^2+b^2\right)+45\right)}{16384a^3b^3\xi^8r^5}$$
$$+ \frac{9\left(-4\xi^6(a+b)^4\left(a^2-4ab+b^2\right)+9\xi^2\left(a^2+b^2\right)-3\right)}{1024a^3b^3\xi^6r^3}$$

3.3.2. ***Case 2,*** $r < a+b, r > b-a, r > a-b$

First scalar mobility function

$$h_0^{(1)} = -\frac{3r^3}{512a^3b^3} + \frac{3}{20}\frac{-a^3-b^3}{a^3b^3} + \frac{9\left(a^2-6ab+b^2\right)(a+b)^6}{2560a^3b^3r^5} + \frac{27\left(a^2-b^2\right)^2}{256a^3b^3r} \tag{3.175}$$
$$- \frac{3\left(a^2-4ab+b^2\right)(a+b)^4}{128a^3b^3r^3} + \frac{9r\left(a^2+b^2\right)}{128a^3b^3}$$

$$
\begin{aligned}
h_1^{(1)} = &-\frac{3r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{3r(a+b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{6\xi^2\left(11a^2-2ab+11b^2\right)+3}{2048\sqrt{\pi}a^3b^3\xi^3} \\
&+ \frac{9(a+b)}{40960\sqrt{\pi}a^3b^3\xi^7r^5}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)-4\xi^4(a+b)^2\left(a^2-6ab+b^2\right)\right. \\
&\left.-6\xi^2\left(29a^2-54ab+29b^2\right)-45\right] + \frac{9}{40960\sqrt{\pi}a^3b^3\xi^7r^4}\left[-8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)\right. \\
&\left.+12\xi^4(a+b)^2\left(a^2-6ab+b^2\right)+30\xi^2\left(5a^2-6ab+5b^2\right)-45\right] \\
&+ \frac{3(a+b)\left(-4\xi^4(a+b)^2\left(17a^2-62ab+17b^2\right)+4\xi^2\left(a^2+14ab+b^2\right)-15\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^3} \\
&+ \frac{3\left(4\xi^4(a+b)^2\left(17a^2-62ab+17b^2\right)-60\xi^2(a-b)^2+75\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^2} \\
&+ \frac{3(a+b)\left(2\xi^2\left(73a^2-118ab+73b^2\right)-15\right)}{10240\sqrt{\pi}a^3b^3\xi^3r}
\end{aligned}
\tag{3.176}
$$

$$
\begin{aligned}
h_2^{(1)} = &-\frac{3r^2}{1024\sqrt{\pi}a^3b^3\xi} - \frac{3r(a+b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{6\xi^2\left(11a^2-2ab+11b^2\right)+3}{2048\sqrt{\pi}a^3b^3\xi^3} \\
&+ \frac{9(a+b)}{40960\sqrt{\pi}a^3b^3\xi^7r^5}\left[-8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)+4\xi^4(a+b)^2\left(a^2-6ab+b^2\right)\right. \\
&\left.+6\xi^2\left(29a^2-54ab+29b^2\right)+45\right] + \frac{9}{40960\sqrt{\pi}a^3b^3\xi^7r^4}\left[-8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)\right. \\
&\left.+12\xi^4(a+b)^2\left(a^2-6ab+b^2\right)+30\xi^2\left(5a^2-6ab+5b^2\right)-45\right] \\
&+ \frac{3(a+b)\left(4\xi^4(a+b)^2\left(17a^2-62ab+17b^2\right)-4\xi^2\left(a^2+14ab+b^2\right)+15\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^3} \\
&+ \frac{3\left(4\xi^4(a+b)^2\left(17a^2-62ab+17b^2\right)-60\xi^2(a-b)^2+75\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^2} \\
&- \frac{3(a+b)\left(2\xi^2\left(73a^2-118ab+73b^2\right)-15\right)}{10240\sqrt{\pi}a^3b^3\xi^3r}
\end{aligned}
\tag{3.177}
$$

$$
\begin{aligned}
h_3^{(1)} = &\frac{3r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{3r(a-b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{-6\xi^2\left(11a^2+2ab+11b^2\right)-3}{2048\sqrt{\pi}a^3b^3\xi^3} \\
&+ \frac{9(a-b)}{40960\sqrt{\pi}a^3b^3\xi^7r^5}\left[8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)-4\xi^4(a-b)^2\left(a^2+6ab+b^2\right)\right. \\
&\left.-6\xi^2\left(29a^2+54ab+29b^2\right)-45\right] + \frac{9}{40960\sqrt{\pi}a^3b^3\xi^7r^4}\left[8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)\right. \\
&\left.-12\xi^4(a-b)^2\left(a^2+6ab+b^2\right)-30\xi^2\left(5a^2+6ab+5b^2\right)+45\right] \\
&+ \frac{3(a-b)\left(-4\xi^4(a-b)^2\left(17a^2+62ab+17b^2\right)+4\xi^2\left(a^2-14ab+b^2\right)-15\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^3} \\
&+ \frac{3\left(-4\xi^4(a-b)^2\left(17a^2+62ab+17b^2\right)+60\xi^2(a+b)^2-75\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^2} \\
&+ \frac{3(a-b)\left(2\xi^2\left(73a^2+118ab+73b^2\right)-15\right)}{10240\sqrt{\pi}a^3b^3\xi^3r}
\end{aligned}
\tag{3.178}
$$

$$h_4^{(1)} = \frac{3r^2}{1024\sqrt{\pi}a^3b^3\xi} - \frac{3r(a-b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{-6\xi^2\left(11a^2+2ab+11b^2\right)-3}{2048\sqrt{\pi}a^3b^3\xi^3} \tag{3.179}$$
$$+ \frac{9(a-b)}{40960\sqrt{\pi}a^3b^3\xi^7r^5}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)+4\xi^4(a-b)^2\left(a^2+6ab+b^2\right)\right.$$
$$\left.+6\xi^2\left(29a^2+54ab+29b^2\right)+45\right] + \frac{9}{40960\sqrt{\pi}a^3b^3\xi^7r^4}\left[8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)\right.$$
$$\left.-12\xi^4(a-b)^2\left(a^2+6ab+b^2\right)-30\xi^2\left(5a^2+6ab+5b^2\right)+45\right]$$
$$+ \frac{3(a-b)\left(4\xi^4(a-b)^2\left(17a^2+62ab+17b^2\right)-4\xi^2\left(a^2-14ab+b^2\right)+15\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^3}$$
$$+ \frac{3\left(-4\xi^4(a-b)^2\left(17a^2+62ab+17b^2\right)+60\xi^2(a+b)^2-75\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^2}$$
$$- \frac{3(a-b)\left(2\xi^2\left(73a^2+118ab+73b^2\right)-15\right)}{10240\sqrt{\pi}a^3b^3\xi^3r}$$

$$h_5^{(1)} = \frac{3r^3}{1024a^3b^3} + \frac{3}{40}\frac{a^3+b^3}{a^3b^3} - \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4r} - \frac{9r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.180}$$
$$+ \frac{9\left(-16\xi^8(a+b)^6\left(a^2-6ab+b^2\right)+360\xi^4\left(a^2-b^2\right)^2+240\xi^2\left(a^2+b^2\right)-45\right)}{81920a^3b^3\xi^8r^5}$$
$$+ \frac{12\xi^6(a+b)^4\left(a^2-4ab+b^2\right)-27\xi^2\left(a^2+b^2\right)+9}{1024a^3b^3\xi^6r^3}$$

$$h_6^{(1)} = -\frac{3r^3}{1024a^3b^3} + \frac{3}{40}\frac{a^3-b^3}{a^3b^3} + \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4r} + \frac{9r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.181}$$
$$+ \frac{9\left(16\xi^8(a-b)^6\left(a^2+6ab+b^2\right)-360\xi^4\left(a^2-b^2\right)^2-240\xi^2\left(a^2+b^2\right)+45\right)}{81920a^3b^3\xi^8r^5}$$
$$+ \frac{-12\xi^6(a-b)^4\left(a^2+4ab+b^2\right)+27\xi^2\left(a^2+b^2\right)-9}{1024a^3b^3\xi^6r^3}$$

$$h_7^{(1)} = -\frac{3r^3}{1024a^3b^3} + \frac{3}{40}\frac{b^3-a^3}{a^3b^3} + \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4r} + \frac{9r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.182}$$
$$+ \frac{9\left(16\xi^8(a-b)^6\left(a^2+6ab+b^2\right)-360\xi^4\left(a^2-b^2\right)^2-240\xi^2\left(a^2+b^2\right)+45\right)}{81920a^3b^3\xi^8r^5}$$
$$+ \frac{-12\xi^6(a-b)^4\left(a^2+4ab+b^2\right)+27\xi^2\left(a^2+b^2\right)-9}{1024a^3b^3\xi^6r^3}$$

$$h_8^{(1)} = \frac{3r^3}{1024a^3b^3} + \frac{3}{40}\frac{-a^3-b^3}{a^3b^3} - \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4r} - \frac{9r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.183}$$
$$+ \frac{9\left(-16\xi^8(a+b)^6\left(a^2-6ab+b^2\right)+360\xi^4\left(a^2-b^2\right)^2+240\xi^2\left(a^2+b^2\right)-45\right)}{81920a^3b^3\xi^8r^5}$$
$$+ \frac{12\xi^6(a+b)^4\left(a^2-4ab+b^2\right)-27\xi^2\left(a^2+b^2\right)+9}{1024a^3b^3\xi^6r^3}$$

Second scalar mobility function

$$h_0^{(2)} = \frac{63r^3}{512a^3b^3} - \frac{9\left(a^2-6ab+b^2\right)(a+b)^6}{512a^3b^3r^5} - \frac{9\left(a^2-4ab+b^2\right)(a+b)^4}{128a^3b^3r^3} - \frac{45r\left(a^2+b^2\right)}{128a^3b^3} + \frac{81\left(a^2-b^2\right)^2}{256a^3b^3r} \tag{3.184}$$

$$\begin{aligned} h_1^{(2)} = {} & \frac{63r^2}{1024\sqrt{\pi}a^3b^3\xi} - \frac{63r(a+b)}{1024\sqrt{\pi}a^3b^3\xi} - \frac{9\left(2\xi^2\left(13a^2-14ab+13b^2\right)+7\right)}{2048\sqrt{\pi}a^3b^3\xi^3} \\ & + \frac{9(a+b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[-8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)+4\xi^4(a+b)^2\left(a^2-6ab+b^2\right)\right. \\ & \left.+6\xi^2\left(29a^2-54ab+29b^2\right)+45\right] + \frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)\right. \\ & \left.-12\xi^4(a+b)^2\left(a^2-6ab+b^2\right)-30\xi^2\left(5a^2-6ab+5b^2\right)+45\right] \\ & + \frac{9(a+b)\left(-4\xi^4(a+b)^2\left(5a^2-22ab+5b^2\right)+4\xi^2(5a-b)(a-5b)+69\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3} \\ & + \frac{9\left(4\xi^2\left(\xi^2(a+b)^2\left(5a^2-22ab+5b^2\right)-11a^2+54ab-11b^2\right)+39\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2} \\ & + \frac{9(a+b)\left(2\xi^2\left(13a^2-14ab+13b^2\right)+21\right)}{2048\sqrt{\pi}a^3b^3\xi^3r} \end{aligned} \tag{3.185}$$

$$\begin{aligned} h_2^{(2)} = {} & \frac{63r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{63r(a+b)}{1024\sqrt{\pi}a^3b^3\xi} - \frac{9\left(2\xi^2\left(13a^2-14ab+13b^2\right)+7\right)}{2048\sqrt{\pi}a^3b^3\xi^3} \\ & + \frac{9(a+b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)-4\xi^4(a+b)^2\left(a^2-6ab+b^2\right)\right. \\ & \left.-6\xi^2\left(29a^2-54ab+29b^2\right)-45\right] + \frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)\right. \\ & \left.-12\xi^4(a+b)^2\left(a^2-6ab+b^2\right)-30\xi^2\left(5a^2-6ab+5b^2\right)+45\right] \\ & + \frac{9(a+b)\left(4\xi^4(a+b)^2\left(5a^2-22ab+5b^2\right)-4\xi^2(a-5b)(5a-b)-69\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3} \\ & + \frac{9\left(4\xi^2\left(\xi^2(a+b)^2\left(5a^2-22ab+5b^2\right)-11a^2+54ab-11b^2\right)+39\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2} \\ & - \frac{9(a+b)\left(2\xi^2\left(13a^2-14ab+13b^2\right)+21\right)}{2048\sqrt{\pi}a^3b^3\xi^3r} \end{aligned} \tag{3.186}$$

$$h_3^{(2)} = -\frac{63r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{63r(b-a)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{18\xi^2\left(13a^2+14ab+13b^2\right)+63}{2048\sqrt{\pi}a^3b^3\xi^3} \tag{3.187}$$
$$+\frac{9(a-b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)+4\xi^4(a-b)^2\left(a^2+6ab+b^2\right)\right.$$
$$\left.+6\xi^2\left(29a^2+54ab+29b^2\right)+45\right]+\frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)\right.$$
$$\left.+12\xi^4(a-b)^2\left(a^2+6ab+b^2\right)+30\xi^2\left(5a^2+6ab+5b^2\right)-45\right]$$
$$+\frac{9(a-b)\left(-4\xi^4(a-b)^2\left(5a^2+22ab+5b^2\right)+4\xi^2(5a+b)(a+5b)+69\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3}$$
$$+\frac{9\left(-4\xi^4(a-b)^2\left(5a^2+22ab+5b^2\right)+4\xi^2\left(11a^2+54ab+11b^2\right)-39\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2}$$
$$+\frac{9(a-b)\left(2\xi^2\left(13a^2+14ab+13b^2\right)+21\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}$$

$$h_4^{(2)} = -\frac{63r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{63r(a-b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{18\xi^2\left(13a^2+14ab+13b^2\right)+63}{2048\sqrt{\pi}a^3b^3\xi^3} \tag{3.188}$$
$$+\frac{9(a-b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)-4\xi^4(a-b)^2\left(a^2+6ab+b^2\right)\right.$$
$$\left.-6\xi^2\left(29a^2+54ab+29b^2\right)-45\right]+\frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)\right.$$
$$\left.+12\xi^4(a-b)^2\left(a^2+6ab+b^2\right)+30\xi^2\left(5a^2+6ab+5b^2\right)-45\right]$$
$$+\frac{9(a-b)\left(4\xi^4(a-b)^2\left(5a^2+22ab+5b^2\right)-4\xi^2(5a+b)(a+5b)-69\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3}$$
$$+\frac{9\left(-4\xi^4(a-b)^2\left(5a^2+22ab+5b^2\right)+4\xi^2\left(11a^2+54ab+11b^2\right)-39\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2}$$
$$-\frac{9(a-b)\left(2\xi^2\left(13a^2+14ab+13b^2\right)+21\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}$$

$$h_5^{(2)} = -\frac{63r^3}{1024a^3b^3} - \frac{324\xi^4\left(a^2-b^2\right)^2+81}{2048a^3b^3\xi^4r} + \frac{45r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.189}$$
$$+\frac{9\left(16\xi^8(a+b)^6\left(a^2-6ab+b^2\right)-360\xi^4\left(a^2-b^2\right)^2-240\xi^2\left(a^2+b^2\right)+45\right)}{16384a^3b^3\xi^8r^5}$$
$$+\frac{9\left(4\xi^6(a+b)^4\left(a^2-4ab+b^2\right)-9\xi^2\left(a^2+b^2\right)+3\right)}{1024a^3b^3\xi^6r^3}$$

$$h_6^{(2)} = \frac{63r^3}{1024a^3b^3} + \frac{324\xi^4\left(a^2-b^2\right)^2+81}{2048a^3b^3\xi^4r} - \frac{45r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.190}$$
$$+\frac{9\left(-16\xi^8(a-b)^6\left(a^2+6ab+b^2\right)+360\xi^4\left(a^2-b^2\right)^2+240\xi^2\left(a^2+b^2\right)-45\right)}{16384a^3b^3\xi^8r^5}$$
$$+\frac{9\left(-4\xi^6(a-b)^4\left(a^2+4ab+b^2\right)+9\xi^2\left(a^2+b^2\right)-3\right)}{1024a^3b^3\xi^6r^3}$$

$$h_7^{(2)} = \frac{63r^3}{1024a^3b^3} + \frac{324\xi^4\left(a^2-b^2\right)^2+81}{2048a^3b^3\xi^4 r} - \frac{45r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.191}$$
$$+ \frac{9\left(-16\xi^8(a-b)^6\left(a^2+6ab+b^2\right)+360\xi^4\left(a^2-b^2\right)^2+240\xi^2\left(a^2+b^2\right)-45\right)}{16384a^3b^3\xi^8r^5}$$
$$+ \frac{9\left(-4\xi^6(a-b)^4\left(a^2+4ab+b^2\right)+9\xi^2\left(a^2+b^2\right)-3\right)}{1024a^3b^3\xi^6r^3}$$

$$h_8^{(2)} = -\frac{63r^3}{1024a^3b^3} - \frac{324\xi^4\left(a^2-b^2\right)^2+81}{2048a^3b^3\xi^4 r} + \frac{45r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.192}$$
$$+ \frac{9\left(16\xi^8(a+b)^6\left(a^2-6ab+b^2\right)-360\xi^4\left(a^2-b^2\right)^2-240\xi^2\left(a^2+b^2\right)+45\right)}{16384a^3b^3\xi^8r^5}$$
$$+ \frac{9\left(4\xi^6(a+b)^4\left(a^2-4ab+b^2\right)-9\xi^2\left(a^2+b^2\right)+3\right)}{1024a^3b^3\xi^6r^3}$$

Third scalar mobility function

$$h_0^{(3)} = -\frac{9r^3}{512a^3b^3} - \frac{9\left(a^2-6ab+b^2\right)(a+b)^6}{512a^3b^3r^5} + \frac{9\left(a^2-4ab+b^2\right)(a+b)^4}{128a^3b^3r^3} \tag{3.193}$$
$$+ \frac{9r\left(a^2+b^2\right)}{128a^3b^3} - \frac{27\left(a^2-b^2\right)^2}{256a^3b^3r}$$

$$h_1^{(3)} = -\frac{9r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{9r(a+b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{18\xi^2\left(3a^2-2ab+3b^2\right)+9}{2048\sqrt{\pi}a^3b^3\xi^3} \tag{3.194}$$
$$+ \frac{9(a+b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[-8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)+4\xi^4(a+b)^2\left(a^2-6ab+b^2\right)\right.$$
$$\left.+6\xi^2\left(29a^2-54ab+29b^2\right)+45\right] + \frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)\right.$$
$$\left.-12\xi^4(a+b)^2\left(a^2-6ab+b^2\right)-30\xi^2\left(5a^2-6ab+5b^2\right)+45\right]$$
$$+ \frac{9(a+b)\left(4\xi^2\left(a^2+\xi^2(3a-b)(a+b)^2(a-3b)-10ab+b^2\right)+21\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3}$$
$$+ \frac{9\left(4\xi^2\left(a^2+6ab+b^2\right)-4\xi^4(a-3b)(3a-b)(a+b)^2-9\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2}$$
$$- \frac{9(a+b)\left(2\xi^2\left(3a^2-2ab+3b^2\right)+3\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}$$

$$
\begin{aligned}
h_2^{(3)} = & -\frac{9r^2}{1024\sqrt{\pi}a^3b^3\xi} - \frac{9r(a+b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{18\xi^2\left(3a^2-2ab+3b^2\right)+9}{2048\sqrt{\pi}a^3b^3\xi^3} \\
& + \frac{9(a+b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)-4\xi^4(a+b)^2\left(a^2-6ab+b^2\right)\right. \\
& \left.-6\xi^2\left(29a^2-54ab+29b^2\right)-45\right] + \frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)\right. \\
& \left.-12\xi^4(a+b)^2\left(a^2-6ab+b^2\right)-30\xi^2\left(5a^2-6ab+5b^2\right)+45\right] \\
& + \frac{9(a+b)\left(-4\xi^2\left(a^2-10ab+b^2\right)-4\xi^4(a-3b)(3a-b)(a+b)^2-21\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3} \\
& + \frac{9\left(4\xi^2\left(a^2+6ab+b^2\right)-4\xi^4(a-3b)(3a-b)(a+b)^2-9\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2} \\
& + \frac{9(a+b)\left(2\xi^2\left(3a^2-2ab+3b^2\right)+3\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}
\end{aligned}
\tag{3.195}
$$

$$
\begin{aligned}
h_3^{(3)} = & \frac{9r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{9r(a-b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{-18\xi^2\left(3a^2+2ab+3b^2\right)-9}{2048\sqrt{\pi}a^3b^3\xi^3} \\
& + \frac{9(a-b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)+4\xi^4(a-b)^2\left(a^2+6ab+b^2\right)\right. \\
& \left.+6\xi^2\left(29a^2+54ab+29b^2\right)+45\right] + \frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)\right. \\
& \left.+12\xi^4(a-b)^2\left(a^2+6ab+b^2\right)+30\xi^2\left(5a^2+6ab+5b^2\right)-45\right] \\
& + \frac{9(a-b)\left(4\xi^2\left(a^2+\xi^2(a-b)^2(3a+b)(a+3b)+10ab+b^2\right)+21\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3} \\
& + \frac{9\left(-4\xi^2\left(a^2-6ab+b^2\right)+4\xi^4(a-b)^2(3a+b)(a+3b)+9\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2} \\
& - \frac{9(a-b)\left(2\xi^2\left(3a^2+2ab+3b^2\right)+3\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}
\end{aligned}
\tag{3.196}
$$

$$
\begin{aligned}
h_4^{(3)} = & \frac{9r^2}{1024\sqrt{\pi}a^3b^3\xi} - \frac{9r(a-b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{-18\xi^2\left(3a^2+2ab+3b^2\right)-9}{2048\sqrt{\pi}a^3b^3\xi^3} \\
& + \frac{9(a-b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)-4\xi^4(a-b)^2\left(a^2+6ab+b^2\right)\right. \\
& \left.-6\xi^2\left(29a^2+54ab+29b^2\right)-45\right] + \frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)\right. \\
& \left.+12\xi^4(a-b)^2\left(a^2+6ab+b^2\right)+30\xi^2\left(5a^2+6ab+5b^2\right)-45\right] \\
& + \frac{9(a-b)\left(-4\xi^2\left(a^2+10ab+b^2\right)-4\xi^4(a-b)^2(3a+b)(a+3b)-21\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3} \\
& + \frac{9\left(-4\xi^2\left(a^2-6ab+b^2\right)+4\xi^4(a-b)^2(3a+b)(a+3b)+9\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2} \\
& + \frac{9(a-b)\left(2\xi^2\left(3a^2+2ab+3b^2\right)+3\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}
\end{aligned}
\tag{3.197}
$$

$$h_5^{(3)} = \frac{9r^3}{1024a^3b^3} + \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4 r} - \frac{9r\left(a^2+b^2\right)}{256a^3b^3} + \frac{9\left(16\xi^8(a+b)^6\left(a^2-6ab+b^2\right)-360\xi^4\left(a^2-b^2\right)^2-240\xi^2\left(a^2+b^2\right)+45\right)}{16384a^3b^3\xi^8r^5} + \frac{9\left(-4\xi^6(a+b)^4\left(a^2-4ab+b^2\right)+9\xi^2\left(a^2+b^2\right)-3\right)}{1024a^3b^3\xi^6r^3} \tag{3.198}$$

$$h_6^{(3)} = -\frac{9r^3}{1024a^3b^3} - \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4 r} + \frac{9r\left(a^2+b^2\right)}{256a^3b^3} + \frac{9\left(-16\xi^8(a-b)^6\left(a^2+6ab+b^2\right)+360\xi^4\left(a^2-b^2\right)^2+240\xi^2\left(a^2+b^2\right)-45\right)}{16384a^3b^3\xi^8r^5} + \frac{9\left(4\xi^6(a-b)^4\left(a^2+4ab+b^2\right)-9\xi^2\left(a^2+b^2\right)+3\right)}{1024a^3b^3\xi^6r^3} \tag{3.199}$$

$$h_7^{(3)} = -\frac{9r^3}{1024a^3b^3} - \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4 r} + \frac{9r\left(a^2+b^2\right)}{256a^3b^3} + \frac{9\left(-16\xi^8(a-b)^6\left(a^2+6ab+b^2\right)+360\xi^4\left(a^2-b^2\right)^2+240\xi^2\left(a^2+b^2\right)-45\right)}{16384a^3b^3\xi^8r^5} + \frac{9\left(4\xi^6(a-b)^4\left(a^2+4ab+b^2\right)-9\xi^2\left(a^2+b^2\right)+3\right)}{1024a^3b^3\xi^6r^3} \tag{3.200}$$

$$h_8^{(3)} = \frac{9r^3}{1024a^3b^3} + \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4 r} - \frac{9r\left(a^2+b^2\right)}{256a^3b^3} + \frac{9\left(16\xi^8(a+b)^6\left(a^2-6ab+b^2\right)-360\xi^4\left(a^2-b^2\right)^2-240\xi^2\left(a^2+b^2\right)+45\right)}{16384a^3b^3\xi^8r^5} + \frac{9\left(-4\xi^6(a+b)^4\left(a^2-4ab+b^2\right)+9\xi^2\left(a^2+b^2\right)-3\right)}{1024a^3b^3\xi^6r^3} \tag{3.201}$$

3.3.3. ***Case 3***, *particle α inside particle β, $r < a+b, r < b-a, r > a-b$*

First scalar mobility function

$$h_0^{(1)} = -\frac{9\left(a^2+b^2\right)}{20r^5} - \frac{3}{10b^3} + \frac{3}{4r^3} \tag{3.202}$$

$$
\begin{aligned}
h_1^{(1)} =& -\frac{3r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{3r(a+b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{6\xi^2\left(11a^2-2ab+11b^2\right)+3}{2048\sqrt{\pi}a^3b^3\xi^3} \\
&+ \frac{9(a+b)}{40960\sqrt{\pi}a^3b^3\xi^7r^5}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)-4\xi^4(a+b)^2\left(a^2-6ab+b^2\right)\right. \\
&\left.-6\xi^2\left(29a^2-54ab+29b^2\right)-45\right] + \frac{9}{40960\sqrt{\pi}a^3b^3\xi^7r^4}\left[-8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)\right. \\
&\left.+12\xi^4(a+b)^2\left(a^2-6ab+b^2\right)+30\xi^2\left(5a^2-6ab+5b^2\right)-45\right] \\
&+ \frac{3(a+b)\left(-4\xi^4(a+b)^2\left(17a^2-62ab+17b^2\right)+4\xi^2\left(a^2+14ab+b^2\right)-15\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^3} \\
&+ \frac{3\left(4\xi^4(a+b)^2\left(17a^2-62ab+17b^2\right)-60\xi^2(a-b)^2+75\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^2} \\
&+ \frac{3(a+b)\left(2\xi^2\left(73a^2-118ab+73b^2\right)-15\right)}{10240\sqrt{\pi}a^3b^3\xi^3r}
\end{aligned} \tag{3.203}
$$

$$
\begin{aligned}
h_2^{(1)} =& -\frac{3r^2}{1024\sqrt{\pi}a^3b^3\xi} - \frac{3r(a+b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{6\xi^2\left(11a^2-2ab+11b^2\right)+3}{2048\sqrt{\pi}a^3b^3\xi^3} \\
&+ \frac{9(a+b)}{40960\sqrt{\pi}a^3b^3\xi^7r^5}\left[-8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)+4\xi^4(a+b)^2\left(a^2-6ab+b^2\right)\right. \\
&\left.+6\xi^2\left(29a^2-54ab+29b^2\right)+45\right] + \frac{9}{40960\sqrt{\pi}a^3b^3\xi^7r^4}\left[-8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)\right. \\
&\left.+12\xi^4(a+b)^2\left(a^2-6ab+b^2\right)+30\xi^2\left(5a^2-6ab+5b^2\right)-45\right] \\
&+ \frac{3(a+b)\left(4\xi^4(a+b)^2\left(17a^2-62ab+17b^2\right)-4\xi^2\left(a^2+14ab+b^2\right)+15\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^3} \\
&+ \frac{3\left(4\xi^4(a+b)^2\left(17a^2-62ab+17b^2\right)-60\xi^2(a-b)^2+75\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^2} \\
&- \frac{3(a+b)\left(2\xi^2\left(73a^2-118ab+73b^2\right)-15\right)}{10240\sqrt{\pi}a^3b^3\xi^3r}
\end{aligned} \tag{3.204}
$$

$$
\begin{aligned}
h_3^{(1)} =& \frac{3r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{3r(a-b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{-6\xi^2\left(11a^2+2ab+11b^2\right)-3}{2048\sqrt{\pi}a^3b^3\xi^3} \\
&+ \frac{9(a-b)}{40960\sqrt{\pi}a^3b^3\xi^7r^5}\left[8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)-4\xi^4(a-b)^2\left(a^2+6ab+b^2\right)\right. \\
&\left.-6\xi^2\left(29a^2+54ab+29b^2\right)-45\right] + \frac{9}{40960\sqrt{\pi}a^3b^3\xi^7r^4}\left[8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)\right. \\
&\left.-12\xi^4(a-b)^2\left(a^2+6ab+b^2\right)-30\xi^2\left(5a^2+6ab+5b^2\right)+45\right] \\
&+ \frac{3(a-b)\left(-4\xi^4(a-b)^2\left(17a^2+62ab+17b^2\right)+4\xi^2\left(a^2-14ab+b^2\right)-15\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^3} \\
&+ \frac{3\left(-4\xi^4(a-b)^2\left(17a^2+62ab+17b^2\right)+60\xi^2(a+b)^2-75\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^2} \\
&+ \frac{3(a-b)\left(2\xi^2\left(73a^2+118ab+73b^2\right)-15\right)}{10240\sqrt{\pi}a^3b^3\xi^3r}
\end{aligned} \tag{3.205}
$$

$$h_4^{(1)} = \frac{3r^2}{1024\sqrt{\pi}a^3b^3\xi} - \frac{3r(a-b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{-6\xi^2\left(11a^2+2ab+11b^2\right)-3}{2048\sqrt{\pi}a^3b^3\xi^3} \tag{3.206}$$
$$+ \frac{9(a-b)}{40960\sqrt{\pi}a^3b^3\xi^7r^5}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)+4\xi^4(a-b)^2\left(a^2+6ab+b^2\right)\right.$$
$$\left.+6\xi^2\left(29a^2+54ab+29b^2\right)+45\right] + \frac{9}{40960\sqrt{\pi}a^3b^3\xi^7r^4}\left[8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)\right.$$
$$\left.-12\xi^4(a-b)^2\left(a^2+6ab+b^2\right)-30\xi^2\left(5a^2+6ab+5b^2\right)+45\right]$$
$$+ \frac{3(a-b)\left(4\xi^4(a-b)^2\left(17a^2+62ab+17b^2\right)-4\xi^2\left(a^2-14ab+b^2\right)+15\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^3}$$
$$+ \frac{3\left(-4\xi^4(a-b)^2\left(17a^2+62ab+17b^2\right)+60\xi^2(a+b)^2-75\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^2}$$
$$- \frac{3(a-b)\left(2\xi^2\left(73a^2+118ab+73b^2\right)-15\right)}{10240\sqrt{\pi}a^3b^3\xi^3r}$$

$$h_5^{(1)} = \frac{3r^3}{1024a^3b^3} + \frac{3}{40}\frac{a^3+b^3}{a^3b^3} - \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4r} - \frac{9r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.207}$$
$$+ \frac{9\left(-16\xi^8(a+b)^6\left(a^2-6ab+b^2\right)+360\xi^4\left(a^2-b^2\right)^2+240\xi^2\left(a^2+b^2\right)-45\right)}{81920a^3b^3\xi^8r^5}$$
$$+ \frac{12\xi^6(a+b)^4\left(a^2-4ab+b^2\right)-27\xi^2\left(a^2+b^2\right)+9}{1024a^3b^3\xi^6r^3}$$

$$h_6^{(1)} = -\frac{3r^3}{1024a^3b^3} + \frac{3}{40}\frac{a^3-b^3}{a^3b^3} + \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4r} + \frac{9r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.208}$$
$$+ \frac{9\left(16\xi^8(a-b)^6\left(a^2+6ab+b^2\right)-360\xi^4\left(a^2-b^2\right)^2-240\xi^2\left(a^2+b^2\right)+45\right)}{81920a^3b^3\xi^8r^5}$$
$$+ \frac{-12\xi^6(a-b)^4\left(a^2+4ab+b^2\right)+27\xi^2\left(a^2+b^2\right)-9}{1024a^3b^3\xi^6r^3}$$

$$h_7^{(1)} = -\frac{3r^3}{1024a^3b^3} + \frac{3}{40}\frac{b^3-a^3}{a^3b^3} + \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4r} + \frac{9r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.209}$$
$$+ \frac{9\left(16\xi^8(a-b)^6\left(a^2+6ab+b^2\right)-360\xi^4\left(a^2-b^2\right)^2-240\xi^2\left(a^2+b^2\right)+45\right)}{81920a^3b^3\xi^8r^5}$$
$$+ \frac{-12\xi^6(a-b)^4\left(a^2+4ab+b^2\right)+27\xi^2\left(a^2+b^2\right)-9}{1024a^3b^3\xi^6r^3}$$

$$h_8^{(1)} = \frac{3r^3}{1024a^3b^3} + \frac{3}{40}\frac{-a^3-b^3}{a^3b^3} - \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4r} - \frac{9r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.210}$$
$$+ \frac{9\left(-16\xi^8(a+b)^6\left(a^2-6ab+b^2\right)+360\xi^4\left(a^2-b^2\right)^2+240\xi^2\left(a^2+b^2\right)-45\right)}{81920a^3b^3\xi^8r^5}$$
$$+ \frac{12\xi^6(a+b)^4\left(a^2-4ab+b^2\right)-27\xi^2\left(a^2+b^2\right)+9}{1024a^3b^3\xi^6r^3}$$

Second scalar mobility function

$$h_0^{(2)} = \frac{9\left(a^2+b^2\right)}{4r^5} + \frac{9}{4r^3} \tag{3.211}$$

$$\begin{aligned} h_1^{(2)} &= \frac{63r^2}{1024\sqrt{\pi}a^3b^3\xi} - \frac{63r(a+b)}{1024\sqrt{\pi}a^3b^3\xi} - \frac{9\left(2\xi^2\left(13a^2-14ab+13b^2\right)+7\right)}{2048\sqrt{\pi}a^3b^3\xi^3} \\ &+ \frac{9(a+b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[-8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)+4\xi^4(a+b)^2\left(a^2-6ab+b^2\right)\right. \\ &\left.+6\xi^2\left(29a^2-54ab+29b^2\right)+45\right] + \frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)\right. \\ &\left.-12\xi^4(a+b)^2\left(a^2-6ab+b^2\right)-30\xi^2\left(5a^2-6ab+5b^2\right)+45\right] \\ &+ \frac{9(a+b)\left(-4\xi^4(a+b)^2\left(5a^2-22ab+5b^2\right)+4\xi^2(5a-b)(a-5b)+69\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3} \\ &+ \frac{9\left(4\xi^2\left(\xi^2(a+b)^2\left(5a^2-22ab+5b^2\right)-11a^2+54ab-11b^2\right)+39\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2} \\ &+ \frac{9(a+b)\left(2\xi^2\left(13a^2-14ab+13b^2\right)+21\right)}{2048\sqrt{\pi}a^3b^3\xi^3r} \end{aligned} \tag{3.212}$$

$$\begin{aligned} h_2^{(2)} &= \frac{63r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{63r(a+b)}{1024\sqrt{\pi}a^3b^3\xi} - \frac{9\left(2\xi^2\left(13a^2-14ab+13b^2\right)+7\right)}{2048\sqrt{\pi}a^3b^3\xi^3} \\ &+ \frac{9(a+b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)-4\xi^4(a+b)^2\left(a^2-6ab+b^2\right)\right. \\ &\left.-6\xi^2\left(29a^2-54ab+29b^2\right)-45\right] + \frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)\right. \\ &\left.-12\xi^4(a+b)^2\left(a^2-6ab+b^2\right)-30\xi^2\left(5a^2-6ab+5b^2\right)+45\right] \\ &+ \frac{9(a+b)\left(4\xi^4(a+b)^2\left(5a^2-22ab+5b^2\right)-4\xi^2(a-5b)(5a-b)-69\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3} \\ &+ \frac{9\left(4\xi^2\left(\xi^2(a+b)^2\left(5a^2-22ab+5b^2\right)-11a^2+54ab-11b^2\right)+39\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2} \\ &- \frac{9(a+b)\left(2\xi^2\left(13a^2-14ab+13b^2\right)+21\right)}{2048\sqrt{\pi}a^3b^3\xi^3r} \end{aligned} \tag{3.213}$$

$$h_3^{(2)} = -\frac{63r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{63r(b-a)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{18\xi^2\left(13a^2+14ab+13b^2\right)+63}{2048\sqrt{\pi}a^3b^3\xi^3} \tag{3.214}$$
$$+ \frac{9(a-b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)+4\xi^4(a-b)^2\left(a^2+6ab+b^2\right)\right.$$
$$\left.+6\xi^2\left(29a^2+54ab+29b^2\right)+45\right] + \frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)\right.$$
$$\left.+12\xi^4(a-b)^2\left(a^2+6ab+b^2\right)+30\xi^2\left(5a^2+6ab+5b^2\right)-45\right]$$
$$+ \frac{9(a-b)\left(-4\xi^4(a-b)^2\left(5a^2+22ab+5b^2\right)+4\xi^2(5a+b)(a+5b)+69\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3}$$
$$+ \frac{9\left(-4\xi^4(a-b)^2\left(5a^2+22ab+5b^2\right)+4\xi^2\left(11a^2+54ab+11b^2\right)-39\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2}$$
$$+ \frac{9(a-b)\left(2\xi^2\left(13a^2+14ab+13b^2\right)+21\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}$$

$$h_4^{(2)} = -\frac{63r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{63r(a-b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{18\xi^2\left(13a^2+14ab+13b^2\right)+63}{2048\sqrt{\pi}a^3b^3\xi^3} \tag{3.215}$$
$$+ \frac{9(a-b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)-4\xi^4(a-b)^2\left(a^2+6ab+b^2\right)\right.$$
$$\left.-6\xi^2\left(29a^2+54ab+29b^2\right)-45\right] + \frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)\right.$$
$$\left.+12\xi^4(a-b)^2\left(a^2+6ab+b^2\right)+30\xi^2\left(5a^2+6ab+5b^2\right)-45\right]$$
$$+ \frac{9(a-b)\left(4\xi^4(a-b)^2\left(5a^2+22ab+5b^2\right)-4\xi^2(5a+b)(a+5b)-69\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3}$$
$$+ \frac{9\left(-4\xi^4(a-b)^2\left(5a^2+22ab+5b^2\right)+4\xi^2\left(11a^2+54ab+11b^2\right)-39\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2}$$
$$- \frac{9(a-b)\left(2\xi^2\left(13a^2+14ab+13b^2\right)+21\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}$$

$$h_5^{(2)} = -\frac{63r^3}{1024a^3b^3} + \frac{9\left(16\xi^8(a+b)^6\left(a^2-6ab+b^2\right)-360\xi^4\left(a^2-b^2\right)^2-240\xi^2\left(a^2+b^2\right)+45\right)}{16384a^3b^3\xi^8r^5} \tag{3.216}$$
$$+ \frac{9\left(4\xi^6(a+b)^4\left(a^2-4ab+b^2\right)-9\xi^2\left(a^2+b^2\right)+3\right)}{1024a^3b^3\xi^6r^3} - \frac{324\xi^4\left(a^2-b^2\right)^2+81}{2048a^3b^3\xi^4r} + \frac{45r\left(a^2+b^2\right)}{256a^3b^3}$$

$$h_6^{(2)} = \frac{63r^3}{1024a^3b^3} + \frac{324\xi^4\left(a^2-b^2\right)^2+81}{2048a^3b^3\xi^4r} - \frac{45r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.217}$$
$$+ \frac{9\left(-16\xi^8(a-b)^6\left(a^2+6ab+b^2\right)+360\xi^4\left(a^2-b^2\right)^2+240\xi^2\left(a^2+b^2\right)-45\right)}{16384a^3b^3\xi^8r^5}$$
$$+ \frac{9\left(-4\xi^6(a-b)^4\left(a^2+4ab+b^2\right)+9\xi^2\left(a^2+b^2\right)-3\right)}{1024a^3b^3\xi^6r^3}$$

$$h_7^{(2)} = \frac{63r^3}{1024a^3b^3} + \frac{324\xi^4\left(a^2-b^2\right)^2+81}{2048a^3b^3\xi^4 r} - \frac{45r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.218}$$
$$+ \frac{9\left(-16\xi^8(a-b)^6\left(a^2+6ab+b^2\right)+360\xi^4\left(a^2-b^2\right)^2+240\xi^2\left(a^2+b^2\right)-45\right)}{16384a^3b^3\xi^8r^5}$$
$$+ \frac{9\left(-4\xi^6(a-b)^4\left(a^2+4ab+b^2\right)+9\xi^2\left(a^2+b^2\right)-3\right)}{1024a^3b^3\xi^6r^3}$$

$$h_8^{(2)} = -\frac{63r^3}{1024a^3b^3} - \frac{324\xi^4\left(a^2-b^2\right)^2+81}{2048a^3b^3\xi^4 r} + \frac{45r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.219}$$
$$+ \frac{9\left(16\xi^8(a+b)^6\left(a^2-6ab+b^2\right)-360\xi^4\left(a^2-b^2\right)^2-240\xi^2\left(a^2+b^2\right)+45\right)}{16384a^3b^3\xi^8r^5}$$
$$+ \frac{9\left(4\xi^6(a+b)^4\left(a^2-4ab+b^2\right)-9\xi^2\left(a^2+b^2\right)+3\right)}{1024a^3b^3\xi^6r^3}$$

Third scalar mobility function

$$h_0^{(3)} = \frac{9\left(a^2+b^2\right)}{4r^5} - \frac{9}{4r^3} \tag{3.220}$$

$$h_1^{(3)} = -\frac{9r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{9r(a+b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{18\xi^2\left(3a^2-2ab+3b^2\right)+9}{2048\sqrt{\pi}a^3b^3\xi^3} \tag{3.221}$$
$$+ \frac{9(a+b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[-8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)+4\xi^4(a+b)^2\left(a^2-6ab+b^2\right)\right.$$
$$\left.+6\xi^2\left(29a^2-54ab+29b^2\right)+45\right] + \frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)\right.$$
$$\left.-12\xi^4(a+b)^2\left(a^2-6ab+b^2\right)-30\xi^2\left(5a^2-6ab+5b^2\right)+45\right]$$
$$+ \frac{9(a+b)\left(4\xi^2\left(a^2+\xi^2(3a-b)(a+b)^2(a-3b)-10ab+b^2\right)+21\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3}$$
$$+ \frac{9\left(4\xi^2\left(a^2+6ab+b^2\right)-4\xi^4(a-3b)(3a-b)(a+b)^2-9\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2}$$
$$- \frac{9(a+b)\left(2\xi^2\left(3a^2-2ab+3b^2\right)+3\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}$$

$$
\begin{aligned}
h_2^{(3)} = & -\frac{9r^2}{1024\sqrt{\pi}a^3b^3\xi} - \frac{9r(a+b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{18\xi^2\left(3a^2-2ab+3b^2\right)+9}{2048\sqrt{\pi}a^3b^3\xi^3} \\
& + \frac{9(a+b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)-4\xi^4(a+b)^2\left(a^2-6ab+b^2\right)\right. \\
& \left.-6\xi^2\left(29a^2-54ab+29b^2\right)-45\right] + \frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)\right. \\
& \left.-12\xi^4(a+b)^2\left(a^2-6ab+b^2\right)-30\xi^2\left(5a^2-6ab+5b^2\right)+45\right] \\
& + \frac{9(a+b)\left(-4\xi^2\left(a^2-10ab+b^2\right)-4\xi^4(a-3b)(3a-b)(a+b)^2-21\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3} \\
& + \frac{9\left(4\xi^2\left(a^2+6ab+b^2\right)-4\xi^4(a-3b)(3a-b)(a+b)^2-9\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2} \\
& + \frac{9(a+b)\left(2\xi^2\left(3a^2-2ab+3b^2\right)+3\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}
\end{aligned} \tag{3.222}
$$

$$
\begin{aligned}
h_3^{(3)} = & \frac{9r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{9r(a-b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{-18\xi^2\left(3a^2+2ab+3b^2\right)-9}{2048\sqrt{\pi}a^3b^3\xi^3} \\
& + \frac{9(a-b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)+4\xi^4(a-b)^2\left(a^2+6ab+b^2\right)\right. \\
& \left.+6\xi^2\left(29a^2+54ab+29b^2\right)+45\right] + \frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)\right. \\
& \left.+12\xi^4(a-b)^2\left(a^2+6ab+b^2\right)+30\xi^2\left(5a^2+6ab+5b^2\right)-45\right] \\
& + \frac{9(a-b)\left(4\xi^2\left(a^2+\xi^2(a-b)^2(3a+b)(a+3b)+10ab+b^2\right)+21\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3} \\
& + \frac{9\left(-4\xi^2\left(a^2-6ab+b^2\right)+4\xi^4(a-b)^2(3a+b)(a+3b)+9\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2} \\
& - \frac{9(a-b)\left(2\xi^2\left(3a^2+2ab+3b^2\right)+3\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}
\end{aligned} \tag{3.223}
$$

$$
\begin{aligned}
h_4^{(3)} = & \frac{9r^2}{1024\sqrt{\pi}a^3b^3\xi} - \frac{9r(a-b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{-18\xi^2\left(3a^2+2ab+3b^2\right)-9}{2048\sqrt{\pi}a^3b^3\xi^3} \\
& + \frac{9(a-b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)-4\xi^4(a-b)^2\left(a^2+6ab+b^2\right)\right. \\
& \left.-6\xi^2\left(29a^2+54ab+29b^2\right)-45\right] + \frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)\right. \\
& \left.+12\xi^4(a-b)^2\left(a^2+6ab+b^2\right)+30\xi^2\left(5a^2+6ab+5b^2\right)-45\right] \\
& + \frac{9(a-b)\left(-4\xi^2\left(a^2+10ab+b^2\right)-4\xi^4(a-b)^2(3a+b)(a+3b)-21\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3} \\
& + \frac{9\left(-4\xi^2\left(a^2-6ab+b^2\right)+4\xi^4(a-b)^2(3a+b)(a+3b)+9\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2} \\
& + \frac{9(a-b)\left(2\xi^2\left(3a^2+2ab+3b^2\right)+3\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}
\end{aligned} \tag{3.224}
$$

$$h_5^{(3)} = \frac{9r^3}{1024a^3b^3} + \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4 r} - \frac{9r\left(a^2+b^2\right)}{256a^3b^3} + \frac{9\left(16\xi^8(a+b)^6\left(a^2-6ab+b^2\right)-360\xi^4\left(a^2-b^2\right)^2-240\xi^2\left(a^2+b^2\right)+45\right)}{16384a^3b^3\xi^8 r^5} + \frac{9\left(-4\xi^6(a+b)^4\left(a^2-4ab+b^2\right)+9\xi^2\left(a^2+b^2\right)-3\right)}{1024a^3b^3\xi^6 r^3} \tag{3.225}$$

$$h_6^{(3)} = -\frac{9r^3}{1024a^3b^3} - \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4 r} + \frac{9r\left(a^2+b^2\right)}{256a^3b^3} + \frac{9\left(-16\xi^8(a-b)^6\left(a^2+6ab+b^2\right)+360\xi^4\left(a^2-b^2\right)^2+240\xi^2\left(a^2+b^2\right)-45\right)}{16384a^3b^3\xi^8 r^5} + \frac{9\left(4\xi^6(a-b)^4\left(a^2+4ab+b^2\right)-9\xi^2\left(a^2+b^2\right)+3\right)}{1024a^3b^3\xi^6 r^3} \tag{3.226}$$

$$h_7^{(3)} = -\frac{9r^3}{1024a^3b^3} - \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4 r} + \frac{9r\left(a^2+b^2\right)}{256a^3b^3} + \frac{9\left(-16\xi^8(a-b)^6\left(a^2+6ab+b^2\right)+360\xi^4\left(a^2-b^2\right)^2+240\xi^2\left(a^2+b^2\right)-45\right)}{16384a^3b^3\xi^8 r^5} + \frac{9\left(4\xi^6(a-b)^4\left(a^2+4ab+b^2\right)-9\xi^2\left(a^2+b^2\right)+3\right)}{1024a^3b^3\xi^6 r^3} \tag{3.227}$$

$$h_8^{(3)} = \frac{9r^3}{1024a^3b^3} + \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4 r} - \frac{9r\left(a^2+b^2\right)}{256a^3b^3} + \frac{9\left(16\xi^8(a+b)^6\left(a^2-6ab+b^2\right)-360\xi^4\left(a^2-b^2\right)^2-240\xi^2\left(a^2+b^2\right)+45\right)}{16384a^3b^3\xi^8 r^5} + \frac{9\left(-4\xi^6(a+b)^4\left(a^2-4ab+b^2\right)+9\xi^2\left(a^2+b^2\right)-3\right)}{1024a^3b^3\xi^6 r^3} \tag{3.228}$$

3.3.4. ***Case 4***, *particle β inside particle α, $r < a+b, r > b-a, r < a-b$*

First scalar mobility function

$$h_0^{(1)} = -\frac{3}{10a^3} - \frac{9\left(a^2+b^2\right)}{20r^5} + \frac{3}{4r^3} \tag{3.229}$$

$$
\begin{aligned}
h_1^{(1)} = &-\frac{3r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{3r(a+b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{6\xi^2\left(11a^2-2ab+11b^2\right)+3}{2048\sqrt{\pi}a^3b^3\xi^3} \\
&+ \frac{9(a+b)}{40960\sqrt{\pi}a^3b^3\xi^7r^5}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)-4\xi^4(a+b)^2\left(a^2-6ab+b^2\right)\right. \\
&\left.-6\xi^2\left(29a^2-54ab+29b^2\right)-45\right] + \frac{9}{40960\sqrt{\pi}a^3b^3\xi^7r^4}\left[-8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)\right. \\
&\left.+12\xi^4(a+b)^2\left(a^2-6ab+b^2\right)+30\xi^2\left(5a^2-6ab+5b^2\right)-45\right] \\
&+ \frac{3(a+b)\left(-4\xi^4(a+b)^2\left(17a^2-62ab+17b^2\right)+4\xi^2\left(a^2+14ab+b^2\right)-15\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^3} \\
&+ \frac{3\left(4\xi^4(a+b)^2\left(17a^2-62ab+17b^2\right)-60\xi^2(a-b)^2+75\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^2} \\
&+ \frac{3(a+b)\left(2\xi^2\left(73a^2-118ab+73b^2\right)-15\right)}{10240\sqrt{\pi}a^3b^3\xi^3r}
\end{aligned} \tag{3.230}
$$

$$
\begin{aligned}
h_2^{(1)} = &-\frac{3r^2}{1024\sqrt{\pi}a^3b^3\xi} - \frac{3r(a+b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{6\xi^2\left(11a^2-2ab+11b^2\right)+3}{2048\sqrt{\pi}a^3b^3\xi^3} \\
&+ \frac{9(a+b)}{40960\sqrt{\pi}a^3b^3\xi^7r^5}\left[-8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)+4\xi^4(a+b)^2\left(a^2-6ab+b^2\right)\right. \\
&\left.+6\xi^2\left(29a^2-54ab+29b^2\right)+45\right] + \frac{9}{40960\sqrt{\pi}a^3b^3\xi^7r^4}\left[-8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)\right. \\
&\left.+12\xi^4(a+b)^2\left(a^2-6ab+b^2\right)+30\xi^2\left(5a^2-6ab+5b^2\right)-45\right] \\
&+ \frac{3(a+b)\left(4\xi^4(a+b)^2\left(17a^2-62ab+17b^2\right)-4\xi^2\left(a^2+14ab+b^2\right)+15\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^3} \\
&+ \frac{3\left(4\xi^4(a+b)^2\left(17a^2-62ab+17b^2\right)-60\xi^2(a-b)^2+75\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^2} \\
&- \frac{3(a+b)\left(2\xi^2\left(73a^2-118ab+73b^2\right)-15\right)}{10240\sqrt{\pi}a^3b^3\xi^3r}
\end{aligned} \tag{3.231}
$$

$$
\begin{aligned}
h_3^{(1)} = &\frac{3r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{3r(a-b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{-6\xi^2\left(11a^2+2ab+11b^2\right)-3}{2048\sqrt{\pi}a^3b^3\xi^3} \\
&+ \frac{9(a-b)}{40960\sqrt{\pi}a^3b^3\xi^7r^5}\left[8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)-4\xi^4(a-b)^2\left(a^2+6ab+b^2\right)\right. \\
&\left.-6\xi^2\left(29a^2+54ab+29b^2\right)-45\right] + \frac{9}{40960\sqrt{\pi}a^3b^3\xi^7r^4}\left[8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)\right. \\
&\left.-12\xi^4(a-b)^2\left(a^2+6ab+b^2\right)-30\xi^2\left(5a^2+6ab+5b^2\right)+45\right] \\
&+ \frac{3(a-b)\left(-4\xi^4(a-b)^2\left(17a^2+62ab+17b^2\right)+4\xi^2\left(a^2-14ab+b^2\right)-15\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^3} \\
&+ \frac{3\left(-4\xi^4(a-b)^2\left(17a^2+62ab+17b^2\right)+60\xi^2(a+b)^2-75\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^2} \\
&+ \frac{3(a-b)\left(2\xi^2\left(73a^2+118ab+73b^2\right)-15\right)}{10240\sqrt{\pi}a^3b^3\xi^3r}
\end{aligned} \tag{3.232}
$$

$$h_4^{(1)} = \frac{3r^2}{1024\sqrt{\pi}a^3b^3\xi} - \frac{3r(a-b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{-6\xi^2\left(11a^2+2ab+11b^2\right)-3}{2048\sqrt{\pi}a^3b^3\xi^3} \tag{3.233}$$
$$+ \frac{9(a-b)}{40960\sqrt{\pi}a^3b^3\xi^7r^5}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)+4\xi^4(a-b)^2\left(a^2+6ab+b^2\right)\right.$$
$$\left.+6\xi^2\left(29a^2+54ab+29b^2\right)+45\right] + \frac{9}{40960\sqrt{\pi}a^3b^3\xi^7r^4}\left[8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)\right.$$
$$\left.-12\xi^4(a-b)^2\left(a^2+6ab+b^2\right)-30\xi^2\left(5a^2+6ab+5b^2\right)+45\right]$$
$$+\frac{3(a-b)\left(4\xi^4(a-b)^2\left(17a^2+62ab+17b^2\right)-4\xi^2\left(a^2-14ab+b^2\right)+15\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^3}$$
$$+\frac{3\left(-4\xi^4(a-b)^2\left(17a^2+62ab+17b^2\right)+60\xi^2(a+b)^2-75\right)}{20480\sqrt{\pi}a^3b^3\xi^5r^2}$$
$$-\frac{3(a-b)\left(2\xi^2\left(73a^2+118ab+73b^2\right)-15\right)}{10240\sqrt{\pi}a^3b^3\xi^3r}$$

$$h_5^{(1)} = \frac{3r^3}{1024a^3b^3} + \frac{3}{40}\frac{a^3+b^3}{a^3b^3} - \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4r} - \frac{9r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.234}$$
$$+\frac{9\left(-16\xi^8(a+b)^6\left(a^2-6ab+b^2\right)+360\xi^4\left(a^2-b^2\right)^2+240\xi^2\left(a^2+b^2\right)-45\right)}{81920a^3b^3\xi^8r^5}$$
$$+\frac{12\xi^6(a+b)^4\left(a^2-4ab+b^2\right)-27\xi^2\left(a^2+b^2\right)+9}{1024a^3b^3\xi^6r^3}$$

$$h_6^{(1)} = -\frac{3r^3}{1024a^3b^3} + \frac{3}{40}\frac{a^3-b^3}{a^3b^3} + \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4r} + \frac{9r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.235}$$
$$+\frac{9\left(16\xi^8(a-b)^6\left(a^2+6ab+b^2\right)-360\xi^4\left(a^2-b^2\right)^2-240\xi^2\left(a^2+b^2\right)+45\right)}{81920a^3b^3\xi^8r^5}$$
$$+\frac{-12\xi^6(a-b)^4\left(a^2+4ab+b^2\right)+27\xi^2\left(a^2+b^2\right)-9}{1024a^3b^3\xi^6r^3}$$

$$h_7^{(1)} = -\frac{3r^3}{1024a^3b^3} + \frac{3}{40}\frac{b^3-a^3}{a^3b^3} + \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4r} + \frac{9r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.236}$$
$$+\frac{9\left(16\xi^8(a-b)^6\left(a^2+6ab+b^2\right)-360\xi^4\left(a^2-b^2\right)^2-240\xi^2\left(a^2+b^2\right)+45\right)}{81920a^3b^3\xi^8r^5}$$
$$+\frac{-12\xi^6(a-b)^4\left(a^2+4ab+b^2\right)+27\xi^2\left(a^2+b^2\right)-9}{1024a^3b^3\xi^6r^3}$$

$$h_8^{(1)} = \frac{3r^3}{1024a^3b^3} + \frac{3}{40}\frac{-a^3-b^3}{a^3b^3} - \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4r} - \frac{9r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.237}$$
$$+\frac{9\left(-16\xi^8(a+b)^6\left(a^2-6ab+b^2\right)+360\xi^4\left(a^2-b^2\right)^2+240\xi^2\left(a^2+b^2\right)-45\right)}{81920a^3b^3\xi^8r^5}$$
$$+\frac{12\xi^6(a+b)^4\left(a^2-4ab+b^2\right)-27\xi^2\left(a^2+b^2\right)+9}{1024a^3b^3\xi^6r^3}$$

Second scalar mobility function

$$h_0^{(2)} = \frac{9\left(a^2+b^2\right)}{4r^5} + \frac{9}{4r^3} \tag{3.238}$$

$$\begin{aligned}
h_1^{(2)} &= \frac{63r^2}{1024\sqrt{\pi}a^3b^3\xi} - \frac{63r(a+b)}{1024\sqrt{\pi}a^3b^3\xi} - \frac{9\left(2\xi^2\left(13a^2-14ab+13b^2\right)+7\right)}{2048\sqrt{\pi}a^3b^3\xi^3} \\
&+ \frac{9(a+b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[-8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)+4\xi^4(a+b)^2\left(a^2-6ab+b^2\right)\right. \\
&\left.+6\xi^2\left(29a^2-54ab+29b^2\right)+45\right] + \frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)\right. \\
&\left.-12\xi^4(a+b)^2\left(a^2-6ab+b^2\right)-30\xi^2\left(5a^2-6ab+5b^2\right)+45\right] \\
&+ \frac{9(a+b)\left(-4\xi^4(a+b)^2\left(5a^2-22ab+5b^2\right)+4\xi^2(5a-b)(a-5b)+69\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3} \\
&+ \frac{9\left(4\xi^2\left(\xi^2(a+b)^2\left(5a^2-22ab+5b^2\right)-11a^2+54ab-11b^2\right)+39\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2} \\
&+ \frac{9(a+b)\left(2\xi^2\left(13a^2-14ab+13b^2\right)+21\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}
\end{aligned} \tag{3.239}$$

$$\begin{aligned}
h_2^{(2)} &= \frac{63r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{63r(a+b)}{1024\sqrt{\pi}a^3b^3\xi} - \frac{9\left(2\xi^2\left(13a^2-14ab+13b^2\right)+7\right)}{2048\sqrt{\pi}a^3b^3\xi^3} \\
&+ \frac{9(a+b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)-4\xi^4(a+b)^2\left(a^2-6ab+b^2\right)\right. \\
&\left.-6\xi^2\left(29a^2-54ab+29b^2\right)-45\right] + \frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)\right. \\
&\left.-12\xi^4(a+b)^2\left(a^2-6ab+b^2\right)-30\xi^2\left(5a^2-6ab+5b^2\right)+45\right] \\
&+ \frac{9(a+b)\left(4\xi^4(a+b)^2\left(5a^2-22ab+5b^2\right)-4\xi^2(a-5b)(5a-b)-69\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3} \\
&+ \frac{9\left(4\xi^2\left(\xi^2(a+b)^2\left(5a^2-22ab+5b^2\right)-11a^2+54ab-11b^2\right)+39\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2} \\
&- \frac{9(a+b)\left(2\xi^2\left(13a^2-14ab+13b^2\right)+21\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}
\end{aligned} \tag{3.240}$$

$$h_3^{(2)} = -\frac{63r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{63r(b-a)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{18\xi^2\left(13a^2+14ab+13b^2\right)+63}{2048\sqrt{\pi}a^3b^3\xi^3} \tag{3.241}$$
$$+ \frac{9(a-b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)+4\xi^4(a-b)^2\left(a^2+6ab+b^2\right)\right.$$
$$\left.+6\xi^2\left(29a^2+54ab+29b^2\right)+45\right] + \frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)\right.$$
$$\left.+12\xi^4(a-b)^2\left(a^2+6ab+b^2\right)+30\xi^2\left(5a^2+6ab+5b^2\right)-45\right]$$
$$+ \frac{9(a-b)\left(-4\xi^4(a-b)^2\left(5a^2+22ab+5b^2\right)+4\xi^2(5a+b)(a+5b)+69\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3}$$
$$+ \frac{9\left(-4\xi^4(a-b)^2\left(5a^2+22ab+5b^2\right)+4\xi^2\left(11a^2+54ab+11b^2\right)-39\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2}$$
$$+ \frac{9(a-b)\left(2\xi^2\left(13a^2+14ab+13b^2\right)+21\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}$$

$$h_4^{(2)} = -\frac{63r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{63r(a-b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{18\xi^2\left(13a^2+14ab+13b^2\right)+63}{2048\sqrt{\pi}a^3b^3\xi^3} \tag{3.242}$$
$$+ \frac{9(a-b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)-4\xi^4(a-b)^2\left(a^2+6ab+b^2\right)\right.$$
$$\left.-6\xi^2\left(29a^2+54ab+29b^2\right)-45\right] + \frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)\right.$$
$$\left.+12\xi^4(a-b)^2\left(a^2+6ab+b^2\right)+30\xi^2\left(5a^2+6ab+5b^2\right)-45\right]$$
$$+ \frac{9(a-b)\left(4\xi^4(a-b)^2\left(5a^2+22ab+5b^2\right)-4\xi^2(5a+b)(a+5b)-69\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3}$$
$$+ \frac{9\left(-4\xi^4(a-b)^2\left(5a^2+22ab+5b^2\right)+4\xi^2\left(11a^2+54ab+11b^2\right)-39\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2}$$
$$- \frac{9(a-b)\left(2\xi^2\left(13a^2+14ab+13b^2\right)+21\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}$$

$$h_5^{(2)} = -\frac{63r^3}{1024a^3b^3} - \frac{324\xi^4\left(a^2-b^2\right)^2+81}{2048a^3b^3\xi^4r} + \frac{45r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.243}$$
$$+ \frac{9\left(16\xi^8(a+b)^6\left(a^2-6ab+b^2\right)-360\xi^4\left(a^2-b^2\right)^2-240\xi^2\left(a^2+b^2\right)+45\right)}{16384a^3b^3\xi^8r^5}$$
$$+ \frac{9\left(4\xi^6(a+b)^4\left(a^2-4ab+b^2\right)-9\xi^2\left(a^2+b^2\right)+3\right)}{1024a^3b^3\xi^6r^3}$$

$$h_6^{(2)} = \frac{63r^3}{1024a^3b^3} + \frac{324\xi^4\left(a^2-b^2\right)^2+81}{2048a^3b^3\xi^4r} - \frac{45r\left(a^2+b^2\right)}{256a^3b^3} \tag{3.244}$$
$$+ \frac{9\left(-16\xi^8(a-b)^6\left(a^2+6ab+b^2\right)+360\xi^4\left(a^2-b^2\right)^2+240\xi^2\left(a^2+b^2\right)-45\right)}{16384a^3b^3\xi^8r^5}$$
$$+ \frac{9\left(-4\xi^6(a-b)^4\left(a^2+4ab+b^2\right)+9\xi^2\left(a^2+b^2\right)-3\right)}{1024a^3b^3\xi^6r^3}$$

$$h_7^{(2)} = \frac{63r^3}{1024a^3b^3} + \frac{324\xi^4\left(a^2-b^2\right)^2+81}{2048a^3b^3\xi^4 r} - \frac{45r\left(a^2+b^2\right)}{256a^3b^3}$$
$$+ \frac{9\left(-16\xi^8(a-b)^6\left(a^2+6ab+b^2\right)+360\xi^4\left(a^2-b^2\right)^2+240\xi^2\left(a^2+b^2\right)-45\right)}{16384a^3b^3\xi^8r^5}$$
$$+ \frac{9\left(-4\xi^6(a-b)^4\left(a^2+4ab+b^2\right)+9\xi^2\left(a^2+b^2\right)-3\right)}{1024a^3b^3\xi^6r^3} \tag{3.245}$$

$$h_8^{(2)} = -\frac{63r^3}{1024a^3b^3} - \frac{324\xi^4\left(a^2-b^2\right)^2+81}{2048a^3b^3\xi^4 r} + \frac{45r\left(a^2+b^2\right)}{256a^3b^3}$$
$$+ \frac{9\left(16\xi^8(a+b)^6\left(a^2-6ab+b^2\right)-360\xi^4\left(a^2-b^2\right)^2-240\xi^2\left(a^2+b^2\right)+45\right)}{16384a^3b^3\xi^8r^5}$$
$$+ \frac{9\left(4\xi^6(a+b)^4\left(a^2-4ab+b^2\right)-9\xi^2\left(a^2+b^2\right)+3\right)}{1024a^3b^3\xi^6r^3} \tag{3.246}$$

Third scalar mobility function

$$h_0^{(3)} = \frac{9\left(a^2+b^2\right)}{4r^5} - \frac{9}{4r^3} \tag{3.247}$$

$$h_1^{(3)} = -\frac{9r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{9r(a+b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{18\xi^2\left(3a^2-2ab+3b^2\right)+9}{2048\sqrt{\pi}a^3b^3\xi^3}$$
$$+ \frac{9(a+b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[-8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)+4\xi^4(a+b)^2\left(a^2-6ab+b^2\right)\right.$$
$$\left.+6\xi^2\left(29a^2-54ab+29b^2\right)+45\right] + \frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)\right.$$
$$\left.-12\xi^4(a+b)^2\left(a^2-6ab+b^2\right)-30\xi^2\left(5a^2-6ab+5b^2\right)+45\right]$$
$$+ \frac{9(a+b)\left(4\xi^2\left(a^2+\xi^2(3a-b)(a+b)^2(a-3b)-10ab+b^2\right)+21\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3}$$
$$+ \frac{9\left(4\xi^2\left(a^2+6ab+b^2\right)-4\xi^4(a-3b)(3a-b)(a+b)^2-9\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2}$$
$$- \frac{9(a+b)\left(2\xi^2\left(3a^2-2ab+3b^2\right)+3\right)}{2048\sqrt{\pi}a^3b^3\xi^3r} \tag{3.248}$$

$$h_2^{(3)} = -\frac{9r^2}{1024\sqrt{\pi}a^3b^3\xi} - \frac{9r(a+b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{18\xi^2\left(3a^2-2ab+3b^2\right)+9}{2048\sqrt{\pi}a^3b^3\xi^3} \tag{3.249}$$
$$+\frac{9(a+b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)-4\xi^4(a+b)^2\left(a^2-6ab+b^2\right)\right.$$
$$\left.-6\xi^2\left(29a^2-54ab+29b^2\right)-45\right]+\frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[8\xi^6(a+b)^4\left(a^2-6ab+b^2\right)\right.$$
$$\left.-12\xi^4(a+b)^2\left(a^2-6ab+b^2\right)-30\xi^2\left(5a^2-6ab+5b^2\right)+45\right]$$
$$+\frac{9(a+b)\left(-4\xi^2\left(a^2-10ab+b^2\right)-4\xi^4(a-3b)(3a-b)(a+b)^2-21\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3}$$
$$+\frac{9\left(4\xi^2\left(a^2+6ab+b^2\right)-4\xi^4(a-3b)(3a-b)(a+b)^2-9\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2}$$
$$+\frac{9(a+b)\left(2\xi^2\left(3a^2-2ab+3b^2\right)+3\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}$$

$$h_3^{(3)} = \frac{9r^2}{1024\sqrt{\pi}a^3b^3\xi} + \frac{9r(a-b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{-18\xi^2\left(3a^2+2ab+3b^2\right)-9}{2048\sqrt{\pi}a^3b^3\xi^3} \tag{3.250}$$
$$+\frac{9(a-b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)+4\xi^4(a-b)^2\left(a^2+6ab+b^2\right)\right.$$
$$\left.+6\xi^2\left(29a^2+54ab+29b^2\right)+45\right]+\frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)\right.$$
$$\left.+12\xi^4(a-b)^2\left(a^2+6ab+b^2\right)+30\xi^2\left(5a^2+6ab+5b^2\right)-45\right]$$
$$+\frac{9(a-b)\left(4\xi^2\left(a^2+\xi^2(a-b)^2(3a+b)(a+3b)+10ab+b^2\right)+21\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3}$$
$$+\frac{9\left(-4\xi^2\left(a^2-6ab+b^2\right)+4\xi^4(a-b)^2(3a+b)(a+3b)+9\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2}$$
$$-\frac{9(a-b)\left(2\xi^2\left(3a^2+2ab+3b^2\right)+3\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}$$

$$h_4^{(3)} = \frac{9r^2}{1024\sqrt{\pi}a^3b^3\xi} - \frac{9r(a-b)}{1024\sqrt{\pi}a^3b^3\xi} + \frac{-18\xi^2\left(3a^2+2ab+3b^2\right)-9}{2048\sqrt{\pi}a^3b^3\xi^3} \tag{3.251}$$
$$+\frac{9(a-b)}{8192\sqrt{\pi}a^3b^3\xi^7r^5}\left[8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)-4\xi^4(a-b)^2\left(a^2+6ab+b^2\right)\right.$$
$$\left.-6\xi^2\left(29a^2+54ab+29b^2\right)-45\right]+\frac{9}{8192\sqrt{\pi}a^3b^3\xi^7r^4}\left[-8\xi^6(a-b)^4\left(a^2+6ab+b^2\right)\right.$$
$$\left.+12\xi^4(a-b)^2\left(a^2+6ab+b^2\right)+30\xi^2\left(5a^2+6ab+5b^2\right)-45\right]$$
$$+\frac{9(a-b)\left(-4\xi^2\left(a^2+10ab+b^2\right)-4\xi^4(a-b)^2(3a+b)(a+3b)-21\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^3}$$
$$+\frac{9\left(-4\xi^2\left(a^2-6ab+b^2\right)+4\xi^4(a-b)^2(3a+b)(a+3b)+9\right)}{4096\sqrt{\pi}a^3b^3\xi^5r^2}$$
$$+\frac{9(a-b)\left(2\xi^2\left(3a^2+2ab+3b^2\right)+3\right)}{2048\sqrt{\pi}a^3b^3\xi^3r}$$

$$h_5^{(3)} = \frac{9r^3}{1024a^3b^3} + \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4r} - \frac{9r\left(a^2+b^2\right)}{256a^3b^3} \\ + \frac{9\left(16\xi^8(a+b)^6\left(a^2-6ab+b^2\right)-360\xi^4\left(a^2-b^2\right)^2-240\xi^2\left(a^2+b^2\right)+45\right)}{16384a^3b^3\xi^8r^5} \\ + \frac{9\left(-4\xi^6(a+b)^4\left(a^2-4ab+b^2\right)+9\xi^2\left(a^2+b^2\right)-3\right)}{1024a^3b^3\xi^6r^3} \tag{3.252}$$

$$h_6^{(3)} = -\frac{9r^3}{1024a^3b^3} - \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4r} + \frac{9r\left(a^2+b^2\right)}{256a^3b^3} \\ + \frac{9\left(-16\xi^8(a-b)^6\left(a^2+6ab+b^2\right)+360\xi^4\left(a^2-b^2\right)^2+240\xi^2\left(a^2+b^2\right)-45\right)}{16384a^3b^3\xi^8r^5} \\ + \frac{9\left(4\xi^6(a-b)^4\left(a^2+4ab+b^2\right)-9\xi^2\left(a^2+b^2\right)+3\right)}{1024a^3b^3\xi^6r^3} \tag{3.253}$$

$$h_7^{(3)} = -\frac{9r^3}{1024a^3b^3} - \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4r} + \frac{9r\left(a^2+b^2\right)}{256a^3b^3} \\ + \frac{9\left(-16\xi^8(a-b)^6\left(a^2+6ab+b^2\right)+360\xi^4\left(a^2-b^2\right)^2+240\xi^2\left(a^2+b^2\right)-45\right)}{16384a^3b^3\xi^8r^5} \\ + \frac{9\left(4\xi^6(a-b)^4\left(a^2+4ab+b^2\right)-9\xi^2\left(a^2+b^2\right)+3\right)}{1024a^3b^3\xi^6r^3} \tag{3.254}$$

$$h_8^{(3)} = \frac{9r^3}{1024a^3b^3} + \frac{108\xi^4\left(a^2-b^2\right)^2+27}{2048a^3b^3\xi^4r} - \frac{9r\left(a^2+b^2\right)}{256a^3b^3} \\ + \frac{9\left(16\xi^8(a+b)^6\left(a^2-6ab+b^2\right)-360\xi^4\left(a^2-b^2\right)^2-240\xi^2\left(a^2+b^2\right)+45\right)}{16384a^3b^3\xi^8r^5} \\ + \frac{9\left(-4\xi^6(a+b)^4\left(a^2-4ab+b^2\right)+9\xi^2\left(a^2+b^2\right)-3\right)}{1024a^3b^3\xi^6r^3} \tag{3.255}$$

3.3.5. ***Case 5***, *self contribution ($\alpha = \beta$ or $r = 0$)*

$$h_0^{(1)} = \left(-\frac{3\left(6a^2\xi^2+1\right)}{80\sqrt{\pi}a^6\xi^3} + \frac{3\left(10a^2\xi^2+1\right)}{80\sqrt{\pi}a^6\xi^3}\,e^{-4a^2\xi^2} - \frac{3}{10a^3}\,\mathrm{erfc}(2a\xi)\right) \tag{3.256}$$

All other scalar coefficients are zero.