Supporting information on "Inertial migration of an electrophoretic rigid sphere in a two-dimensional Poiseuille flow"

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In this supplementary material, we provide a detailed derivation of Navier-Stokes equation in inertial frame of reference, and provide an explanation for neglect of temporal changes in the problem. Next, we provide the details of derivation of second reflection of the velocity field ($\tilde{\boldsymbol{v}}_2^{(0)}$). Furthermore, we illustrate the procedure to ease the computation of lift force integral (see (4.12) of the main text). This is done by (i) using transformations to reduce the number of integrals involved in reflected velocity components, and (ii) using orthogonal relations to overcome apparent divergence during the evaluation of volume integral.

I. MOMENTUM EQUATION IN AN INERTIAL FRAME OF REFERENCE

In a fixed frame of reference, the equation governing the momentum is:

$$\rho' \left(\frac{\partial \widehat{\boldsymbol{U}}'}{\partial \widehat{t}} + \widehat{\boldsymbol{U}}' \cdot \widehat{\nabla}' \widehat{\boldsymbol{U}}' \right) = \mu' \widehat{\nabla}'^2 \widehat{\boldsymbol{U}}' - \widehat{\nabla}' \widehat{P}' \tag{S1}$$

The symbol $\widehat{}$ denotes the variables in a fixed frame of reference and $'$ denotes the dimensional variables. For clarity we drop the prime ($'$) notation.

We now shift the frame of reference to the particle center. Since we are analyzing the migration in the weak inertial regime, the particle is assumed to move (translate or migrate) with a constant velocity $\boldsymbol{U}_s$. A similar assumption has been made by Becker et al., (1996, p. 209), Magnaudet et al., (2003, p. 119), and Ho and Leal (1974) in the context of weak inertia. The assumption of constant migration velocity might breakdown if the particle is very near the wall (i.e. particle –wall distance to particle size ratio is $d/a \sim O(1)$) (see Magnaudet et al., 2003 pg. 147). Since we address the $d/a \ll 1$ regime, we assume the migration velocity to be steady and therefore analyze the problem in an inertial frame of reference. Since time is a Galilean invariant in this frame, we write:

$$t = \widehat{t} \tag{S2}$$

The shift in coordinate system is governed by the following invertible relation:

$$\boldsymbol{\xi} = \widehat{\boldsymbol{\xi}} - \widehat{t}\boldsymbol{U}_s \quad \Leftrightarrow \quad \widehat{\boldsymbol{\xi}} = \boldsymbol{\xi} + t\boldsymbol{U}_s. \tag{S3}$$

The velocity field in the moving frame of reference is related to that in the fixed frame as:

$$\boldsymbol{U}(\boldsymbol{\xi}, t) = \widehat{\boldsymbol{U}} \left(\widehat{\boldsymbol{\xi}}(\boldsymbol{\xi}, t), \widehat{t}(t) \right) - \boldsymbol{U}_s. \tag{S4}$$

Here, $\widehat{\boldsymbol{\xi}}$ represents the coordinates in a fixed reference frame (i.e. $\widehat{\xi}_1 = \widehat{x}$, $\widehat{\xi}_2 = \widehat{y}$, $\widehat{\xi}_3 = \widehat{z}$), and $\boldsymbol{\xi}$ represents the coordinates in the reference frame of the moving particle (i.e. $\xi_1 = x$, $\xi_2 = y$, $\xi_3 = z$).

Taking partial derivative of (S4) in $\xi_1 = x$ direction:

$$\frac{\partial \boldsymbol{U}}{\partial x} = \frac{\partial \widehat{\boldsymbol{U}}}{\partial \widehat{x}}\frac{\partial \widehat{x}}{\partial x} + \frac{\partial \widehat{\boldsymbol{U}}}{\partial \widehat{y}}\frac{\partial \widehat{y}}{\partial x} + \frac{\partial \widehat{\boldsymbol{U}}}{\partial \widehat{z}}\frac{\partial \widehat{z}}{\partial x} + \frac{\partial \widehat{\boldsymbol{U}}}{\partial \widehat{t}}\frac{\partial \widehat{t}}{\partial x} = \frac{\partial \widehat{\boldsymbol{U}}}{\partial \widehat{x}}. \tag{S5}$$

We conclude that the partial derivative in all the spatial directions are unchanged (i.e. $\nabla \boldsymbol{U} = \widehat{\nabla}\widehat{\boldsymbol{U}}$).

Performing a partial differentiation in time on (S4), we obtain:

$$\frac{\partial \boldsymbol{U}}{\partial t} = \frac{\partial \widehat{\boldsymbol{U}}}{\partial \widehat{x}}\frac{\partial \widehat{x}}{\partial t} + \frac{\partial \widehat{\boldsymbol{U}}}{\partial \widehat{y}}\frac{\partial \widehat{y}}{\partial t} + \frac{\partial \widehat{\boldsymbol{U}}}{\partial \widehat{z}}\frac{\partial \widehat{z}}{\partial t} + \frac{\partial \widehat{\boldsymbol{U}}}{\partial \widehat{t}}\frac{\partial \widehat{t}}{\partial t} \tag{S6}$$

Using (S2), (S3) and (S5) in the above equation, we find:

$$\frac{\partial \boldsymbol{U}}{\partial t} = \frac{\partial \boldsymbol{U}}{\partial x}\boldsymbol{U}_{sx} + \frac{\partial \boldsymbol{U}}{\partial y}\boldsymbol{U}_{sy} + \frac{\partial \boldsymbol{U}}{\partial z}\boldsymbol{U}_{sz} + \frac{\partial \widehat{\boldsymbol{U}}}{\partial \widehat{t}} \tag{S7}$$

The above relation can be represented as:

$$\frac{\partial \boldsymbol{U}}{\partial t} = \boldsymbol{U}_s \cdot \nabla \boldsymbol{U} + \frac{\partial \widehat{\boldsymbol{U}}}{\partial \widehat{t}} \quad \Rightarrow \quad \frac{\partial \widehat{\boldsymbol{U}}}{\partial \widehat{t}} = \frac{\partial \boldsymbol{U}}{\partial t} - \boldsymbol{U}_s \cdot \nabla \boldsymbol{U} \tag{S8}$$

Substituting (S8) and (S4) in (S1) and restoring the dimensional notation $'$, we write:

$$\rho' \left(\left(\frac{\partial \boldsymbol{U}'}{\partial t'} - \boldsymbol{U}_s{}' \cdot \nabla' \boldsymbol{U}' \right) + \boldsymbol{U}' \cdot \nabla' \boldsymbol{U}' + \boldsymbol{U}_s{}' \cdot \nabla' \boldsymbol{U}' \right) = \mu' \nabla'^2 \boldsymbol{U}' - \nabla' P' \tag{S9}$$

Simplifying the above relation, we obtain the dimensional momentum equation in the inertial frame of reference:

$$\rho' \left(\frac{\partial \boldsymbol{U}'}{\partial t'} + \boldsymbol{U}' \cdot \nabla' \boldsymbol{U}' \right) = \mu' \nabla'^2 \boldsymbol{U}' - \nabla' P' \tag{S10}$$

We represent the system in non-dimensional variables using a', $\kappa U'_{max}$, $\mu' \kappa U'_{max}/a'$ as the characteristic scales of length, velocity and pressure respectively. We find that the hydrodynamics of the current problem has three time scales: (i) viscous time scale ($t'_{visc} \sim a'^2/\nu'$), (ii) convective time scale of the flow ($t'_{conv} \sim a'/\kappa U'_{max}$) and (iii) migration (or geometric) time scale ($t'_{mig} \sim a'/U'_{mig}$). Since the time dependence enters the system through the migration time scale, we choose a'/U'_{mig} as the characteristic time scale. The momentum equation (S10) is represented in dimensionless form as:

$$\left(\frac{\rho' a' \boldsymbol{U}'_{mig}}{\mu'} \right) \left(\frac{\partial \boldsymbol{U}}{\partial t} \right) + \left(\frac{\rho' a' \kappa \boldsymbol{U}'_{max}}{\mu'} \right) (\boldsymbol{U} \cdot \nabla \boldsymbol{U}) = \mu \nabla^2 \boldsymbol{U} - \nabla P \tag{S11}$$

On comparing the coefficients multiplying the temporal and convective terms, we find that the temporal variations can be assumed to be negligible if the migration time scale is much larger than the convective time scale (or $U'_{mig} \ll \kappa U'_{max}$). For problems involving lateral migration in the presence of weak inertia, a similar assumption has been made by: Becker et al., (1996, p. 209); Hogg (1994, p. 295); Magnaudet et al., (2003, p. 119), (Ho and Leal, 1974). The flow is therefore governed by continuity and *quasi-steady* Navier-Stokes equation (the term quasi-steady implies that the variables depend only on the instantaneous geometric configuration, provided $U'_{mig} \ll \kappa U'_{max}$):

$$\nabla \cdot \boldsymbol{U} = 0 \quad Re_p\,(\boldsymbol{U} \cdot \nabla \boldsymbol{U}) = \mu \nabla^2 \boldsymbol{U} - \nabla P \tag{S12}$$

Here, Re_p is the particle Reynolds number: $\frac{\rho' a' \kappa U'_{max}}{\mu'}$. The above equation is used to model the hydrodynamics in our analysis.

II. SOLUTION TO $\tilde{\boldsymbol{v}}_2^{(0)}$

Here, we provide the details of the evaluation of $\boldsymbol{v}_2^{(0)}$ (see §3.2 in the main text). The solution is determined by the form of non-homogeneity in the boundary condition at the walls (see (3.12)). Following the procedure described in §3.1, the non-homogeneities ($HaZ_w \nabla \psi_1$, $HaZ_w \nabla \psi_2$ and $\boldsymbol{v}_1^{(0)}$) are represented into the outer scale coordinates before applying Faxén's integral transformation. ψ_2 is already defined in the integral form (see (3.6)), whereas ψ_1 and $\boldsymbol{v}_1^{(0)}$ have been defined in the particle scale ((3.4) and (3.14), respectively). Upon performing Faxén transformation of the non-homogeneities, we find that $\tilde{\nabla}\tilde{\psi}_2$ has a different integral form in comparison to $\tilde{\boldsymbol{v}}_1^{(0)}$ and $\tilde{\nabla}\tilde{\psi}_1$. Therefore, we use superposition and seek $\tilde{\boldsymbol{v}}_2^{(0)}$ as $\tilde{\boldsymbol{v}}_{2\,(i)}^{(0)} + \tilde{\boldsymbol{v}}_{2\,(ii)}^{(0)}$. These components satisfy the following boundary conditions (c.f. boundary condition in (3.12)):

$$\tilde{\boldsymbol{v}}_{2\,(i)}^{(0)} = HaZ_w \kappa \tilde{\nabla}(\tilde{\psi_1}) - \tilde{\boldsymbol{v}}_1^{(0)} \quad \text{at the walls,} \tag{S13}$$

$$\tilde{\boldsymbol{v}}_{2\,(ii)}^{(0)} = HaZ_w \kappa \tilde{\nabla}(\tilde{\psi_2}) \quad \text{at the walls.} \tag{S14}$$

Solution to $\tilde{\boldsymbol{v}}_{2\,(i)}$: In view of the particle boundary condition in (S13), ψ_1 (3.4) and $\boldsymbol{v}_1^{(0)}$ (3.14) are represented in the outer coordinates ($\tilde{\psi}_1$ and $\tilde{\boldsymbol{v}}_1^{(0)}$) and then Faxén transformation is applied. $\tilde{\psi}_1$ has already been represented in outer coordinates and transformed into integral form (see (A8)); $\boldsymbol{v}_1^{(0)}$ is represented into outer coordinates and then Faxén's transformation is applied. Various terms present in the expression for $\boldsymbol{v}_1^{(0)}$ (such as: $1/r$, $x^2/r^3, \cdots$ in (3.14)) are transformed into Faxén's integral form. Upon deriving each term, we obtain the RHS of wall boundary condition (S13) as:

$$HaZ_w \kappa \tilde{\nabla}\psi_1 - \tilde{\boldsymbol{v}}_1^{(0)} = -\frac{1}{2\pi}\begin{bmatrix} \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \mathrm{e}^{\left(\mathrm{i}\Theta - \frac{\lambda|Z|}{2}\right)} \left(\ell_1 + (\xi^2/\lambda^2)\left(\ell_2 + \frac{\lambda|Z|}{2}\ell_3\right)\right) d\xi d\eta \\ \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \mathrm{e}^{\left(\mathrm{i}\Theta - \frac{\lambda|Z|}{2}\right)} \left(\ell_2 + \frac{\lambda|Z|}{2}\ell_3\right) ((\eta\xi)/\lambda^2)\, d\xi d\eta \\ \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \mathrm{e}^{\left(\mathrm{i}\Theta - \frac{\lambda|Z|}{2}\right)} \left(\ell_1 + \ell_2 + \left(1 + \frac{\lambda|Z|}{2}\right)\ell_3\right) ((\mathrm{i}\xi)/\lambda)\frac{Z}{|Z|}\, d\xi d\eta \end{bmatrix} \tag{S15}$$

Here the terms ℓ_1, ℓ_2, and ℓ_3 are given by:

$$\ell_1 = \frac{A_1\kappa}{\lambda} + \frac{\kappa^2}{4}\left(C_1 + \frac{D_1}{3}\right)\frac{Z}{|Z|},\ \ell_2 = \frac{-A_1\kappa}{2\lambda} - \frac{B_1\lambda\kappa^3}{8} + \frac{HaZ_w b_1\lambda\kappa^3}{2},\ \ell_3 = \frac{-A_1\kappa}{2\lambda} - \frac{D_1\kappa^2}{12}\frac{Z}{|Z|} \tag{S16}$$

Following the procedure carried out in §3.1, we assume the form of $\tilde{\boldsymbol{v}}^{(0)}_{2\,(i)} = \{\tilde{u}^{(0)}_{2\,(i)},\ \tilde{v}^{(0)}_{2\,(i)},\ \tilde{w}^{(0)}_{2\,(i)}\}$ as:

$$\tilde{u}^{(0)}_{2(i)} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\Theta}\begin{pmatrix} \mathrm{e}^{\left(\frac{-\lambda Z}{2}\right)}\left(\ell_4 + \frac{\xi^2}{\lambda^2}\left(\ell_5 + \frac{\lambda Z}{2}\ell_6\right)\right) \\ +\,\mathrm{e}^{\left(\frac{+\lambda Z}{2}\right)}\left(\ell_7 + \frac{\xi^2}{\lambda^2}\left(\ell_8 - \frac{\lambda Z}{2}\ell_9\right)\right)\end{pmatrix}\mathrm{d}\xi\mathrm{d}\eta \tag{S17}$$

$$\tilde{v}^{(0)}_{2(i)} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\Theta}\begin{pmatrix} \mathrm{e}^{\left(\frac{-\lambda Z}{2}\right)}\left(\ell_5 + \frac{\lambda Z}{2}\ell_6\right) \\ +\,\mathrm{e}^{\left(\frac{+\lambda Z}{2}\right)}\left(\ell_8 - \frac{\lambda Z}{2}\ell_9\right)\end{pmatrix}\left(\frac{\xi\eta}{\lambda^2}\right)\mathrm{d}\xi\mathrm{d}\eta \tag{S18}$$

$$\tilde{w}^{(0)}_{2(i)} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\Theta}\begin{pmatrix} \mathrm{e}^{\left(\frac{-\lambda Z}{2}\right)}\left(\ell_4 + \ell_5 + \left(1 + \frac{\lambda Z}{2}\right)\ell_6\right) \\ -\,\mathrm{e}^{\left(\frac{+\lambda Z}{2}\right)}\left(\ell_7 + \ell_8 + \left(1 - \frac{\lambda Z}{2}\right)\ell_9\right)\end{pmatrix}\left(\frac{\mathrm{i}\xi}{\lambda}\right)\mathrm{d}\xi\mathrm{d}\eta \tag{S19}$$

Here, the terms $\ell_4,\ \ell_5,\cdots\ell_9$ are functions of the Fourier variable λ and coefficients ($A_1,\ B_1,\ C_1$ and D_1) defined in (3.15). The terms $\ell_4,\ \ell_5,\cdots\ell_9$ in the above equations can be expressed in terms of the known $\ell_1,\ \ell_2$, and ℓ_3. Towards this, we form a system of six equations by substituting (S15) into RHS of (S13). The LHS of (S13) is represented by (S17)-(S19). Since the integrals on both the sides are identical, we obtain the following linear system of equations:

$$\begin{bmatrix} e^{\frac{s\lambda}{2}} & e^{\frac{s\lambda}{2}}\xi^2/\lambda^2 & -e^{\frac{s\lambda}{2}}s\xi^2/2\lambda & e^{-\frac{s\lambda}{2}} & e^{-\frac{s\lambda}{2}}\xi^2/\lambda^2 & e^{-\frac{s\lambda}{2}}s\xi^2/2\lambda \\ 0 & e^{\frac{s\lambda}{2}} & -e^{\frac{s\lambda}{2}}s\lambda/2 & 0 & e^{-\frac{s\lambda}{2}} & e^{-\frac{s\lambda}{2}}s\lambda/2 \\ e^{\frac{s\lambda}{2}} & e^{\frac{s\lambda}{2}} & e^{\frac{s\lambda}{2}}\left(1 - s\lambda/2\right) & -e^{-\frac{s\lambda}{2}} & -e^{-\frac{s\lambda}{2}} & -e^{-\frac{s\lambda}{2}}\left(1 + s\lambda/2\right) \\ e^{-\frac{1}{2}(1-s)\lambda} & e^{-\frac{1}{2}(1-s)\lambda}\xi^2/\lambda^2 & e^{-\frac{1}{2}(1-s)\lambda}\left(1-s\right)\xi^2/2\lambda & e^{\frac{1}{2}(1-s)\lambda} & e^{\frac{1}{2}(1-s)\lambda}\xi^2/\lambda^2 & -e^{\frac{1}{2}(1-s)\lambda}\left(1-s\right)\xi^2/2\lambda \\ 0 & e^{-\frac{1}{2}(1-s)\lambda} & e^{-\frac{1}{2}(1-s)\lambda}\left(1-s\right)\lambda/2 & 0 & e^{\frac{1}{2}(1-s)\lambda} & -e^{\frac{1}{2}(1-s)\lambda}\left(1-s\right)\lambda/2 \\ e^{-\frac{1}{2}(1-s)\lambda} & e^{-\frac{1}{2}(1-s)\lambda} & e^{-\frac{1}{2}(1-s)\lambda}\left(1 + \left(1-s\right)\lambda/2\right) & -e^{\frac{1}{2}(1-s)\lambda} & -e^{\frac{1}{2}(1-s)\lambda} & -e^{\frac{1}{2}(1-s)\lambda}\left(1 - \left(1-s\right)\lambda/2\right) \end{bmatrix}\begin{bmatrix}\ell_4\\ \ell_5\\ \ell_6\\ \ell_7\\ \ell_8\\ \ell_9\end{bmatrix}$$

$$= \begin{bmatrix} -e^{-\frac{s\lambda}{2}}\left(\ell_{1b} + \left(\ell_2 + \frac{\ell_{3b}s\lambda}{2}\right)\xi^2/\lambda^2\right) \\ -e^{-\frac{s\lambda}{2}}\left(\ell_2 + \ell_{3b}s\lambda/2\right) \\ e^{-\frac{s\lambda}{2}}\left(\ell_{1b} + \ell_2 + \ell_{3b}\left(1 + s\lambda/2\right)\right) \\ -e^{-\frac{1}{2}(1-s)\lambda}\left(\ell_{1t} + \left(\ell_2 + \frac{1}{2}\ell_{3t}\left(1-s\right)\lambda\right)\xi^2/\lambda^2\right) \\ -e^{-\frac{1}{2}(1-s)\lambda}\left(\ell_2 + \frac{1}{2}\ell_{3t}\left(1-s\right)\lambda\right) \\ -e^{-\frac{1}{2}(1-s)\lambda}\left(\ell_{1t} + \ell_2 + \ell_{3t}\left(1 + \left(1-s\right)\lambda/2\right)\right) \end{bmatrix} \tag{S20}$$

Here, ℓ_{1b}, ℓ_{3b} and ℓ_{1t}, ℓ_{3t} correspond to the boundary condition at the bottom wall $Z/|Z| < 0$ and the top wall $Z/|Z| > 0$, respectively.

Solution to $\tilde{\boldsymbol{v}}^{(0)}_{2\,(ii)}$: Upon substituting $\tilde{\psi}_2$ in (S14), we obtain $\tilde{\boldsymbol{v}}^{(0)}_{2\,(ii)} = \{\tilde{u}^{(0)}_{2\,(ii)},\ \tilde{v}^{(0)}_{2\,(ii)},\ \tilde{w}^{(0)}_{2\,(ii)}\}$:

$$\tilde{u}^{(0)}_{2(ii)} = \frac{HaZ_w\kappa^3}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\Theta}\left(\mathrm{e}^{\left(-\frac{\lambda Z}{2}\right)}b_2 + \mathrm{e}^{\left(+\frac{\lambda Z}{2}\right)}b_3\right)\left(\frac{i^2\xi^2}{2\lambda}\right)\mathrm{d}\xi\mathrm{d}\eta \tag{S21}$$

$$\tilde{v}^{(0)}_{2(ii)} = \frac{HaZ_w\kappa^3}{2\pi}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\Theta}\left(\mathrm{e}^{\left(-\frac{\lambda Z}{2}\right)}b_2 + \mathrm{e}^{\left(+\frac{\lambda Z}{2}\right)}b_3\right)\left(\frac{i^2\xi\eta}{2\lambda}\right)\mathrm{d}\xi\mathrm{d}\eta \tag{S22}$$

$$\tilde{w}^{(0)}_{2(ii)} = \frac{HaZ_w\kappa^3}{2\pi}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\Theta}\left(-\mathrm{e}^{\left(-\frac{\lambda Z}{2}\right)}b_2 + \mathrm{e}^{\left(+\frac{\lambda Z}{2}\right)}b_3\right)\left(\frac{\lambda Z}{2}\right)\left(\frac{\mathrm{i}\xi}{\lambda}\right)\mathrm{d}\xi\mathrm{d}\eta \tag{S23}$$

Combining (S17)-(S19) and (S21)-(S23), we obtain (3.17)-(3.19).

III. REDUCTION OF NESTED INTEGRALS

Here, we show the procedure for reducing the number of nested integrals. The test field (see (A20) in the main text) is taken for the illustration:

$$\tilde{u}^t_2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i\Theta^t}\left(e^{\left(\frac{-\lambda^t Z}{2}\right)}\left(\frac{h_3 Z}{2}+\frac{h_4}{\lambda^t}\right)+e^{\left(\frac{\pm\lambda^t Z}{2}\right)}\left(\frac{h_5 Z}{2}-\frac{h_6}{\lambda^t}\right)\right) i\xi^t d\xi^t d\eta^t \tag{S24}$$

The expression for current derivation can be simply identified as:

$$\tilde{u}^t_2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i\Theta^t} f\left(\lambda^t, Z\right) i\xi^t \mathrm{d}\xi^t \mathrm{d}\eta^t \tag{S25}$$

Substituting the transformation: $\xi_t = \lambda^t\cos\theta$ and $\eta^t = \lambda^t\sin\theta$ in the above equation, we get:

$$\tilde{u}^t_2 = \frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi} exp\left\{\frac{\mathrm{i}\lambda^{\mathrm{t}}}{2}\left(Xcos\theta + Ysin\theta\right)\right\} f\left(\lambda^t, Z\right)\mathrm{i}\lambda^{\mathrm{t}}\mathrm{cos}\theta\lambda^{\mathrm{t}}\mathrm{d}\theta\mathrm{d}\lambda^{\mathrm{t}} \tag{S26}$$

Substituting $X = \rho\cos\varphi$ and $Y = \rho\sin\varphi$ in the above expression, and simplifying we get:

$$\tilde{u}^t_2 = \frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi} exp\left\{\frac{\mathrm{i}\lambda^{\mathrm{t}}}{2}\left(\rho cos\left(\theta-\varphi\right)\right)\right\}\mathrm{i}\,\mathfrak{F}\left(\lambda^{\mathrm{t}}, \mathrm{Z}\right)\mathrm{cos}\theta\mathrm{d}\theta\mathrm{d}\lambda^{\mathrm{t}} \tag{S27}$$

Here, $\mathfrak{F}(\lambda^t, Z) = \lambda^{t\,2} f(\lambda^t, Z)$. We shift the polar integral limit by φ:

$$\tilde{u}^t_2 = \frac{1}{2\pi}\int_0^{\infty}\left(\int_{-\varphi}^{2\pi-\varphi} exp\left\{\frac{\mathrm{i}\lambda^{\mathrm{t}}}{2}\left(\rho cos\theta\right)\right\}\mathrm{i}\mathrm{cos}\left(\theta+\varphi\right)\mathfrak{F}\left(\lambda^{\mathrm{t}}, \mathrm{Z}\right)\mathrm{d}\theta\mathrm{d}\lambda^{\mathrm{t}}\right) \tag{S28}$$

By splitting the integral, we get:

$$\tilde{u}^t_2 = \frac{1}{2\pi}\int_0^{\infty}\left[\begin{array}{c}\left(\int_{-\varphi}^{0} exp\left\{\frac{\mathrm{i}\lambda^{\mathrm{t}}}{2}\left(\rho cos\theta\right)\right\}\mathrm{i}\mathrm{cos}\left(\theta+\varphi\right)\mathfrak{F}\left(\lambda^{\mathrm{t}}, \mathrm{Z}\right)\mathrm{d}\theta\right)\\ +\left(\int_0^{2\pi}\ldots''\ldots\right)+\left(\int_{2\pi-\varphi}^{2\pi}\ldots''\ldots\right)\end{array}\right]\mathrm{d}\lambda^{\mathrm{t}} \tag{S29}$$

Here, $\ldots''\ldots$ denotes the second and third integrands are identical to the first integrand. The first and third term in the above equation cancel each other owing to periodicity of cosine function. The second term is an integral form of the Bessel's function J_n (Weisstein, 2013):

$$\tilde{u}^t_2 = \frac{1}{2\pi}\int_0^{\infty}\left[-2\pi J_1\left(\frac{\lambda^t\rho}{2}\right)cos\varphi\right]\left(\lambda^t, Z\right)d\lambda^t \tag{S30}$$

Upon re-transforming (ρ, φ to X, Y), we get:

$$\tilde{u}^t_2 = \frac{-X}{\sqrt{X^2+Y^2}}\int_0^{\infty} J_1\left(\lambda^t\sqrt{X^2+Y^2}\Big/2\right)\mathfrak{F}\left(\lambda^t, Z\right)d\lambda^t \tag{S31}$$

Therefore, the complete expression for $\tilde{u}_2^t$ is:

$$\tilde{u}_2^t = \frac{-X}{\sqrt{X^2+Y^2}} \int_0^\infty J_1\left(\tau^t\right)\left(e^{\left(\frac{-\lambda^t Z}{2}\right)}\left(\frac{h_3 Z}{2}+\frac{h_4}{\lambda^t}\right)+e^{\left(\frac{+\lambda^t Z}{2}\right)}\left(\frac{h_5 Z}{2}-\frac{h_6}{\lambda^t}\right)\right)\left(\lambda^t\right)^2 d\lambda^t \quad \text{(S32)}$$

Similarly, the expressions can be reduced for other components of test field $\tilde{\boldsymbol{u}}_2^t$:

$$\tilde{v}_2^t = \frac{-Y}{\sqrt{X^2+Y^2}} \int_0^\infty J_1\left(\tau^t\right)\left(e^{\left(\frac{-\lambda^t Z}{2}\right)}\left(\frac{h_3 Z}{2}+\frac{h_4}{\lambda^t}\right)+e^{\left(\frac{+\lambda^t Z}{2}\right)}\left(\frac{h_5 Z}{2}-\frac{h_6}{\lambda^t}\right)\right)\left(\lambda^t\right)^2 d\lambda^t \quad \text{(S33)}$$

$$\tilde{w}_2^t = -\int_0^\infty J_0\left(\tau^t\right)\left(e^{\left(\frac{-\lambda^t Z}{2}\right)}\left(h_3\left(1+\lambda^t Z/2\right)+h_4\right)+e^{\left(\frac{+\lambda^t Z}{2}\right)}\left(h_5\left(1-\lambda^t Z/2\right)+h_6\right)\right)d\lambda^t \quad \text{(S34)}$$

Here, $\tau^t = \lambda^t\sqrt{X^2+Y^2}\Big/2$.

In a similar manner, we derive the reduced expressions for the disturbance velocity $\tilde{\boldsymbol{v}}_2^{(0)}$ is:

$$\begin{aligned}\tilde{u}_2^{(0)} = {} & \int_0^\infty\left\{J_0(\tau)\begin{bmatrix} e^{\left(\frac{-\lambda Z}{2}\right)}\left(2\ell_4+\ell_5+\ell_6\left(\lambda Z/2\right)\right) \\ +e^{\left(\frac{+\lambda Z}{2}\right)}\left(2\ell_4+\ell_5-\ell_6\left(\lambda Z/2\right)\right)\end{bmatrix} - \frac{X^2-Y^2}{X^2+Y^2}J_2(\tau)\begin{bmatrix} e^{\left(\frac{-\lambda Z}{2}\right)}\left(\ell_5+\ell_6\left(\lambda Z/2\right)\right) \\ +e^{\left(\frac{+\lambda Z}{2}\right)}\left(\ell_5-\ell_6\left(\lambda Z/2\right)\right)\end{bmatrix}\right\}\frac{\lambda}{2}d\lambda \\ & +Ha\mathrm{Z}_w\kappa^3\int_0^\infty\left\{\left(\frac{J_1(\tau)\left(X^2-Y^2\right)}{\left(X^2+Y^2\right)^{3/2}}-\frac{J_0(\tau)X^2\lambda}{2\left(X^2+Y^2\right)}\right)\left[b_2 e^{\left(\frac{-\lambda Z}{2}\right)}+b_3 e^{\left(\frac{+\lambda Z}{2}\right)}\right]\right\}\lambda d\lambda\end{aligned} \quad \text{(S35)}$$

$$\begin{aligned}\tilde{v}_2^{(0)} = {} & \frac{-XY}{X^2+Y^2}\int_0^\infty J_2(\tau)\left(e^{\left(\frac{-\lambda Z}{2}\right)}\left(\ell_5+\ell_6\left(\lambda Z/2\right)\right)+e^{\left(\frac{+\lambda Z}{2}\right)}\left(\ell_8-\ell_9\left(\lambda Z/2\right)\right)\right)\lambda d\lambda \\ & +Ha\mathrm{Z}_w\kappa^3\frac{XY}{2\left(X^2+Y^2\right)}\int_0^\infty J_2(\tau)\left(b_2 e^{\left(\frac{-\lambda Z}{2}\right)}+b_3 e^{\left(\frac{+\lambda Z}{2}\right)}\right)\lambda^2 d\lambda\end{aligned} \quad \text{(S36)}$$

$$\begin{aligned}\tilde{w}_2^{(0)} = {} & \frac{-X}{\sqrt{X^2+Y^2}}\int_0^\infty J_1(\tau)\left(e^{\left(\frac{-\lambda Z}{2}\right)}\left(\ell_4+\ell_5+\ell_6\left(1+\lambda Z/2\right)\right)+e^{\left(\frac{+\lambda Z}{2}\right)}\left(\ell_7+\ell_8+\ell_9\left(1-\lambda Z/2\right)\right)\right)\lambda d\lambda \\ & -Ha\mathrm{Z}_w\kappa^3\frac{XZ}{2\sqrt{X^2+Y^2}}\int_0^\infty J_1(\tau)\left(-b_2 e^{\left(\frac{-\lambda Z}{2}\right)}+b_3 e^{\left(\frac{+\lambda Z}{2}\right)}\right)\lambda^2 d\lambda.\end{aligned} \quad \text{(S37)}$$

Here, $\tau = \lambda\sqrt{X^2+Y^2}\Big/2$.

IV. OVERCOMING AN APPARENT DIVERGENCE

The volume integral with the following integrand:

$$\int_0^{2\pi}\int_0^{\infty}\int_{-s||k}^{-\kappa||1-s}\int_0^{\infty}\int_0^{\infty}\tilde{\mathbf{u}}_2^t\cdot\left[\tilde{\mathbf{v}}_2^{(0)}\cdot\tilde{\nabla}\tilde{\mathbf{V}}_\infty+\tilde{\mathbf{V}}_\infty\cdot\tilde{\nabla}\tilde{\mathbf{v}}_2^{(0)}\right]\kappa^{-2}R_C d\lambda^t d\lambda dZ_C dR_C d\theta_C. \quad \text{(S38)}$$

Here || stands for 'or'. Upon substitution of the velocity fields, we obtain the following form:

$$\int_0^{2\pi}\int_0^{\infty}\int_{-s||k}^{-\kappa||1-s}\int_0^{\infty}\int_0^{\infty}\begin{pmatrix} R_C\lambda^t\lambda J_1\left(R_C\lambda^t/2\right)J_1\left(R_C\lambda/2\right) \\ f\left(\lambda^t,Z_C,\theta_C\right)g\left(\lambda,Z_C,\theta_C\right)\end{pmatrix}d\lambda^t d\lambda dZ_C dR_C d\theta_C. \quad \text{(S39)}$$

Numerically, this integral shows apparent divergence. Using the orthogonal property of Bessel's functions (Abramowitz and Stegun, 1972), we obtain:

$$\int_0^{2\pi}\int_{-s||k}^{-\kappa||1-s}\int_0^{\infty}\int_0^{\infty}\lambda^t\lambda\frac{4\delta\left(\lambda^t-\lambda\right)}{\lambda^t}f\left(\lambda^t,Z_C,\theta_C\right)g\left(\lambda,Z_C,\theta_C\right)d\lambda^t d\lambda dZ_C d\theta_C. \quad \text{(S40)}$$

Further, upon use of the property of Dirac-Delta functions we arrive at a converged form:

$$\int_0^{2\pi}\int_{-s||k}^{-\kappa||1-s}\int_0^{\infty}4\lambda f\left(\lambda,Z_C,\theta_C\right)g\left(\lambda,Z_C,\theta_C\right)d\lambda dZ_C d\theta_C. \quad \text{(S41)}$$

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