Supplementary material for "A bulk-interface correspondence for equatorial waves"

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The aim of these supplementary material is to check the possible remaining modes at the interface. The last case to study is $v \neq 0$ and $k_x^2 = \omega^2$, equations (4.4) and (4.5) become

$$\big(\epsilon\partial_{yy} - \frac{k_x}{\omega}\partial_y + (\pm f - \epsilon k_x^2)\big)v = 0 \tag{0.1}$$

$$\big(\epsilon\partial_{yy} + \frac{k_x}{\omega}\partial_y + (\pm f - \epsilon k_x^2)\big)u = -\frac{\mathrm{i}}{\omega}(\partial_{yy} + \omega^2)v \tag{0.2}$$

from which we infer

$$v_{\uparrow/\downarrow}(y) = A_{\uparrow/\downarrow}\mathrm{e}^{r_{\uparrow/\downarrow+}y} + B_{\uparrow/\downarrow}\mathrm{e}^{r_{\uparrow/\downarrow-}y} \tag{0.3}$$

$$u_{\uparrow/\downarrow}(y) = \alpha_{\uparrow/\downarrow+}A_{\uparrow/\downarrow}\mathrm{e}^{r_{\uparrow/\downarrow+}y} + \alpha_{\uparrow/\downarrow-}B_{\uparrow/\downarrow}\mathrm{e}^{r_{\uparrow/\downarrow-}y} + C_{\uparrow/\downarrow}\mathrm{e}^{q_{\uparrow/\downarrow+}y} + D_{\uparrow/\downarrow}\mathrm{e}^{q_{\uparrow/\downarrow-}y} \tag{0.4}$$

where

$$r_{\uparrow/\downarrow} = \frac{1}{2\epsilon}\big(\frac{k_x}{\omega} \pm \sqrt{1 + 4\epsilon(\epsilon k_x^2 - \pm f)}\big) \tag{0.5}$$

with the last $\pm$ sign on the right hand side refers to $\uparrow / \downarrow$. The roots $q_{\uparrow/\downarrow\pm}$ are the one from (**??**) and (**??**) and furthermore

$$\alpha_{\uparrow/\downarrow\pm} = -\frac{i}{\omega}(r_{\uparrow/\downarrow\pm}^2 + \omega^2)\big(\epsilon r_{\uparrow/\downarrow\pm}^2 + \frac{k_x}{\omega}r_{\uparrow/\downarrow\pm} + \pm f - \epsilon k_x^2\big)^{-1} \tag{0.6}$$

where $\pm f$ refers to $\uparrow / \downarrow$.

Case 1: $k_x = \omega$. One has $r_{\uparrow+} \geqslant 0$ for all k_x and $r_{\uparrow-} \geqslant 0$ for $|k_x| \leqslant k_0 = \sqrt{f/\nu}$, so that $v_\uparrow(y) = B_\uparrow \mathrm{e}^{r_{\uparrow-}y}$ for $|k_x| > k_0$ and vanishes otherwise. On the lower half-plane $r_{\downarrow+} > 0$ and $r_{\downarrow-} < 0$ for all k_x so that $v_\downarrow(y) = A_\downarrow \mathrm{e}^{r_{\downarrow+}y}$. The gluing condition (**??**) implies $B_\uparrow = A_\downarrow = 0$ so that $v \equiv 0$, which is forbidden by assumption (equivalently we are back to the Kelvin solution). There is no extra interface mode in that case.

Case 2: $k_x = -\omega$. One has $r_{\uparrow-} < 0$ for all k_x and $r_{\uparrow+} < 0$ for $|k_x| < k_0$ whereas $r_{\downarrow-} < 0$ and $r_{\downarrow+} > 0$ for all k_x, so that

$$v_\uparrow(y) = A_\uparrow \mathrm{e}^{r_{\uparrow+}y} + B_\uparrow \mathrm{e}^{r_{\uparrow-}y} \tag{0.7}$$

$$v_\downarrow(y) = A_\downarrow \mathrm{e}^{r_{\downarrow+}y} \tag{0.8}$$

with $A_\uparrow = 0$ for $|k_x| \geqslant k_0$. Similarly

$$u_\uparrow(y) = \alpha_{\uparrow+}A_\uparrow \mathrm{e}^{r_{\uparrow+}y} + \alpha_{\uparrow-}B_\uparrow \mathrm{e}^{r_{\uparrow-}y} + C_\uparrow \mathrm{e}^{q_{\uparrow+}y} \tag{0.9}$$

$$u_\downarrow(y) = \alpha_{\downarrow+}A_\downarrow \mathrm{e}^{r_{\downarrow+}y} + D_\downarrow \mathrm{e}^{q_{\downarrow-}y} \tag{0.10}$$

with $C_\uparrow = 0$ for $|k_x| \leqslant k_0$ since $q_{\uparrow +} \geqslant 0$ in that case, see (**??**). For $|k_x| \geqslant k_0$ the gluing condition (**??**) implies $B_\uparrow = A_\downarrow = 0$ so that $v \equiv 0$ which is forbidden by assumption. For $|k_x| < k_0$ the four free parameter $A_\uparrow, B_\uparrow, A_\downarrow$ and $D_\downarrow$ are constrained by the four gluing conditions (**??**). There exist a non-trivial solution only if $\det N(k_x, -k_x) = 0$ with

$$N = \begin{pmatrix} 1 & 1 & -1 & 0 \\ r_{\uparrow +} & r_{\uparrow -} & -s_{\downarrow +} & 0 \\ \alpha_{\uparrow +} & \alpha_{\uparrow -} & -\alpha_{\downarrow +} & -1 \\ \alpha_{\uparrow +} r_{\uparrow +} & \alpha_{\uparrow -} r_{\uparrow -} & -\alpha_{\downarrow +} s_{\downarrow +} & -q_{\downarrow -} \end{pmatrix} \tag{0.11}$$

One can check numerically that along $(k_x, -k_x)$ this occurs only twice, and exactly at the crossing with the Yanai wave dispersion relation. This ensures the continuity of the latter and confirms that there is no other mode in that case.