Supplementary Materials to : "An inviscid analysis of the Prandtl azimuthal mass-transport during swirl-type sloshing"

Odd M. Faltinsen[1]† and Alexander N. Timokha[1,2]

[1]Centre for Autonomous Marine Operations and Systems & Department of Marine Technology, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway

[2]Institute of Mathematics, National Academy of Sciences of Ukraine, 01004 Kiev, Ukraine

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Appendix A. Explicit form of the Craik-Leibovich equation

Explicit form of the continuity equation (3.2) in cylindrical coordinate system can be written as

$$\partial \Omega_1^E + \partial \Omega_3^E = -\frac{1}{r}\left(\Omega_1^E + \partial \Omega_1^E\right). \tag{A 1}$$

Using (A 1) and definition of the Stokes drift (3.9) utilise the derivation line

$$\begin{aligned}
\nabla \times \left[\boldsymbol{w}^S \times \boldsymbol{\Omega}^E\right] &= \nabla \times \left[\hat{\boldsymbol{r}}\left(w\Omega_2^E\right) - \hat{\boldsymbol{z}}\left(w\Omega_1^E\right)\right] = -\frac{1}{r}w\left[\hat{\boldsymbol{r}}\partial_\theta \Omega_1^E + \hat{\boldsymbol{z}}\partial_\theta \Omega_3^E\right] \\
&+ \hat{\boldsymbol{\theta}}\left[\partial_z\left(w\Omega_3^E\right) + \partial_r\left(w\Omega_1^E\right)\right] = -\Xi e^{2z}\frac{1}{r}w^S\left[\hat{\boldsymbol{r}}\partial_\theta \Omega_1^E + \hat{\boldsymbol{z}}\partial_\theta \Omega_3^E\right] \\
&+ \hat{\boldsymbol{\theta}}\left[2w\Omega_3^E + w\left(\partial_z \Omega_3^E + \partial_r \Omega_1^E\right) + \Omega_1^E \partial_r w\right] \\
&= -\Xi e^{2z}\left(\frac{1}{r}w^S\,\partial_\theta \boldsymbol{\Omega}^E - \hat{\boldsymbol{\theta}}\left[\left((w^S)' - \frac{1}{r}w^S\right)\Omega_1^E + 2w^S\,\Omega_3^E\right]\right).
\end{aligned}$$

Adopting the Bessel function definition $J_1'' = -r^{-1}J_1' - J_1 + r^{-2}J_1$ and employing the Maplesoft analytical manipulator derive

$$(w^S)' - \frac{1}{r}w^S = -\frac{3}{r}\left[\frac{1}{r}\left(J_1' - \frac{J_1}{r}\right)^2\right] = -\frac{3}{r}\left[2w^S - w^s\right].$$

Appendix B. Nonlinear (Eulerian-mean) boundary-layer effect

The boundary layer analysis starts with adopting, as in the main paper body, the characteristic length r_0/k and time $1/\sigma$. This mathematically computes the corresponding Reynolds number $Re_s = (r_0^2\sigma)/(\nu k) \gg 1$, where ν is the kinematic viscosity. Transition and turbulent flow is neglected that implies an upper bound for applicable Reynolds numbers (Eq. (6.2.3) in Faltinsen & Timokha 2009).

† Email address for correspondence: odd.faltinsen@ntnu.no

B.1. *Nonlinear boundary-layer equations*

We consider, in parallel way, the non-dimensional inviscid ambient velocity field $\boldsymbol{v}(r,\theta,z,t) = v_1\hat{\boldsymbol{r}} + v_2\hat{\boldsymbol{\theta}} + v_3\hat{\boldsymbol{z}}$ containing the non-zero mean-flow component $\boldsymbol{w}^E(r,\theta,z) = \langle \boldsymbol{v}\rangle = w_1^E\hat{\boldsymbol{r}} + w_2^E\hat{\boldsymbol{\theta}} + w_3^E\hat{\boldsymbol{z}}$, which is associated with (generally, unknown *a priori*) steady streaming, and the non-dimensional velocity field $\boldsymbol{V}(r,\theta,z,t) = U\hat{\boldsymbol{r}} + V\hat{\boldsymbol{\theta}} + W\hat{\boldsymbol{z}}$, which is affected by the viscous boundary layer at the vertical wall. We need also the ambient pressure field p and P, which is associated with the viscous velocity field, forgetting on the first stage that these are the same in the lowest-order approximation (this fact will be shown later, mathematically).

The viscous flow is governed by the continuity equation

$$(rU)_r + V_\theta + r\,W_z = 0, \tag{B 1}$$

and the nondimensional Navier–Stokes equation

$$\dot{U} + UU_r + \frac{VU_\theta}{r} - \frac{V^2}{r} + WU_z = -P_r + \delta^2\left[\frac{(rU_r)_r}{r} - \frac{U}{r^2} + \frac{U_{\theta\theta}}{r^2} - \frac{2V_\theta}{r^2} + U_{zz}\right] + \cos\theta\ddot{\eta}_1 + \sin\theta\ddot{\eta}_2, \tag{B 2a}$$

$$\dot{V} + UV_r + \frac{VV_\theta}{r} + \frac{UV}{r} + WV_z = -\frac{P_\theta}{r} + \delta^2\left[\frac{(rV_r)_r}{r} - \frac{V}{r^2} + \frac{V_{\theta\theta}}{r^2} + \frac{2U_\theta}{r^2} + V_{zz}\right] - \sin\theta\ddot{\eta}_1 + \cos\theta\ddot{\eta}_2, \tag{B 2b}$$

$$\dot{W} + UW_r + \frac{VW_\theta}{r} + WW_z = -P_z + \delta^2\left[\frac{r(W_r)_r}{r} + \frac{W_{\theta\theta}}{r^2} + W_{zz}\right], \tag{B 2c}$$

where $\delta = \sqrt{1/Re_s}$ is an asymptotic measure of the boundary layer thickness δ at the vertical wall and, therefore, δ is small parameter, which is assumed be smaller than the forcing amplitude (2.5).

The viscous-flow velocity field must satisfy the *no-slip condition* at the wall and tend to the inviscid velocity field (*including* the steady streaming component $\boldsymbol{w}^E$) away from the boundary layer. These two conditions can mathematically be formalised as

$$\boldsymbol{V} = \boldsymbol{0} \ \text{ at } \ r = k \quad \text{and} \quad ||\boldsymbol{V} - \boldsymbol{v}|| = O(\delta) \ \text{ as } \ k - r \gg O(\delta). \tag{B 3}$$

The forthcoming asymptotic derivations will be done in terms of the *differences*

$$\boldsymbol{\mathcal{V}} = \boldsymbol{V} - \boldsymbol{v} = (R, \varTheta, Z) = (U-u, V-v, W-w) \text{ and } \mathcal{P} = P - p, \tag{B 4}$$

between viscous and ambient flow parameters.

Because the ambient flow satisfies (B 1), (B 2) *with* $\delta = 0$, the governing equations for the differences take the form

$$(rR)_r + \varTheta_\theta + r\,Z_z = 0, \tag{B 5a}$$

$$\dot{R} + RR_r + \frac{\varTheta R_\theta}{r} - \frac{\varTheta^2}{r} + ZR_z + [uR_r + Ru_r] + \frac{1}{r}[vR_\theta + \varTheta u_\theta] - \frac{2\varTheta v}{r} + [Zu_z + wR_z] = -\mathcal{P}_r + \delta^2\left[\frac{(rR_r)_r}{r} - \frac{R}{r^2} + \frac{R_{\theta\theta}}{r^2} - \frac{2\varTheta_\theta}{r^2} + R_{zz}\right] + \delta^2\left[\frac{(ru_r)_r}{r} - \frac{u}{r^2} + \frac{u_{\theta\theta}}{r^2} - \frac{2v_\theta}{r^2} + w_{zz}\right], \tag{B 5b}$$

$$\dot{\Theta}+R\Theta_r+\frac{\Theta\Theta_\theta}{r}+\frac{R\Theta}{r}+Z\Theta_z+[u\Theta_r+Rv_r]+\frac{1}{r}[v\Theta_\theta+\Theta v_\theta]+\frac{1}{r}[u\Theta+Rv]+[w\Theta_z+Zv_z]=$$
$$-\frac{\mathcal{P}_\theta}{r}+\delta^2\left[\frac{(r\Theta_r)_r}{r}-\frac{\Theta}{r^2}+\frac{\Theta_{\theta\theta}}{r^2}+\frac{2R_\theta}{r^2}+\Theta_{zz}\right]+\delta^2\left[\frac{(rv_r)_r}{r}-\frac{v}{r^2}+\frac{v_{\theta\theta}}{r^2}+\frac{2u_\theta}{r^2}+v_{zz}\right], \tag{B 5c}$$

$$\dot{Z}+RZ_r+\frac{\Theta Z_\theta}{r}+ZZ_z+[uZ_r+Rw_r]+\frac{1}{r}[vZ_\theta+\Theta w_\theta]+[wZ_z+Zw_z]$$
$$=-\mathcal{P}_z+\delta^2\left[\frac{r(Z_r)_r}{r}+\frac{Z_{\theta\theta}}{r^2}+Z_{zz}\right]+\delta^2\left[\frac{r(w_r)_r}{r}+\frac{w_{\theta\theta}}{r^2}+w_{zz}\right]. \tag{B 5d}$$

The no-slip condition (B 3) transforms to

$$\boldsymbol{\mathcal{V}}=(R,\Theta,Z)=\boldsymbol{V}-\boldsymbol{v}=(-v,-u,-w)\quad\text{at}\quad r=k, \tag{B 6}$$

but the closeness of $\boldsymbol{V}$ and $\boldsymbol{v}$ far from the boundary layer will be rewritten as

$$||\boldsymbol{\mathcal{V}}||=O(\delta)\quad\text{for}\quad(k-r)\gg O(\delta);\quad 0\leqslant r<k. \tag{B 7}$$

To clarify what is the mathematical infinity for the closed domain, we introduce the boundary-layer spatial variable ξ as

$$r=k-\delta\xi,\quad 0<\xi=O(1) \tag{B 8}$$

and consider the differences as functions of ξ,t and θ,z, i.e., $R=R(\xi,t;\theta,z),\Theta=\Theta(\xi,t;\theta,z),Z=Z(\xi,t;\theta,z)$ and $\mathcal{P}=\mathcal{P}(\xi,t;\theta,z)$. Furthermore, we look for the asymptotic solution of (B 5)-(B 7)

$$R=\delta R_1+...,\ \Theta=\Theta_0+\delta\Theta_1+...,\ Z=Z_0+\delta Z_1+...,\ \mathcal{P}=\delta\mathcal{P}_1+\dots. \tag{B 9}$$

One must note that $R_0=0$ because the normal velocity is zero at $r=k$, but the zero-order pressure difference $\mathcal{P}_0=0$ (the ambient pressure is continuous through the boundary layer) is according to (B 5b) rewritten in the $\xi,t;\ \theta,z$ coordinates.

Utilising the rule $(\cdot)_\xi=-\delta(\cdot)_r$ for R, Θ and Z and keeping only the $O(1)$ terms derive

$$R_{1\xi}=Z_{0z}+\Theta_{0\theta}/k \tag{B 10}$$

from (B 5a), but (B 5c) and (B 5d) transform to the two equations

$$\dot{\Theta}_0-\Theta_{0\xi\xi}-R_1\Theta_{0\xi}+\frac{\Theta_0\Theta_{0\theta}}{k}+Z_0\Theta_{0z}+\xi\bar{u}_r\Theta_{0\xi}+\frac{1}{k}[\bar{v}\Theta_{0\theta}+\bar{v}_\theta\Theta_0]+[\bar{w}\Theta_{0z}+\bar{v}_zZ_0]=0, \tag{B 11a}$$

$$\dot{Z}_0-Z_{0\xi\xi}-R_1Z_{0\xi}+\frac{\Theta_0Z_{0\theta}}{k}+Z_0Z_{0z}+\xi\bar{u}_rZ_{0\xi}+\frac{1}{k}[\bar{v}Z_{0\theta}+\bar{w}_\theta\Theta_0]+[\bar{w}Z_{0z}+\bar{w}_zZ_0]=0, \tag{B 11b}$$

in which the bars denote projections of the vector-function $\boldsymbol{v}$, and its derivatives, on the wall (these are simply expanded in a Taylor series by δ) so that all coefficients in (B 11) become the known time-depending functions, which parametrically depend on θ and z,

$$\bar{u}_r(t;\theta,z)=u_r(k,\theta,z,t),\ \bar{v}(t;\theta,z)=v(k,\theta,z,t),\ \bar{v}_\theta(t;\theta,z)=v_\theta(k,\theta,z,t),$$
$$\bar{w}(t;\theta,z)=w(k,\theta,z,t),\ \bar{v}_z(t;\theta,z)=w_z(k,\theta,z,t).$$

Eqs. (B 4), (B 5) are, in fact, nonlinear boundary-layer equations, which are written in terms of the *differences* between viscous and inviscid (including steady streaming) components. According to (B 6), the solution of these 'difference field' equations (B 11) satisfies the *inhomogeneous* boundary conditions

$$R_1 = 0, \quad \Theta_0 = -\bar{v}, \quad Z_0 = -\bar{w} \ \text{ at } \ \xi = 0 \tag{B 12}$$

where the right-hand side is the minus projection of tangential components of the inviscid ambient flow.

Because Θ_0, $Z_0 = O(1)$, but R_1 corresponds to the first-order approximation in (B 9), the asymptotic condition (B 7) transforms to the form

$$|\Theta_0| + |Z_0| \to 0 \text{ and } |R_1| \to O(1) \text{ as } \ \xi \to +\infty. \tag{B 13}$$

B.2. *Asymptotic solution of the nonlinear boundary-layer problem*

The steady-state wave solution by Faltinsen *et al.* (2016) implies an asymptotic representation of the inviscid (ambient) velocity field by the small parameter $O(\epsilon^{1/3})$ where the lowest-order component takes the form (2.3) but the second-order approximation includes the steady-streaming component and is defined by (3.3). Because the nonlinear boundary-layer problem (B 10)–(B 13) governs the $O(1)$ approximation on the $O(\delta)$ scale and the asymptotic condition (2.5) is satisfied, one can consider an asymptotic approximation in terms of $O(\epsilon^{1/3})$ as follows

$$\begin{aligned} &\Theta_0 = \Theta_0^{(1/3)} + \Theta_0^{(2/3)} + O(\epsilon)..., \quad Z_0 = Z_0^{(1/3)} + Z_0^{(2/3)} + O(\epsilon)..., \\ &R_1 = R_1^{(1/3)} + R_1^{(2/3)} + O(\epsilon)... \end{aligned} \tag{B 14}$$

for $\xi > 0$, $-\infty < t < \infty$ and $z < 0$, $-\pi \leqslant \theta < \pi$.

B.2.1. *The $O(\epsilon^{1/3})$ component*

Taking (2.3) derives that the first-order approximation of (B 10)–(B 12) comes from the *linear* parabolic problems ($\xi > 0$, $-\infty < t < \infty$):

$$\dot{\Theta}_0^{(1/3)} - \Theta_{0\xi\xi}^{(1/3)} = 0, \ \Theta_0^{(1/3)}(0, t; \theta, z) = -\frac{J_1(k)}{k} e^z \left[\cos t\, \theta_c'(\theta) + \sin t\, \theta_s'(\theta)\right], \tag{B 15a}$$

$$\dot{Z}_0^{(1/3)} - Z_{0\xi\xi}^{(1/3)} = 0, \ \ Z_0^{(1/3)}(0, t; \theta, z) = -J_1(k)\, e^z \left[\cos t\, \theta_c(\theta) + \sin t\, \theta_s(\theta)\right], \tag{B 15b}$$

which consists of the two independent linear Stokes boundary-layer equations (Batchelor 2000) parametrically dependent on $z < 0$, $-\pi \leqslant \theta < \pi$. The exact time-periodic solution of (B 15) reads, according to § 3.1.1 in Polyanin & Nasaikinskii (2015), as

$$\Theta_0^{(1/3)}(\xi, t; \theta, z) = -\frac{J_1(k)}{k}\, e^{z-\alpha\xi} \left[\theta_c'(\theta)\cos(t-\alpha\xi) + \theta_s'(\theta)\sin(t-\alpha\xi)\right], \tag{B 16a}$$

$$Z_0^{(1/3)}(\xi, t; \theta, z) = -J_1(k)\, e^{z-\alpha\xi} \left[\theta_c(\theta)\cos(t-\alpha\xi) + \theta_s(\theta)\sin(t-\alpha\xi)\right], \tag{B 16b}$$

($\alpha = 1/\sqrt{2}$), where the $e^{-\alpha\xi}$-multiplier corresponds to $e^{-\alpha(k-r)/\delta}$ in the original nondimensional (r, θ, z)-coordinates, which exponentially decays and becomes small as $(k-r) = O(1)$.

Substituting (B 16) into the continuity equation (B 10) and using the first boundary condition of (B 12) gives

$$\begin{aligned} R_1^{(1/3)}(\xi, t; \theta, z) = \int_0^\xi (Z_{0z}^{(1/3)} + \Theta_{0\theta}^{(1/3)}/k)\, d\xi = -\frac{1}{2\alpha} J_1(k)\, e^z \left(1 - \frac{1}{k^2}\right) \\ \times \Big\{\theta_c(\theta)\big[\sin t + \cos t - e^{-\alpha\xi}(\sin(t-\alpha\xi) + \cos(t-\alpha\xi))\big] \end{aligned}$$

$$+\,\theta_s(\theta)\big[\sin t-\cot t-e^{-\alpha\xi}(\sin(t-\alpha\xi)-\cos(t-\alpha\xi))\big]\Big\}. \quad (\mathrm{B}\,17)$$

One can see that $|R_1^{(1/3)}|\to O(\epsilon^{1/3})$ and $\Theta_0^{(1/3)}\sim Z_0^{(1/3)}\to 0$ as $\xi\to+\infty$, in what follows, the asymptotic condition (B 13) is automatically satisfied.

Eqs. (B 16), (B 17) present the well-known solution of the linear boundary-layer problem given in term of the *differences* between viscous and inviscid velocity fields. To restore the viscous velocity field $\boldsymbol{V}$, one should take this solution, the lowest-order inviscid flow component (2.3), substitute $\xi=(k-r)/\delta$. This gives

$$U^{(1/3)}=v_1^{(1/3)},\ V^{(1/3)}=v_2^{(1/3)}+\Theta_0^{(1/3)}((k-r)/\delta,t;\theta,z),$$
$$W^{(1/3)}=v_3^{(1/3)}+Z_0^{(1/3)}((k-r)/\delta,t;\theta,z).$$

This time-periodic solution is zero on the tank surface and rapidly converges to $\boldsymbol{v}^{(1/3)}$ away from the boundary layer. It does not contain a steady-flow component, which is expected in the second-order approximation.

B.2.2. *The $O(\epsilon^{2/3})$ component; steady streaming*

Inserting (B 16) and (B 17) into (B 11) leads to the inhomogeneous parabolic equations with respect to $\Theta_0^{(2/3)}$ and $Z_0^{(2/3)}$

$$\dot\Theta_0^{(2/3)}-\Theta_{0\xi\xi}^{(2/3)}=R_1^{(1/3)}\Theta_{0\xi}^{(1/3)}-\frac{\Theta_0^{(1/3)}\Theta_{0\theta}^{(1/3)}}{k}-Z_0^{(1/3)}\Theta_{0z}^{(1/3)}-\xi\bar u_r^{(1/3)}\Theta_{0\xi}^{(1/3)}$$
$$-\frac{1}{k}\left[\bar v^{(1/3)}\Theta_{0\theta}^{(1/3)}+\bar v_\theta^{(1/3)}\Theta_0^{(1/3)}\right]-\left[\bar w^{(1/3)}\Theta_{0z}^{(1/3)}+\bar v_z^{(1/3)}Z_0^{(1/3)}\right], \quad (\mathrm{B}\,18a)$$

$$\dot Z_0^{(2/3)}-Z_{0\xi\xi}^{(2/3)}=R_1^{(1/3)}Z_{0\xi}^{(1/3)}-\frac{\Theta_0^{(1/3)}Z_{0\theta}^{(1/3)}}{k}-Z_0^{(1/3)}Z_{0z}^{(1/3)}-\xi\bar u_r^{(1/3)}Z_{0\xi}^{(1/3)}$$
$$-\frac{1}{k}\left[\bar v^{(1/3)}Z_{0\theta}^{(1/3)}+\bar w_\theta^{(1/3)}\Theta_0^{(1/3)}\right]-\left[\bar w^{(1/3)}Z_{0z}^{(1/3)}+\bar w_z^{(1/3)}Z_0^{(1/3)}\right], \quad (\mathrm{B}\,18b)$$

where the right-hand sides are explicitly-given functions.

Because the steady streaming effect is formally included into the ambient flow, the time-periodic solution of (B 18) (governing the differences!) should *obligatory* decay at the infinity,

$$\Theta_0^{(2/3)}(\xi,t;\theta,z)\to 0\ \text{ and }\ Z_0^{(2/3)}(\xi,t;\theta,z)\to 0\ \text{ as }\ \xi\to+\infty. \quad (\mathrm{B}\,19)$$

In addition, the second-order differences should satisfy the boundary conditions (B 12) $\Rightarrow$

$$\Theta_0^{(2/3)}(0,t;\theta,z)=-\bar v^{(2/3)}, \quad (\mathrm{B}\,20a)$$

$$Z_0^{(2/3)}(0,t;\theta,z)=-\bar w^{(2/3)}, \quad (\mathrm{B}\,20b)$$

where, according to (3.3), the right-hand sides include, formally, the *time-independent projections* of $\boldsymbol{w}^E=O(\epsilon^{2/3})$ on $r=k$.

When solving the *linear inhomogeneous* problem (B 18)-(B 20) with respect to the unknowns $\Theta_0^{(2/3)}$ and $Z^{(2/3)}$, we should distinguish the time-independent (steady) quantities as well as the $\cos 2t$ and $\sin 2t$ harmonics.

Huge derivations show that a unique solution exists for the $\cos 2t$ and $\sin 2t$ components. This means that these components of the difference field exist only in the boundary layer but vanish away from it.

However, considering the time-averaged (time-independent, steady) difference field yields a requirement on $\boldsymbol{w}^E$. Indeed, derivations show that the time-averaged component of (B 18), (B 19) (*without* the boundary conditions (B 20)!!!) has the following unique solution

$$\left\langle \Theta_0^{(2/3)} \right\rangle (\xi;\theta,z) = \frac{(k^2-1)J_1^2(k)}{4k^3\alpha^2} e^{-\alpha\xi+2z} \Big\{ (ab-\bar{a}\bar{b}) \\ \times \left[-\tfrac{1}{2}e^{-\alpha\xi} + (\alpha\xi-1)\sin\alpha\xi + (\alpha\xi+2)\cos\alpha\xi \right] \\ + \left[\tfrac{1}{2}e^{-\alpha\xi} + (\alpha\xi+4)\sin\alpha\xi + (1-\alpha\xi)\cos(\alpha\xi) \right] \\ \times \left[(a\bar{b}+\bar{a}b)\cos 2\theta + \tfrac{1}{2}(b^2+\bar{b}^2-a^2-\bar{a}^2)\sin 2\theta \right] \Big\}, \quad \text{(B 21}a\text{)}$$

$$\left\langle Z_0^{(2/3)} \right\rangle (\xi;\theta,z) = \frac{J_1^2(k)}{8k^2\alpha^2}\, e^{-\alpha\xi+2z} \Big\{ \tfrac{1}{2}(a^2+\bar{a}^2+b^2+\bar{b}^2) \\ \times \left[(k^2+1)e^{-\alpha\xi} + 2[k^2(\alpha\xi+4)-\alpha\xi]\sin\alpha\xi - 2(k^2-1)(\alpha\xi-1)\cos\alpha\xi \right] \\ + \left[\tfrac{1}{2}(a^2+\bar{a}^2-b^2-\bar{b}^2)\cos 2\theta + (a\bar{b}+\bar{a}b)\sin 2\theta \right] \\ \times (k^2-1)\left(e^{-\alpha\xi} + 2(\alpha\xi+4)\sin\alpha\xi + 2(1-\alpha\xi)\cos\alpha\xi \right). \quad \text{(B 21}b\text{)}$$

The time-averaging in the *remaining* boundary conditions (B 20) transforms them to

$$\left\langle \Theta_0^{(2/3)}(0,t;\theta,z) \right\rangle = -w_2^E(k,\theta,z), \quad \left\langle Z_0^{(2/3)}(0,t;\theta,z) \right\rangle = -w_3^E(k,\theta,z). \quad \text{(B 22)}$$

Using (B 21) with $\xi = 0$ derives from (B 22) the tangential boundary condition for the mean Eulerian $\boldsymbol{w}^E$:

$$w_2^E(k,\theta,z) = -\left\langle \Theta_0^{(2/3)} \right\rangle (0;\theta,z) = -\frac{3(k^2-1)J_1^2(k)}{4k^3} e^{2z} \Big\{ \boxed{\underbrace{(ab-\bar{a}\bar{b})}_{\Xi}} \\ + (a\bar{b}+\bar{a}b)\cos 2\theta + \tfrac{1}{2}(b^2+\bar{b}^2-a^2-\bar{a}^2)\sin 2\theta \Big\}, \quad \text{(B 23}a\text{)}$$

$$w_3^E(k,\theta,z) = -\left\langle Z_0^{(2/3)} \right\rangle (0;\theta,z) = -\frac{J_1^2(k)}{4k^2}\, e^{2z} \Big\{ \boxed{\tfrac{1}{2}(a^2+\bar{a}^2+b^2+\bar{b}^2)(3k^2-1)} \\ + 3(k^2-1)\left[\tfrac{1}{2}(a^2+\bar{a}^2-b^2-\bar{b}^2)\cos 2\theta + (a\bar{b}+\bar{a}b)\sin 2\theta \right] \Big\} \quad \text{(B 23}b\text{)}$$

on the vertical wall.

Appendix C. The Stokes drift in a rectangular channel

For the incompressible wave flow $\mathbf{v}$, the first-order Lagrangian displacement is $\mathbf{d} = \int \mathbf{v} dt$; it is also solenoidal. The Stokes drift velocity (in the second-order approximation) equals to (see, equation (3.9))

$$\mathbf{w}^S = \frac{1}{2}\nabla \times \langle \mathbf{v} \times \mathbf{d} \rangle . \quad \text{(C 1)}$$

Assume that $\mathbf{v}$ implies a three-dimensional progressive wave in the Oy direction in a rectangular channel confined by the vertical walls at $x = \pm a$, the bottom $z = -h$, and the mean free surface at $z = 0$. We consider the cross-sectional plane at $y = 0$, which intersects the time-changing two-dimensional cross-sectional area $C(t)$ confined by the solid part (walls and bottom, γ_0), and the free-surface curve $\gamma(t)$ by $z = \zeta(x,t)$. Within the framework of the first-order (linear) approximation, ζ, due to the linear kinematic boundary condition, is linked with the first-order Lagrangian displacements $\mathbf{d}$ as follows

$$z = \zeta(x,t) = d_3(x,0,0,t) \tag{C 2}$$

(fluid particles are kept on the free surface).

We assume that $\mathbf{w}^S$, $C(t)$, and boundaries γ_0 and $\gamma(t)$ satisfy assumptions of the Stokes integration theorem. Keeping only quadratic terms and taking into account that normal velocities (and Lagrange displacements) are zero on the solid parts (walls and bottom) gives

$$\begin{aligned} M^S &= \int_{\langle C\rangle} \frac{1}{2}\nabla \times \langle \mathbf{v}\times\mathbf{d}\rangle \cdot \hat{\mathbf{y}}\mathrm{d}y\mathrm{d}z = \int_{-a}^{a} \frac{1}{2}\left\langle (v_2 d_3 - v_3 d_2)|_{z=0,y=0}\right\rangle \mathrm{d}x \\ &= \int_{-a}^{a} \left\langle (v_2 d_3)|_{z=0,y=0}\right\rangle \mathrm{d}x = \int_{-a}^{a} \left\langle (v_2)|_{z=0,y=0}\zeta\right\rangle \mathrm{d}x = \left\langle \int_{-a}^{a}\int_{-h}^{\zeta} v_2|_{y=0}\mathrm{d}z\mathrm{d}x \right\rangle. \end{aligned} \tag{C 3}$$

The backward reading of the derivation line (C 3) shows that, in the second-order approximation, the mass-flux through the plane $y = 0$ due to moving free surface is the same as the Stokes mass-transport. The latter fact may be violated if there is an inflow/outflow though the vertical walls as in § 4, i.e. the first-order horizontal Lagrangian displacements are not zero, e.g., at $x = -a$. The cross-displacements cause an extra non-zero quantity of the non-Eulerian mean nature in (C 3) as it happened in (4.6).

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