Supplementary material for 'Tracking vortex surfaces frozen in the virtual velocity in non-ideal flows'

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1. Proofs of virtual conservation theorems

THEOREM 1 (VIRTUAL KELVIN'S THEOREM). *Let $\mathcal{C}(t)$ be a closed curve moving with a globally smooth virtual velocity $\boldsymbol{v}$. Then the circulation Γ is a virtual Lagrangian scalar as*

$$\frac{\mathrm{D}_v \Gamma}{\mathrm{D}_v t} = 0. \tag{1.1}$$

Proof. Expanding the virtual material derivative of Γ in (1.1) and considering the rate of change of a line element

$$\frac{\mathrm{D}_v \mathrm{d}\boldsymbol{l}}{\mathrm{D}_v t} = \mathrm{d}\boldsymbol{l} \cdot \boldsymbol{\nabla} \boldsymbol{v}, \tag{1.2}$$

we have

$$\frac{\mathrm{D}_v \Gamma}{\mathrm{D}_v t} = \oint_{\mathcal{C}(t)} \left[\frac{\partial \boldsymbol{u}}{\partial t} \cdot \mathrm{d}\boldsymbol{l} + (\boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{u}) \cdot \mathrm{d}\boldsymbol{l} + (\mathrm{d}\boldsymbol{l} \cdot \boldsymbol{\nabla} \boldsymbol{v}) \cdot \boldsymbol{u} \right]. \tag{1.3}$$

Applying vector identities to last two terms in the integrand yields

$$\frac{\mathrm{D}_v \Gamma}{\mathrm{D}_v t} = \oint_{\mathcal{C}(t)} \left[\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla}(\boldsymbol{v} \cdot \boldsymbol{u}) - \boldsymbol{v} \times \boldsymbol{\omega} \right] \cdot \mathrm{d}\boldsymbol{l}. \tag{1.4}$$

According to the Stokes theorem and vorticity transport equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{\omega}) = 0, \tag{1.5}$$

where $\boldsymbol{\omega}$ is transported in $\boldsymbol{v}$, we have

$$\frac{\mathrm{D}_v \Gamma}{\mathrm{D}_v t} = \int_{\mathcal{S}(t)} \left[\frac{\partial \boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{\omega}) \right] \cdot \boldsymbol{n} \, \mathrm{d}\mathcal{S} = 0, \tag{1.6}$$

where $\mathcal{S}(t)$ is a surface bounded by $\mathcal{C}(t)$ with the surface normal $\boldsymbol{n}$. □

THEOREM 2 (VIRTUAL HELMHOLTZ'S THEOREM). *(1) Let a vortex tube move with a globally smooth virtual velocity, and then the vorticity flux*

$$\Phi = \int_{\mathcal{S}} \boldsymbol{\omega} \cdot \boldsymbol{n} \, \mathrm{d}\mathcal{S} \tag{1.7}$$

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through cross section $\mathcal{S}$ of the vortex tube is a virtual Lagrangian scalar as

$$\frac{\mathrm{D}_v \varPhi}{\mathrm{D}_v t} = 0. \tag{1.8}$$

(2) Let a vortex line move with a globally smooth virtual velocity, and then the line elements moving with the virtual velocity lying on the vortex line at some instant continue to lie on that vortex line, i.e.

$$\frac{\mathrm{D}_v}{\mathrm{D}_v t}\left(\frac{\boldsymbol{\omega}}{\rho} \times \delta \boldsymbol{l}\right) = 0 \tag{1.9}$$

is satisfied with the initial condition

$$\frac{\boldsymbol{\omega}}{\rho} \times \delta \boldsymbol{l} = 0. \tag{1.10}$$

Proof. (1) We apply Stokes' theorem to (1.1) in virtual Kelvin's theorem and then obtain

$$\frac{\mathrm{D}_v \varPhi}{\mathrm{D}_v t} = \frac{\mathrm{D}_v}{\mathrm{D}_v t}\int_{\mathcal{S}} \boldsymbol{\omega}\cdot\boldsymbol{n}\,\mathrm{d}\mathcal{S} = \frac{\mathrm{D}_v}{\mathrm{D}_v t}\oint_{\mathcal{C}} \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{l} = 0. \tag{1.11}$$

(2) First we expand the left hand side (l.h.s.) of (1.9) as

$$\frac{\mathrm{D}_v}{\mathrm{D}_v t}\left(\frac{\boldsymbol{\omega}}{\rho} \times \delta \boldsymbol{l}\right) = \frac{\mathrm{D}_v}{\mathrm{D}_v t}\left(\frac{\boldsymbol{\omega}}{\rho}\right) \times \delta \boldsymbol{l} + \frac{\boldsymbol{\omega}}{\rho} \times \frac{\mathrm{D}_v(\delta \boldsymbol{l})}{\mathrm{D}_v t}. \tag{1.12}$$

Then we derive the virtual transport equation for $\boldsymbol{\omega}/\rho$

$$\frac{\mathrm{D}_v}{\mathrm{D}_v t}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \frac{1}{\rho}\frac{\mathrm{D}_v \boldsymbol{\omega}}{\mathrm{D}_v t} - \frac{\boldsymbol{\omega}}{\rho^2}\frac{\mathrm{D}_v \rho}{\mathrm{D}_v t}, \tag{1.13}$$

and re-express (1.5) as the virtual transport equation for vorticity

$$\frac{\mathrm{D}_v \boldsymbol{\omega}}{\mathrm{D}_v t} = \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\boldsymbol{v} - (\boldsymbol{\nabla}\cdot\boldsymbol{v})\boldsymbol{\omega}. \tag{1.14}$$

Substituting (1.14) and the continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{u}) = 0 \tag{1.15}$$

into the first and second terms on the r.h.s of (1.13), respectively, we obtain

$$\frac{\mathrm{D}_v}{\mathrm{D}_v t}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \frac{\boldsymbol{\omega}}{\rho}\cdot\boldsymbol{\nabla}\boldsymbol{v} + \chi\frac{\boldsymbol{\omega}}{\rho} \tag{1.16}$$

with a scalar

$$\chi \equiv -\frac{1}{\rho}\boldsymbol{v}_d\cdot\boldsymbol{\nabla}\rho - \boldsymbol{\nabla}\cdot\boldsymbol{v}_d. \tag{1.17}$$

Then substituting (1.16) and (1.2) into the r.h.s. of (1.12) yields

$$\frac{\mathrm{D}_v}{\mathrm{D}_v t}\left(\frac{\boldsymbol{\omega}}{\rho} \times \delta \boldsymbol{l}\right) = \left(\frac{\boldsymbol{\omega}}{\rho}\cdot\boldsymbol{\nabla}\boldsymbol{v}\right) \times \delta \boldsymbol{l} + \frac{\boldsymbol{\omega}}{\rho} \times (\delta \boldsymbol{l}\cdot\boldsymbol{\nabla}\boldsymbol{v}) + \chi\left(\frac{\boldsymbol{\omega}}{\rho} \times \delta \boldsymbol{l}\right). \tag{1.18}$$

At the initial time $t = t_0$, a virtual material line element $\delta \boldsymbol{l}$ governed by (1.2) coincides with the local vector $\boldsymbol{\omega}/\rho$. The initial condition (1.10) is equivalent to $\varrho\delta\boldsymbol{l}_0 = \boldsymbol{\omega}_0/\rho_0$ with a scalar ϱ, so the first two terms on the r.h.s. of (1.18) cancel each other at $t = t_0$.

In addition, (1.10) implies that the last term on the r.h.s. of (1.18) is also vanishing at $t = t_0$. Thus (1.18) is reduced to

$$\frac{\mathrm{D}_v}{\mathrm{D}_v t}\left(\frac{\boldsymbol{\omega}}{\rho} \times \delta \boldsymbol{l}\right) = 0 \tag{1.19}$$

at $t = t_0$ with the initial condition (1.10). Finally, (1.9) is valid because all the terms on the r.h.s. of (1.18) are always vanishing with $(\boldsymbol{\omega}/\rho) \times \delta \boldsymbol{l} = 0$. □

THEOREM 3 (VIRTUAL ERTEL'S THEOREM). *Let ϕ be a virtual Lagrangian scalar convected by a globally smooth virtual velocity as*

$$\frac{\mathrm{D}_v \phi}{\mathrm{D}_v t} = 0 \tag{1.20}$$

with

$$\frac{\boldsymbol{\omega}}{\rho} \cdot \boldsymbol{\nabla} \phi = 0 \tag{1.21}$$

at the initial time, and then $(\boldsymbol{\omega}/\rho) \cdot \boldsymbol{\nabla}\phi$ is also a virtual Lagrangian scalar as

$$\frac{\mathrm{D}_v}{\mathrm{D}_v t}\left(\frac{\boldsymbol{\omega}}{\rho} \cdot \boldsymbol{\nabla} \phi\right) = 0. \tag{1.22}$$

Proof. First we expand the l.h.s. of (1.22) as

$$\frac{\mathrm{D}_v}{\mathrm{D}_v t}\left(\frac{\boldsymbol{\omega}}{\rho} \cdot \boldsymbol{\nabla} \phi\right) = \boldsymbol{\nabla}\phi \cdot \frac{\mathrm{D}_v}{\mathrm{D}_v t}\left(\frac{\boldsymbol{\omega}}{\rho}\right) + \frac{\boldsymbol{\omega}}{\rho} \cdot \frac{\mathrm{D}_v(\boldsymbol{\nabla}\phi)}{\mathrm{D}_v t}. \tag{1.23}$$

Taking the gradient of (1.20), we have

$$\frac{\mathrm{D}_v(\boldsymbol{\nabla}\phi)}{\mathrm{D}_v t} = \boldsymbol{\nabla}\left(\frac{\mathrm{D}_v \phi}{\mathrm{D}_v t}\right) - \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\boldsymbol{v} - \boldsymbol{\nabla}\phi \times (\boldsymbol{\nabla} \times \boldsymbol{v}). \tag{1.24}$$

Substituting (1.16) and (1.24) into (1.23) yields

$$\frac{\mathrm{D}_v}{\mathrm{D}_v t}\left(\frac{\boldsymbol{\omega}}{\rho} \cdot \boldsymbol{\nabla} \phi\right) = \chi\left(\frac{\boldsymbol{\omega}}{\rho} \cdot \boldsymbol{\nabla} \phi\right) = 0, \tag{1.25}$$

where the r.h.s. vanishes when (1.21) is satisfied at the initial time. □

THEOREM 4 (VIRTUAL CONSERVATION OF HELICITY). *Let a volume $\mathcal{V}(t)$ enclosed by a vortex surface move with a smooth virtual velocity, and then the helicity*

$$H \equiv \int_{\mathcal{V}} \boldsymbol{\omega} \cdot \boldsymbol{u} \, d\mathcal{V} \tag{1.26}$$

is a virtual Lagrangian scalar as

$$\frac{\mathrm{D}_v H}{\mathrm{D}_v t} = 0. \tag{1.27}$$

Proof. First we derive the virtual transport equation for the helicity density as

$$\frac{\mathrm{D}_v h}{\mathrm{D}_v t} = \boldsymbol{\omega} \cdot \frac{\mathrm{D}_v \boldsymbol{u}}{\mathrm{D}_v t} + \boldsymbol{u} \cdot \frac{\mathrm{D}_v \boldsymbol{\omega}}{\mathrm{D}_v t}. \tag{1.28}$$

Substituting

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = \boldsymbol{\nabla} \mathit{\Pi} + \boldsymbol{F} \tag{1.29}$$

and (1.14) into (1.28) and expanding the r.h.s. of (1.28), we have

$$\frac{\mathrm{D}_v h}{\mathrm{D}_v t} = (\boldsymbol{\nabla} \mathit{\Pi} + \boldsymbol{F} - \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{u}) \cdot \boldsymbol{\omega} + [\boldsymbol{\omega} \cdot \boldsymbol{\nabla} \boldsymbol{v} - (\boldsymbol{\nabla} \cdot \boldsymbol{v})\boldsymbol{\omega}] \cdot \boldsymbol{u}. \tag{1.30}$$

Substituting the constraint of the virtual velocity

$$\boldsymbol{v}_d \times \boldsymbol{\omega} = \boldsymbol{F} + \boldsymbol{\nabla} \mathit{\Psi} \tag{1.31}$$

into (1.30) and rearranging terms on the r.h.s. of (1.30), we have

$$\frac{\mathrm{D}_v h}{\mathrm{D}_v t} = -\boldsymbol{\nabla} \left(\mathit{\Pi} + \mathit{\Psi} + \frac{u^2}{2} \right) \cdot \boldsymbol{\omega} + (\boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{\nabla} \boldsymbol{v} \cdot \boldsymbol{u}) \cdot \boldsymbol{\omega} - (\boldsymbol{\nabla} \cdot \boldsymbol{v}) h \tag{1.32}$$

Applying vector identities to the r.h.s. of (1.32) yields

$$\frac{\mathrm{D}_v h}{\mathrm{D}_v t} = \boldsymbol{\nabla} \cdot \left[\left(\mathit{\Pi} + \mathit{\Psi} + \frac{u^2}{2} - \boldsymbol{u} \cdot \boldsymbol{v} \right) \boldsymbol{\omega} \right] - (\boldsymbol{\nabla} \cdot \boldsymbol{v}) h. \tag{1.33}$$

Applying the Reynolds transport theorem to (1.26) yields

$$\frac{\mathrm{D}_v H}{\mathrm{D}_v t} = \int_{\mathcal{V}(t)} \left[\frac{\mathrm{D}_v h}{\mathrm{D}_v t} + (\boldsymbol{\nabla} \cdot \boldsymbol{v})\, h \right] \mathrm{d}\mathcal{V}. \tag{1.34}$$

Then substituting (1.33) into (1.34) yields

$$\frac{\mathrm{D}_v H}{\mathrm{D}_v t} = \int_{\mathcal{V}(t)} \boldsymbol{\nabla} \cdot \left[\left(\mathit{\Pi} + \mathit{\Psi} + \frac{u^2}{2} - \boldsymbol{u} \cdot \boldsymbol{v} \right) \boldsymbol{\omega} \right] \mathrm{d}\mathcal{V}, \tag{1.35}$$

From the divergence theorem, we have

$$\frac{\mathrm{D}_v H}{\mathrm{D}_v t} = \int_{\mathcal{S}(t)} \left[\left(\mathit{\Pi} + \mathit{\Psi} + \frac{u^2}{2} - \boldsymbol{u} \cdot \boldsymbol{v} \right) \boldsymbol{\omega} \right] \cdot \boldsymbol{n}\, \mathrm{d}\mathcal{S} = 0, \tag{1.36}$$

where $\boldsymbol{n}$ denotes the surface normal and the boundary $\mathcal{S}(t)$ of $\mathcal{V}(t)$ is a vortex surface with $\boldsymbol{\omega} \cdot \boldsymbol{n} = 0$. $\square$