Internal wave energy flux from density perturbations in nonlinear stratifications - Supplementary material: Evaluation of the tanh profile Green's function

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1. Cancellation of Legendre functions

This section is included as an appendix to briefly outline the steps that were necessary to numerically evaluate the analytic Green's function for the tanh N^2 case. Direct evaluation of the expression as given in the main body of the paper is not possible because it is not suited for it; machine precision issues make it unusable. The Green's function is given by

$$\bar{G}(y,y') = \frac{-1}{D_T\,W}\begin{cases}\Big(\mathit{\Phi}_2 P_\nu^\mu(y') + \mathit{\Pi}_2 Q_\nu^\mu(y')\Big)\Big(\mathit{\Phi}_1 P_\nu^\mu(y) + \mathit{\Pi}_1 Q_\nu^\mu(y)\Big), & y < y' \\ \Big(\mathit{\Phi}_1 P_\nu^\mu(y') + \mathit{\Pi}_1 Q_\nu^\mu(y')\Big)\Big(\mathit{\Phi}_2 P_\nu^\mu(y) + \mathit{\Pi}_2 Q_\nu^\mu(y)\Big), & y > y', \end{cases} \tag{1.1}$$

where

$$D_T = -\begin{vmatrix}\mathit{\Pi}_1 & \mathit{\Pi}_2 \\ \mathit{\Phi}_1 & \mathit{\Phi}_2\end{vmatrix}, \qquad W = 2^{2\mu}\frac{\Gamma(\frac{\nu+\mu+2}{2})\Gamma(\frac{\nu+\mu+1}{2})}{\Gamma(\frac{\nu-\mu+2}{2})\Gamma(\frac{\nu-\mu+1}{2})}, \tag{1.2}$$

$$\mathit{\Pi}_{1,2} = \frac{dP_\nu^\mu}{dy}(y_{0,h}) - \frac{N_{1,2}^2}{2\,g\,(1-y_{0,h}^2)}P_\nu^\mu(y_{0,h}), \tag{1.3}$$

and

$$\mathit{\Phi}_{1,2} = -\frac{dQ_\nu^\mu}{dy}(y_{0,h}) + \frac{N_{1,2}^2}{2\,g\,(1-y_{0,h}^2)}Q_\nu^\mu(y_{0,h}). \tag{1.4}$$

However, the Green's function in this form is unsuitable for direct numerical evaluation because of the factors $1-y_{0,h}^2$ in (1.3) and (1.4). Because y_0 and y_h are extremely close to -1 and 1, respectively, $1-y_0^2$ and $1-y_h^2$ are extremely close to 0. Thus we must

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somehow remove these factors from the expression. We note that we can use the following recurrence relation (Abramowitz & Stegun 8.5.4),

$$\frac{dP^\mu_\nu}{dy}(y) = \frac{(\nu+\mu)P^\mu_{\nu-1}(y) - \nu\, y\, P^\mu_\nu(y)}{(1-y^2)}, \tag{1.5}$$

which works for both P and Q, to remove the derivative and combine the terms in (1.3) and (1.4) to give

$$\Pi_{1,2} = \frac{(\nu+\mu)P^\mu_{\nu-1}(y_{0,h}) - (\nu\, y_{0,h} + N^2_{1,2}/2g)\, P^\mu_\nu(y_{0,h})}{(1-y^2_{0,h})}, \tag{1.6}$$

$$\Phi_{1,2} = \frac{-(\nu+\mu)Q^\mu_{\nu-1}(y_{0,h}) + (\nu\, y_{0,h} + N^2_{1,2}/2g)\, Q^\mu_\nu(y_{0,h})}{(1-y^2_{0,h})}. \tag{1.7}$$

We can then define the following to rescale our parameters,

$$\Pi_{1,2} = \frac{1}{(1-y^2_{0,h})}\pi_{1,2} \qquad \Phi_{1,2} = \frac{1}{(1-y^2_{0,h})}\varphi_{1,2}, \tag{1.8}$$

which means that then the denominator rescales like the following,

$$D_T = \begin{vmatrix} \Pi_1 & \Pi_2 \\ \Phi_1 & \Phi_2 \end{vmatrix} = \frac{1}{(1-y_0^2)(1-y_h^2)} \begin{vmatrix} \pi_1 & \pi_2 \\ \varphi_1 & \varphi_2 \end{vmatrix} = \frac{d}{(1-y_0^2)(1-y_h^2)}. \tag{1.9}$$

Then the near-zero factors $(1-y_0^2)(1-y_h^2)$ can be cancelled from the numerator and denominator of the Green's function (1.1) to give:

$$\bar{G}(z,z') = \frac{-1}{d\,W} \begin{cases} \Big(\varphi_2 P^\mu_\nu(y') + \pi_2 Q^\mu_\nu(y')\Big)\Big(\varphi_1 P^\mu_\nu(y) + \pi_1 Q^\mu_\nu(y)\Big), & y < y' \\ \Big(\varphi_1 P^\mu_\nu(y') + \pi_1 Q^\mu_\nu(y')\Big)\Big(\varphi_2 P^\mu_\nu(y) + \pi_2 Q^\mu_\nu(y)\Big), & y > y'. \end{cases} \tag{1.10}$$

However, the expression is still not suited for numerical computation because P and Q are being evaluated at coordinates where the terms become really large. To this end, we will express Q in terms of P and cancel some terms in

$$\varphi_1 P^\mu_\nu(y) + \pi_1 Q^\mu_\nu(y). \tag{1.11}$$

Using Gradshteyn and Ryzhik 8.705,

$$Q^\mu_\nu(y) = \frac{\pi}{2\sin(\mu\pi)}\left[P^\mu_\nu(y)\cos(\mu\pi) - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}P^{-\mu}_\nu(y)\right], \tag{1.12}$$

we can express (1.11) in terms of just P:

$$\begin{aligned}
&P_\nu^\mu(y)\varphi_1 + Q_\nu^\mu(y)\pi_1 \\
&\quad = P_\nu^\mu(y)\left[-(\nu+\mu)Q_{\nu-1}^\mu(y_0) + \left(\nu\, y_0 + \frac{N_1^2}{2g}\right) Q_\nu^\mu(y_0)\right] \\
&\qquad + Q_\nu^\mu(y)\left[(\nu+\mu)P_{\nu-1}^\mu(y_0) - \left(\nu\, y_0 + \frac{N_1^2}{2g}\right) P_\nu^\mu(y_0)\right] && (1.13)\\
&\quad = -P_\nu^\mu(y)\frac{(\nu+\mu)\pi}{2\sin(\mu\pi)}\left[P_{\nu-1}^\mu(y_0)\cos(\mu\pi) - \frac{\Gamma(\nu+\mu)}{\Gamma(\nu-\mu)}P_{\nu-1}^{-\mu}(y_0)\right] \\
&\qquad + P_\nu^\mu(y)\,\frac{\left(\nu\, y_0 + \frac{N_1^2}{2g}\right)\pi}{2\sin(\mu\pi)}\left[P_\nu^\mu(y_0)\cos(\mu\pi) - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}P_\nu^{-\mu}(y_0)\right] \\
&\qquad + \frac{(\nu+\mu)\pi}{2\sin(\mu\pi)}\left[P_\nu^\mu(y)\cos(\mu\pi) - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}P_\nu^{-\mu}(y)\right]P_{\nu-1}^\mu(y_0) \\
&\qquad - \frac{\left(\nu\, y_0 + \frac{N_1^2}{2g}\right)\pi}{2\sin(\mu\pi)}\left[P_\nu^\mu(y)\cos(\mu\pi) - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}P_\nu^{-\mu}(y)\right]P_\nu^\mu(y_0). && (1.14)
\end{aligned}$$

In (1.14), the first and fifth, and the third and seventh terms cancel. Then it becomes

$$\begin{aligned}
&P_\nu^\mu(y)\varphi_1 + Q_\nu^\mu(y)\pi_1 \\
&\quad = \frac{(\nu+\mu)\pi}{2\sin(\mu\pi)}\left[\frac{\Gamma(\nu+\mu)}{\Gamma(\nu-\mu)}P_\nu^\mu(y)P_{\nu-1}^{-\mu}(y_0) - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}P_{\nu-1}^\mu(y_0)P_\nu^{-\mu}(y)\right] \\
&\qquad + \frac{\left(\nu\, y_0 + \frac{N_1^2}{2g}\right)\pi}{2\sin(\mu\pi)}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\left[P_\nu^\mu(y_0)P_\nu^{-\mu}(y) - P_\nu^\mu(y)P_\nu^{-\mu}(y_0)\right]. && (1.15)
\end{aligned}$$

The same cancellations occur for $P_\nu^\mu(y)\varphi_2 + Q_\nu^\mu(y)\pi_2$. However, there are still very large terms present in the expression.

2. Exponential expansion

We can see the exponential behavior if we do the appropriate transformations. We express the P's in terms of the Gauss hypergeometric function ${}_2F_1$ and get the explicit exponential behavior by undoing our coordinate transform,

$$y(z) = \tanh(\alpha(z-z_t)) = \frac{1-e^{-2\alpha(z-z_t)}}{1+e^{-2\alpha(z-z_t)}} = \frac{e^{2\alpha(z-z_t)}-1}{e^{2\alpha(z-z_t)}+1}. \tag{2.1}$$

If $z \sim h$, then we have

$$P_\nu^\mu(y) = \frac{1}{\Gamma(1-\mu)}\left(\frac{1+y}{1-y}\right)^{\mu/2} F\left(-\nu, \nu+1; 1-\mu; \frac{1-y}{2}\right), \tag{2.2}$$

$$P_\nu^\mu(y) = \frac{e^{\mu\alpha(z-z_t)}}{\Gamma(1-\mu)}F\left(-\nu, \nu+1; 1-\mu; f^+(z)\right), \tag{2.3}$$

where

$$f^+(z) = \frac{1}{1+e^{2\alpha(z-z_t)}} \approx 0. \tag{2.4}$$

We note that "${}_2F_1$" is now being written as "F" for brevity. The hypergeometric function evaluated at this point converges to 1. However, if $z \sim 0$, the hypergeometric function tends to diverge and we need to use a linear transformation formula (Abramowitz & Stegun 15.3.6) to shift the argument and bring the divergent behavior out into an exponential factor. This gives

$$P_\nu^\mu(y) = \frac{\left(\frac{1+y}{1-y}\right)^{\mu/2}}{\Gamma(1-\mu)} F\left(-\nu, \nu+1; 1-\mu; \frac{1-y}{2}\right) \tag{2.5}$$

$$\begin{aligned} &= \frac{\left(\frac{1+y}{1-y}\right)^{\mu/2}}{\Gamma(1-\mu)} \left[\frac{\Gamma(1-\mu)\Gamma(-\mu)}{\Gamma(1-\mu+\nu)\Gamma(-\mu-\nu)} F\left(-\nu, \nu+1; 1+\mu; \frac{1+y}{2}\right) \right. \\ &\quad \left. + \left(\frac{1+y}{2}\right)^{-\mu} \frac{\Gamma(1-\mu)\Gamma(\mu)}{\Gamma(-\nu)\Gamma(\nu+1)} F\left(1-\mu+\nu, -\mu-\nu; 1-\mu; \frac{1+y}{2}\right) \right] \end{aligned} \tag{2.6}$$

$$\begin{aligned} P_\nu^\mu(y) =& \frac{\Gamma(-\mu) e^{\mu\alpha(z-z_t)}}{\Gamma(1-\mu+\nu)\Gamma(-\mu-\nu)} F(-\nu, \nu+1; 1+\mu; f^-(z)) \\ &+ \frac{\Gamma(\mu)(e^{\alpha(z-z_t)} + e^{-\alpha(z-z_t)})^\mu}{\Gamma(-\nu)\Gamma(\nu+1)} F(1-\mu+\nu, -\mu-\nu; 1-\mu; f^-(z)), \end{aligned} \tag{2.7}$$

where

$$f^-(z) = \frac{1}{1+e^{-2\alpha(z-z_t)}} \approx 0. \tag{2.8}$$

Then, using all this, we can get the exact exponential behavior of our terms that look like $P_\nu^\mu(y)\varphi_{1,2} + Q_\nu^\mu(y)\pi_{1,2}$. For $z \approx h$ we have

$$\begin{aligned} P_\nu^\mu(y)\varphi_1 + Q_\nu^\mu(y)\pi_1 =& a_1 e^{\alpha\mu z} + a_2(e^{-\alpha z} + e^{-\alpha(z-2z_t)})^{-\mu} \\ &+ a_3 e^{-\alpha\mu z} + a_4(e^{-\alpha z} + e^{-\alpha(z-2z_t)})^\mu, \end{aligned} \tag{2.9}$$

$$P_\nu^\mu(y)\varphi_2 + Q_\nu^\mu(y)\pi_2 = b_1 e^{-\alpha\mu(z-h)} + b_2 e^{\alpha\mu(z-h)}. \tag{2.10}$$

Here, the first terms are the largest, and the various factors like a_1 are comprised of gamma functions and hypergeometric functions. For $z \approx 0$ we have

$$\begin{aligned} P_\nu^\mu(y)\varphi_1 + Q_\nu^\mu(y)\pi_1 =& c_1 (e^{\alpha z} + e^{-\alpha(z-2z_t)})^\mu + c_2 (e^{-\alpha z} + e^{-\alpha(z-2z_t)})^\mu \\ &+ c_3 \left(\frac{e^{-\alpha z_t} + e^{\alpha z_t}}{e^{\alpha(z-z_t)} + e^{-\alpha(z-z_t)}}\right)^\mu + c_4 \left(\frac{e^{\alpha(z-z_t)} + e^{-\alpha(z-z_t)}}{e^{-\alpha z_t} + e^{\alpha z_t}}\right)^\mu \\ &+ c_5\, e^{\alpha\mu z} + c_6\, e^{-\alpha\mu z} \\ &+ c_7 (e^{-\alpha z} + e^{-\alpha(z-2z_t)})^{-\mu} + c_8 (e^{\alpha z} + e^{-\alpha(z-2z_t)})^{-\mu}, \end{aligned} \tag{2.11}$$

$$\begin{aligned} P_\nu^\mu(y)\varphi_2 + Q_\nu^\mu(y)\pi_2 =& d_1\, e^{-\alpha\mu(z-h)} + d_2 (e^{\alpha(z-h)} + e^{-\alpha(z+h-2z_t)})^{-\mu} \\ &+ d_3\, e^{\alpha\mu(z-h)} + d_4 (e^{\alpha(z-h)} + e^{-\alpha(z+h-2z_t)})^\mu. \end{aligned} \tag{2.12}$$

For (2.11), the first two terms are extremely large, have opposite sign, and have almost the same magnitude, giving a result that is extremely small that produces errors because of the limit of machine precision. Thus for numerical evaluation we take c_1 and c_2 to be zero. The determinant looks like the following:

$$\begin{aligned} D_T =& m_1\, e^{\alpha\mu h} + m_2 (e^{-\alpha h} + e^{-\alpha(h-2z_t)})^{-\mu} \\ &+ m_3\, e^{-\alpha\mu h} + m_4 (e^{-\alpha(h)} + e^{-\alpha(h-2z_t)})^\mu. \end{aligned} \tag{2.13}$$

This means we can divide out $e^{\alpha\mu h}$ from both the numerator and denominator of the Green's function to reduce term sizes. Also, different combinations of (2.9) - (2.12) should be used depending on what values z and z' take. For instance, if $z \approx 0$ and $z' = z + \epsilon$, then we will multiply (2.12) evaluated at z' with (2.11) evaulated at z. However, if $z \approx h$ and $z' \approx 0$, then we will multiply (2.10) evaluated at z with (2.11) evaluated at z'. The following values for a_i, b_i, c_i, d_i, and m_i have been simplified using the following expressions from Abramowitz & Stegun 6.1.15 and 6.1.17,

$$\Gamma(x+1) = x\,\Gamma(x), \tag{2.14}$$

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \tag{2.15}$$

Also, to save space we simplify the notation:

$$\gamma_{1,2} = \nu y_{0,h} + \frac{N_{1,2}^2}{2\,g}, \tag{2.16}$$

$$F(a,b;c;f^{\pm}(z)) = F^{a,b}_{c,\pm}(z), \tag{2.17}$$

$$\delta_{\pm} = \nu \pm \mu. \tag{2.18}$$

$$a_1 = \Gamma^2(\mu)\frac{\sin\delta_-\pi}{2\pi}\left[\delta_+ F^{1-\nu,\nu}_{1-\mu,-}(0) + \gamma_1 F^{-\nu,1+\nu}_{1-\mu,-}(0)\right] F^{-\nu,1+\nu}_{1-\mu,+}(z) \tag{2.19}$$

$$a_2 = \frac{\Gamma(\delta_+)}{\Gamma(\delta_-)}\frac{\sin\nu\pi}{2\sin\mu\pi}\delta_+\left[\frac{-1}{\mu}F^{\delta_+,1-\delta_-}_{1+\mu,-}(0) + \frac{\gamma_1}{\delta_-}F^{1+\delta_+,-\delta_-}_{1+\mu,-}(0)\right] F^{-\nu,1+\nu}_{1-\mu,+}(z) \tag{2.20}$$

$$a_3 = \frac{\Gamma^2(\delta_+)}{\Gamma^2(\delta_-)\Gamma^2(\mu)}\frac{\pi\sin\delta_+\pi}{2\sin^2\mu\pi}\frac{\delta_+^2}{\mu^2\delta_-}\left[F^{1-\nu,\nu}_{1+\mu,-}(0) + \frac{\gamma_1}{\delta_-}F^{-\nu,1+\nu}_{1+\mu,-}(0)\right] F^{-\nu,1+\nu}_{1+\mu,+}(z) \tag{2.21}$$

$$a_4 = \frac{\Gamma(\delta_+)}{\Gamma(\delta_-)}\frac{\sin\nu\pi}{2\sin\mu\pi}\frac{\delta_+}{\mu\delta_-}\left[-\delta_+ F^{\delta_-,1-\delta_+}_{1-\mu,-}(0) - \gamma_1 F^{1+\delta_-,-\delta_+}_{1-\mu,-}(0)\right] F^{-\nu,1+\nu}_{1+\mu,+}(z) \tag{2.22}$$

$$b_1 = \frac{\Gamma(\delta_+)}{\Gamma(\delta_-)}\frac{\delta_+}{2\mu\delta_-}\left[-\delta_+ F^{1-\nu,\nu}_{1-\mu,+}(h) + \gamma_2 F^{-\nu,1+\nu}_{1-\mu,+}(h)\right] F^{-\nu,1+\nu}_{1+\mu,+}(z) \tag{2.23}$$

$$b_2 = \frac{\Gamma(\delta_+)}{\Gamma(\delta_-)}\frac{\delta_+}{2\mu}\left[F^{1-\nu,\nu}_{1+\mu,+}(h) - \frac{\gamma_2}{\delta_-}F^{-\nu,1+\nu}_{1+\mu,+}(h)\right] F^{-\nu,1+\nu}_{1-\mu,+}(z) \tag{2.24}$$

$$c_1 = \Gamma^2(\mu)\frac{\sin\delta_-\pi\sin\nu\pi}{2\pi\sin\mu\pi}\left[-\delta_+ F^{1-\nu,\nu}_{1-\mu,-}(0) - \gamma_1 F^{-\nu,1+\nu}_{1-\mu,-}(0)\right] F^{1+\delta_-,-\delta_+}_{1-\mu,-}(z) \tag{2.25}$$

$$c_2 = \Gamma^2(\mu)\frac{\sin\delta_-\pi\sin\nu\pi}{2\pi\sin\mu\pi}\left[\delta_+ F^{\delta_-,1-\delta_+}_{1-\mu,-}(0) + \gamma_1 F^{1+\delta_-,-\delta_+}_{1-\mu,-}(0)\right] F^{-\nu,1+\nu}_{1-\mu,-}(z) \tag{2.26}$$

$$c_3 = \frac{\Gamma(\delta_+)}{\Gamma(\delta_-)}\frac{\sin^2\nu\pi}{2\sin^2\mu\pi}\frac{\delta_+}{\mu\delta_-}\left[-\delta_+ F^{\delta_-,1-\delta_+}_{1-\mu,-}(0) - \gamma_1 F^{1+\delta_-,-\delta_+}_{1-\mu,-}(0)\right] F^{1+\delta_+,-\delta_-}_{1+\mu,-}(z) \tag{2.27}$$

$$c_4 = \frac{\Gamma(\delta_+)}{\Gamma(\delta_-)}\frac{\sin^2\nu\pi}{2\sin^2\mu\pi}\frac{\delta_+}{\mu}\left[F^{\delta_+,1-\delta_-}_{1+\mu,-}(0) + \frac{\gamma_1}{\delta_-}F^{1+\delta_+,-\delta_-}_{1+\mu,-}(0)\right] F^{1+\delta_-,-\delta_+}_{1-\mu,-}(z) \tag{2.28}$$

$$c_5 = \frac{\Gamma(\delta_+)}{\Gamma(\delta_-)}\frac{\sin\delta_-\pi\sin\delta_+\pi}{2\sin^2\mu\pi}\frac{\delta_+}{\mu\delta_-}\left[\delta_+ F^{1-\nu,\nu}_{1-\mu,-}(0) + \gamma_1 F^{-\nu,1+\nu}_{1-\mu,-}(0)\right] F^{-\nu,1+\nu}_{1+\mu,-}(z) \tag{2.29}$$

$$c_6 = \frac{\Gamma(\delta_+)}{\Gamma(\delta_-)} \frac{\sin\delta_-\pi \sin\delta_+\pi}{2\sin^2\mu\pi} \frac{\delta_+}{\mu} \left[-F^{1-\nu,\nu}_{1+\mu,-}(0) - \frac{\gamma_1}{\delta_-} F^{-\nu,1+\nu}_{1+\mu,-}(0) \right] F^{-\nu,1+\nu}_{1-\mu,-}(z) \tag{2.30}$$

$$c_7 = \frac{\Gamma^2(\delta_+)}{\Gamma^2(\delta_-)\Gamma^2(\mu)} \frac{\pi\sin\delta_+\pi \sin\nu\pi}{2\sin^3\mu\pi} \frac{\delta_+^2}{\mu^2\delta_-} \times \left[-F^{\delta_+,1-\delta_-}_{1+\mu,-}(0) - \frac{\gamma_1}{\delta_-} F^{1+\delta_+,-\delta_-}_{1+\mu,-}(0) \right] F^{-\nu,1+\nu}_{1+\mu,-}(z) \tag{2.31}$$

$$c_8 = \frac{\Gamma^2(\delta_+)}{\Gamma^2(\delta_-)\Gamma^2(\mu)} \frac{\pi\sin\delta_+\pi \sin\nu\pi}{2\sin^3\mu\pi} \frac{\delta_+^2}{\mu^2\delta_-} \times \left[F^{1-\nu,\nu}_{1+\mu,-}(0) + \frac{\gamma_1}{\delta_-} F^{-\nu,1+\nu}_{1+\mu,-}(0) \right] F^{1+\delta_+,-\delta_-}_{1+\mu,-}(z) \tag{2.32}$$

$$d_1 = \Gamma^2(\mu) \frac{\sin\delta_-\pi}{2\pi} \left[\delta_+ F^{1-\nu,\nu}_{1-\mu,+}(h) - \gamma_2 F^{-\nu,1+\nu}_{1-\mu,+}(h) \right] F^{-\nu,1+\nu}_{1-\mu,-}(z) \tag{2.33}$$

$$d_2 = \frac{\Gamma(\delta_+)}{\Gamma(\delta_-)} \frac{\sin\nu\pi}{2\sin\mu\pi} \frac{\delta_+}{\mu\delta_-} \left[-\delta_+ F^{1-\nu,\nu}_{1-\mu,+}(h) + \gamma_2 F^{-\nu,1+\nu}_{1-\mu,+}(h) \right] F^{1+\delta_+,-\delta_-}_{1+\mu,-}(z) \tag{2.34}$$

$$d_3 = \frac{\Gamma^2(\delta_+)}{\Gamma^2(\delta_-)\Gamma^2(\mu)} \frac{\pi\sin\delta_+\pi}{2\sin^2\mu\pi} \frac{\delta_+^2}{\mu^2\delta_-} \left[F^{1-\nu,\nu}_{1+\mu,+}(h) - \frac{\gamma_2}{\delta_-} F^{-\nu,1+\nu}_{1-\mu,+}(h) \right] F^{-\nu,1+\nu}_{1+\mu,-}(z) \tag{2.35}$$

$$d_4 = \frac{\Gamma(\delta_+)}{\Gamma(\delta_-)} \frac{\sin\nu\pi}{2\sin\mu\pi} \frac{\delta_+}{\mu} \left[-F^{1-\nu,\nu}_{1+\mu,+}(h) + \frac{\gamma_2}{\delta_-} F^{-\nu,1+\nu}_{1-\mu,+}(h) \right] F^{1+\delta_-,-\delta_+}_{1-\mu,-}(z) \tag{2.36}$$

$$m_1 = \Gamma^2(\mu) \frac{\sin\delta_-\pi}{2\pi} \delta_+ \left[\frac{\Gamma(\delta_+)}{\Gamma(\delta_-)} \delta_+ F^{1-\nu,\nu}_{1-\mu,-}(0)\, F^{1-\nu,\nu}_{1-\mu,+}(h) - \gamma_2 F^{1-\nu,\nu}_{1-\mu,-}(0)\, F^{-\nu,1+\nu}_{1-\mu,+}(h) \right. \\ \left. + \gamma_1 F^{-\nu,1+\nu}_{1-\mu,-}(0)\, F^{1-\nu,\nu}_{1-\mu,+}(h) + \frac{\Gamma(\delta_+)}{\Gamma(\delta_-)} \frac{\mu\gamma_1\gamma_2}{\delta_-} F^{-\nu,1+\nu}_{1-\mu,-}(0)\, F^{-\nu,1+\nu}_{1-\mu,+}(h) \right] \tag{2.37}$$

$$m_2 = \frac{\Gamma(\delta_+)}{\Gamma(\delta_-)} \frac{\sin\nu\pi}{2\sin\mu\pi} \frac{\delta_+}{\mu} \left[-\delta_+ F^{\delta_+,1-\delta_-}_{1+\mu,-}(0)\, F^{1-\nu,\nu}_{1-\mu,+}(h) + \gamma_2 F^{\delta_+,1-\delta_-}_{1+\mu,-}(0)\, F^{-\nu,1+\nu}_{1-\mu,+}(h) \right. \\ \left. - \frac{\delta_+\gamma_1}{\delta_-} F^{1+\delta_+,-\delta_-}_{1+\mu,-}(0)\, F^{1-\nu,\nu}_{1-\mu,+}(h) + \frac{\gamma_1\gamma_2}{\delta_-} F^{1+\delta_+,-\delta_-}_{1+\mu,-}(0)\, F^{-\nu,1+\nu}_{1-\mu,+}(h) \right] \tag{2.38}$$

$$m_3 = \frac{\Gamma^2(\delta_+)}{\Gamma^2(\delta_-)\Gamma^2(\mu)} \frac{\pi\sin\delta_+\pi}{2\sin^2\mu\pi} \frac{\delta_+^2}{\mu^2} \left[F^{1-\nu,\nu}_{1+\mu,-}(0)\, F^{1-\nu,\nu}_{1+\mu,+}(h) + \frac{\gamma_1}{\delta_-} F^{-\nu,1+\nu}_{1+\mu,-}(0)\, F^{1-\nu,\nu}_{1+\mu,+}(h) \right. \\ \left. - \frac{\gamma_2}{\delta_-} F^{1-\nu,\nu}_{1+\mu,-}(0)\, F^{-\nu,1+\nu}_{1+\mu,+}(h) - \frac{\gamma_1\gamma_2}{\delta_-^2} F^{-\nu,1+\nu}_{1+\mu,-}(0)\, F^{-\nu,1+\nu}_{1+\mu,+}(h) \right] \tag{2.39}$$

$$m_4 = \frac{\Gamma(\delta_+)}{\Gamma(\delta_-)} \frac{\sin\nu\pi}{2\sin\mu\pi} \frac{\delta_+}{\mu} \left[-\delta_+ F^{\delta_-,1-\delta_+}_{1-\mu,-}(0) F^{1-\nu,\nu}_{1+\mu,+}(h) - \gamma_1 F^{1+\delta_+,-\delta_+}_{1-\mu,-}(0) F^{1-\nu,\nu}_{1+\mu,+}(h) \right. \\ \left. + \frac{\delta_+\gamma_2}{\delta_-} F^{\delta_-,1-\delta_+}_{1-\mu,-}(0) F^{-\nu,1+\nu}_{1+\mu,+}(h) + \frac{\gamma_1\gamma_2}{\delta_-} F^{1+\delta_-,-\delta_+}_{1-\mu,-}(0) F^{-\nu,1+\nu}_{1+\mu,+}(h) \right] \tag{2.40}$$