# Supplementary Material

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## 1. Faraday waves on a cylindrical filament

The derivation provided in this section is a special case of the generalised derivation provided in section 2 of the manuscript, for a cylindrical fluid filament.

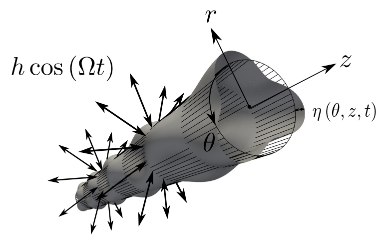



Figure 1: Cylindrical filament with radial forcing in both fluids. The surface indicated with lines is the unperturbed filament.

As shown in figure 1, the cylindrical filament of radius $R_0$ is subject to radial time periodic body force in both the fluids. For the unperturbed filament, the tangential coordinates are $x_{||}^{(1)} = z, \quad x_{||}^{(2)} = \theta$ and the normal coordinate is $x_\perp = r$. A time periodic body force $G(r,t)$ acts radially on the filament of radius $R_0$ and thus the radius of curvature in the base state $\mathcal{R} = R_0$. The interface separates two inviscid, immiscible, quiescent fluids of density $\rho^{\mathcal{I}}$ (inner fluid) and $\rho^{\mathcal{O}}$ with surface tension coefficient $T$. We study the (linear) stability of this time-periodic base state, by imposing an interfacial perturbation in the form of a standing wave at $t = 0$. The equation governing the amplitude $a(t)$ of the perturbation is obtained. All flow field related dependent variables (indicated with a tilde) can be written as base (with subscript b) plus perturbation with subsequent linearisation in the perturbed variables.

$$\tilde{\phi}^{\mathcal{I}}(r,z,\theta,t) = 0 + \phi^{\mathcal{I}}(r,z,\theta,t), \quad \tilde{p}^{\mathcal{I}}(r,z,\theta,t) = P_b^{\mathcal{I}}(r,t) + p^{\mathcal{I}}(r,z,\theta,t),$$
$$\tilde{\phi}^{\mathcal{O}}(r,z,\theta,t) = 0 + \phi^{\mathcal{O}}(r,z,\theta,t), \quad \tilde{p}^{\mathcal{O}}(r,z,\theta,t) = P_b^{\mathcal{O}}(r,t) + p^{\mathcal{O}}(r,z,\theta,t),$$

The equation of filament in the base state is $r = R_0$ and thus $x_\perp^0 = R_0$. The base state momentum equations in the (normal) radial direction are (base state velocity is zero in both fluids)

$$-\frac{\partial P_b^{\mathcal{I}}}{\partial r} + \rho^{\mathcal{I}} G(r,t) = 0, \quad -\frac{\partial P_b^{\mathcal{O}}}{\partial r} + \rho^{\mathcal{O}} G(r,t) = 0 \tag{1.1}$$

Integrating these equations along $r$ we obtain (one of the integration limits is $R_0$)

$$P_b^{\mathcal{I}}(r,t) = P_b^{\mathcal{I}}(R_0,t) - \rho^{\mathcal{I}} \int_r^{R_0} G(r',t)dr', \quad P_b^{\mathcal{O}}(r,t) = P_b^{\mathcal{O}}(R_0,t) + \rho^{\mathcal{O}} \int_{R_0}^{r} G(r',t)dr'. \tag{1.2}$$

Due to surface tension coefficient $T$, the pressure jump boundary condition across the interface in the base state is $P_b^{\mathcal{I}}(R_0,t) - P_b^{\mathcal{O}}(R_0,t) = T/R_0$. The perturbation velocity potential satisfies the Laplace equation

$$\nabla^2\phi^{\mathcal{I}} = \nabla^2\phi^{\mathcal{O}} = 0 \tag{1.3}$$

With $\eta(\theta,z,t)$ defined as the difference between the perturbed and the unperturbed surfaces (see fig. 1), The linearised kinematic boundary condition is

$$\frac{\partial\eta}{\partial t} = \left(\nabla\phi^{\mathcal{I}}\cdot\hat{\mathbf{e}}_r\right)_{r=R_0} = \left(\nabla\phi^{\mathcal{O}}\cdot\hat{\mathbf{e}}_r\right)_{r=R_0}, \tag{1.4}$$

where $\hat{\mathbf{e}}_r$ being the unit vector in the radial direction of the unperturbed filament. Using variable separability, the interfacial perturbation and the perturbation velocity potentials are chosen to have the form of standing wave (see Prosperetti (1976); Farsoiya *et al.* (2017) for similar forms in other coordinate systems)

$$\eta(z,\theta,t) = a(t)F(z,\theta) \tag{1.5}$$

$$\phi^{\mathcal{I}}(r,z,\theta,t) = L^{\mathcal{I}}(r)F(z,\theta)\dot{a}(t), \quad \phi^{\mathcal{O}}(r,z,\theta,t) = L^{\mathcal{O}}(r)F(z,\theta)\dot{a}(t) \tag{1.6}$$

where $F(z,\theta)$, $L^{\mathcal{I}}(r)$, $L^{\mathcal{O}}(r)$ and $a(t)$ are yet unknown functions of their argument. In cylindrical coordinates we can write the Laplacian operator as a sum of a horizontal ($\nabla^2_{||}$) and a vertical operator $\nabla^2_{\perp}$,

$$\nabla^2 = \nabla^2_{||} + \nabla^2_{\perp} \equiv \left(\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right). \tag{1.7}$$

$F(z,\theta)$ is chosen to satisfy the eigenvaue problem

$$\nabla^2_{||}F \equiv \left(\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}\right)F = \lambda F \tag{1.8}$$

It is easily verified that the solution to the eigenavalue problem 1.8 is given by $F(z,\theta) = \cos(m\theta)\cos(kz)$ and $\lambda = -\left(\frac{m^2}{r^2} + k^2\right)$. Substituting this form of $F$ in equation 1.6, using decomposition 1.7 in equation 1.3, we obtain equations for $L$. i.e.

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)L^{\mathcal{I}}(r) - \left(\frac{m^2}{r^2} + k^2\right)L^{\mathcal{I}}(r) = 0, \quad \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)L^{\mathcal{O}}(r) - \left(\frac{m^2}{r^2} + k^2\right)L^{\mathcal{O}}(r) = 0. \tag{1.9}$$

It can be verified that the solutions to equations 1.9 are (see expression 6.2.28 in Prosperetti (2011))

$$L^{\mathcal{I}}(r) = C^{\mathcal{I}}I_m(kr) + D^{\mathcal{I}}K_m(kr), \quad L^{\mathcal{O}}(r) = C^{\mathcal{O}}I_m(kr) + D^{\mathcal{O}}K_m(kr) \tag{1.10}$$

where $I_m(kr), K_m(kr)$ are the modified Bessel functions of $mth$ order, of the first and second kind respectively. We would like the solution to stay bounded at $r=\infty$ and $r=0$. This implies that (Weisstein 2017*b*,*c*) $L^{\mathcal{I}}(r) = C^{\mathcal{I}}I_m(kr)$ and $L^{\mathcal{O}}(r) = D^{\mathcal{O}}K_m(kr)$. In order to determine $C^{\mathcal{I}}$ and $D^{\mathcal{O}}$, we use this in equation 1.4 alongwith 1.6 to obtain ($\nabla\phi\cdot\hat{\mathbf{e}}_r$ in cylindrical coordinates is $\frac{\partial\phi}{\partial r}$),

$$1 = \left(\frac{dL^{\mathcal{I}}}{dr}\right)_{r=R_0} = \left(\frac{dL^{\mathcal{O}}}{dr}\right)_{r=R_0}, \tag{1.11}$$

using which we obtain

$$L^{\mathcal{I}}(r) = \frac{I_m(kr)}{kI'_m(kR_0)} \quad \text{and} \quad L^{\mathcal{O}}(r) = \frac{K_m(kr)}{kK'_m(kR_0)}. \tag{1.12}$$

In the presence of an interfacial perturbation, the difference of total pressure (base + perturbation) on either sides of the interface equals surface tension coefficient $T$ times the surface divergence of the unit normal at the perturbed interface (Prosperetti 1976, 1981),

$$\begin{aligned} &P_b^{\mathcal{I}}(R_0+\eta,t)+p^{\mathcal{I}}(\theta,z,R_0,t) \\ &-P_b^{\mathcal{O}}(R_0+\eta,t)-p^{\mathcal{O}}(\theta,z,R_0,t)=T\left(\boldsymbol{\nabla}\cdot\mathbf{q}\right)_{r=R_0+\eta}, \end{aligned} \tag{1.13}$$

where $\mathbf{q}$ is the unit normal to the *perturbed* cylinder and the perturbation pressure $p$ is evaluated at the unperturbed cylinder radius $r=R_0$ in order to retain terms only upto $O(a(0))$ only. In order to compute curvature at the perturbed interface we define (Bush 2013)

$$Q(\theta,z,r,t)\equiv r-R_0-\eta(\theta,z,t)=r-R_0-a(t)F(z,\theta) \tag{1.14}$$

At a linear approximation, the divergence of $\mathbf{q}$ equals the laplacian of $Q$ (see equations 5.40 and 5.41 in Bush (2013)). Thus from equation 1.14, expressions 1.5, decomposition 1.7 and 1.8 upto linear order in $a(0)$ we have,

$$\begin{aligned} \left(\boldsymbol{\nabla}\cdot\mathbf{q}\right)_{r=R_0+\eta}\approx\left(\nabla^2 Q\right)_{r=R_0+\eta}&=\chi(R_0+\eta)-a(t)\nabla^2_{||}F(z,\theta) \\ &\approx\chi(R_0)+\left(\frac{d\chi}{dr}\right)_{R_0}\eta(\theta,z,t)-a(t)\lambda(R_0)F(z,\theta). \end{aligned} \tag{1.15}$$

where $\chi(r)\equiv\nabla^2_{\perp}(r)$ and $\chi(R_0)$ is an $O(1)$ term arising from base state curvature with $\frac{1}{R_0}=\chi(R_0)$ through $\eta$. In writing the second line after equation 1.15, we have Taylor expanded $\chi(r)$ about $R_0$ retaining terms upto $\mathcal{O}(a(0))$. An expression for the difference of base pressures on the left hand side of equation 1.13 can be obtained using equation 1.2. After some simple algebraic manipulations, this is

$$P_b^{\mathcal{I}}(R_0+\eta,t)-P_b^{\mathcal{O}}(R_0+\eta,t)=\frac{T}{R_0}-\left(\rho^{\mathcal{I}}-\rho^{\mathcal{O}}\right)\int_{R_0+\eta}^{R_0}G(r',t)dr' \tag{1.16}$$

where we have used the base state pressure-jump condition to replace $P_b^{\mathcal{I}}(R_0,t)-P_b^{\mathcal{O}}(R_0,t)$ with $T/R_0$ (see below equation 1.2). Defining $G(r,t)\equiv\frac{\partial\Psi(r,t)}{\partial r}$ and using Taylor series expansion, we find

$$\begin{aligned} \int_{R_0+\eta}^{R_0}G(r',t)dr'=\Psi(R_0,t)-\Psi(R_0+\eta,t)&=-\left(\frac{\partial\Psi}{\partial r}\right)_{R_0}\eta(\theta,z,t)+\mathcal{O}(\eta^2) \\ &\approx -G(R_0,t)\eta(\theta,z,,t). \end{aligned} \tag{1.17}$$

Using 1.13 with 1.15, 1.16, 1.17 (after cancelling out the base state contribution to pressure using $\frac{1}{R_0}=\chi(R_0)$), we obtain,

$$\left(\rho^{\mathcal{I}}-\rho^{\mathcal{O}}\right)G(R_0,t)\eta(\theta,z,t)+\left(p^{\mathcal{I}}-p^{\mathcal{O}}\right)_{r=R_0}=T\left[\left(\frac{d\chi}{dr}\right)_{R_0}-\lambda(R_0)\right]a(t)F(z,\theta) \tag{1.18}$$

The (linearised) Bernoulli's equation for the perturbation pressure at the linearised interface (Farsoiya *et al.* 2017)

$$\left(p^{\mathcal{I}}-p^{\mathcal{O}}\right)_{r=R_0}=\left(\rho^{\mathcal{O}}\frac{\partial\phi^{\mathcal{O}}}{\partial t}-\rho^{\mathcal{I}}\frac{\partial\phi^{\mathcal{I}}}{\partial t}\right)_{r=R_0} \tag{1.19}$$

Substituting the expression for perturbation pressures $p$ from equation 1.19 in equation

1.18, using 1.6 and setting the periodic forcing in both fluids to be of the form $G(r,t) = -h\cos(\Omega t)\frac{r}{R_0}$ (Adou & Tuckerman 2016), we obtain a Mathieu equation of the form

$$\ddot{a}(t) + f(t)a(t) = 0 \tag{1.20}$$

with

$$f(t) \equiv \frac{1}{R_0}\left[\frac{T}{R_0^2}\frac{kR_0\left(k^2R_0^2 + m^2 - 1\right)}{\rho^{\mathcal{I}}\frac{I_m(kR_0)}{I_m'(kR_0)} - \rho^{\mathcal{O}}\frac{K_m(kR_0)}{K_m'(kR_0)}} + \left(\frac{\left(\rho^{\mathcal{I}} - \rho^{\mathcal{O}}\right)kR_0}{\rho^{\mathcal{I}}\frac{I_m(kR_0)}{I_m'(kR_0)} - \rho^{\mathcal{O}}\frac{K_m(kR_0)}{K_m'(kR_0)}}\right)h\cos(\Omega t)\right] \tag{1.21}$$

which is the Mathieu equation governing Faraday waves on a cylindrical filament. Expression 1.21 is a special case of expression 1.23 below, derived in the main manuscript.

$$\ddot{a}(t) + f(t)a(t) = 0 \tag{1.22}$$

where

$$f(t) \equiv \left\{\frac{T\left(\frac{d\chi}{dx_\perp} - \lambda(x_\perp)\right) - \left(\rho^{\mathcal{I}} - \rho^{\mathcal{O}}\right)G(x_\perp, t)}{\rho^{\mathcal{I}}L^{\mathcal{I}}(x_\perp) - \rho^{\mathcal{O}}L^{\mathcal{O}}(x_\perp)}\right\}_{x_\perp = x_\perp^0} \tag{1.23}$$

Equation 1.22 can also be used for obtaining Faraday waves in Cartesian, cylindrical and spherical geometry recovering the results derived in Benjamin & Ursell (1954) and Adou & Tuckerman (2016) as shown in section 3.

## 2. Generalised dispersion relation for free perturbations

Consider the generalised equation derived in the paper.

$$\begin{aligned} &\ddot{a}(t) + f(t)a(t) = 0 \\ &\text{with} \quad f(t) \equiv \left\{\frac{T\left(\frac{d\chi}{dx_\perp} - \lambda(x_\perp)\right) - \left(\rho^{\mathcal{I}} - \rho^{\mathcal{O}}\right)G(x_\perp, t)}{\rho^{\mathcal{I}}L^{\mathcal{I}}(x_\perp) - \rho^{\mathcal{O}}L^{\mathcal{O}}(x_\perp)}\right\}_{x_\perp = x_\perp^0} \end{aligned} \tag{2.1}$$

We apply equation 2.1 to rectangular, spherical and cylindrical geometries showing that dispersion relations for free perturbations (i.e. with $G(x_\perp, t) = 0$) are obtainable from it, using simple algebra. The Mathieu equations for each base state are then obtained from 2.1.

### 2.1. *Rectangular cartesian, standing capillary waves*

As shown in Fig. 2a, we consider capillary oscillations of a 2D planar interface arising on a base state of rest at the interface between two fluids. Both fluids are taken to be infinitely deep and unbounded horizontally.

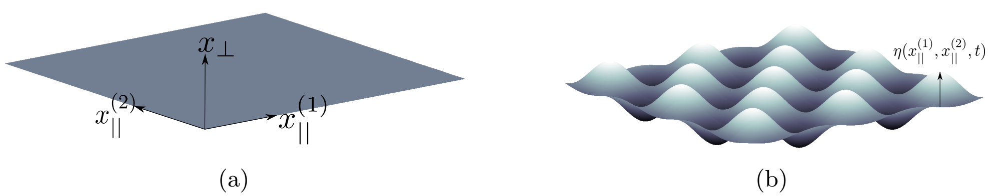



Figure 2: a) Unperturbed interface b) Perturbed interface

For this example, the horizontal coordinates tangential to the unperturbed interface are $x_{||}^{(1)} = x$ and $x_{||}^{(2)} = y$ while the vertical coordinate is $x_\perp = z$. The unperturbed interface has the vertical coordinate $x_\perp^0 = z = 0$. The Laplacian operator in this coordinate system is,

$$\nabla^2 = \nabla_{||}^2 + \nabla_\perp^2 \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{\partial^2}{\partial z^2}.$$

$\nabla_{||}^2$ satisfies

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\cos(kx)\cos(ly) = -\left(k^2 + l^2\right)\cos(kx)\cos(ly).$$

Hence for this problem, the eigenvalue $\lambda = -\left(k^2 + l^2\right)$ is independent of $z$. The equation governing $L(z)$ is,

$$\frac{d^2 L^{\mathcal{I}}}{dz^2} - (k^2 + l^2)L^{\mathcal{I}}(z) = 0, \quad \frac{d^2 L^{\mathcal{O}}}{dz^2} - (k^2 + l^2)L^{\mathcal{O}}(z) = 0$$

which has the general solution $L^{\mathcal{I}}(z) = C^{\mathcal{I}} \exp[\sqrt{k^2 + l^2} z]$ and $L^{\mathcal{O}}(z) = C^{\mathcal{O}} \exp[-\sqrt{k^2 + l^2} z]$ due to boundedness requirements at $\pm\infty$. To determine $C^{\mathcal{I}}$ and $C^{\mathcal{O}}$, we use

$$\left(\frac{dL^{\mathcal{I}}}{dz}\right)_{z=0} = \left(\frac{dL^{\mathcal{O}}}{dz}\right)_{z=0} = 1,$$

implying

$$L^{\mathcal{I}}(z) = \frac{\exp[\sqrt{k^2 + l^2} z]}{\sqrt{k^2 + l^2}}, \text{and} \quad L^{\mathcal{O}}(z) = -\frac{\exp[-\sqrt{k^2 + l^2} z]}{\sqrt{k^2 + l^2}}$$

For this problem $\chi(z) = (\nabla_\perp^2)z = 0$. We can now obtain the value of $f(t) \equiv \omega^2$ from the generalized formula 2.1

$$\omega^2 \equiv T \left\{\frac{\frac{d\chi}{dz} - \lambda(z)}{\rho^{\mathcal{I}} L^{\mathcal{I}}(z) - \rho^{\mathcal{O}} L^{\mathcal{O}}(z)}\right\}_{z=0},$$

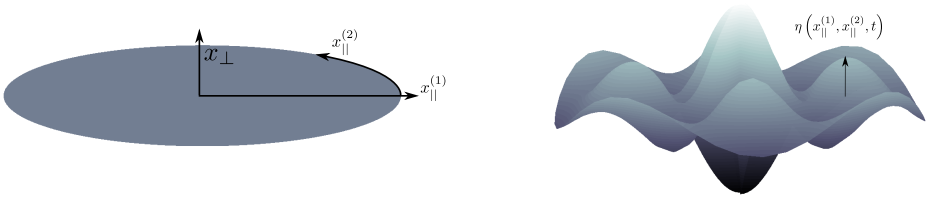
to obtain

$$\boxed{\omega^2(k,l) = \frac{T\left(k^2 + l^2\right)^{3/2}}{\rho^{\mathcal{I}} + \rho^{\mathcal{O}}} = \frac{T|\mathbf{k}|^3}{\rho^{\mathcal{I}} + \rho^{\mathcal{O}}}, \quad \mathbf{k} \equiv (k,l)}$$

which is the well known dispersion relation for capillary standing waves on fluid of infinite depth. With little modification to the above algebra, the dispersion relation for capillary waves on a pool of finite depth can also be obtained.

### 2.2. *Standing capillary waves on a cylindrical pool*

As seen in Fig. 3a, we impose axisymmetric, radial perturbations on the fluid interface which is taken to be radially unbounded. Standing capillary waves form as a result.




(a) (b)

Figure 3: a) Unperturbed interface - 3D visualisation. b) Perturbed interface - 3D.

For this problem the horizontal coordinates tangential to the unperturbed interface are $x_{||}^{(1)} = r$, $x_{||}^{(2)} = \theta$, the vertical coordinate $x_\perp = z$ and the unperturbed interface has the vertical coordinate $x_\perp^0 = 0$. The Laplacian operator in this coordinate system is given by (Weisstein 2017*a*)

$$\nabla^2 = \nabla_{||}^2 + \nabla_\perp^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

It is known that $\nabla_{||}^2$ satisfies (see eqn. 12.2.2 in Prosperetti (2011))

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right] J_m(kr)\cos(m\theta) = -k^2 J_m(kr)\cos(m\theta)$$

where $J_m(kr)$ is the $m$th order Bessel function of the first kind. Hence for this problem the eigenvalue $\lambda = -k^2$ and is independent of the vertical coordinate $z$. The equation governing $L(z)$ is,

$$\left(\frac{d^2 L^{\mathcal{I}}}{dz^2}\right) - k^2 L^{\mathcal{I}}(z) = 0, \quad \left(\frac{d^2 L^{\mathcal{O}}}{dz^2}\right) - k^2 L^{\mathcal{O}}(z) = 0$$

which have the solution $L^{\mathcal{I}}(z) = C^{\mathcal{I}}\exp[kz]$ and $L^{\mathcal{O}}(z) = C^{\mathcal{O}}\exp[-kz]$. From boundedness requirements at $z \to \pm\infty$. To determine $C^{\mathcal{I}}$ and $C^{\mathcal{O}}$, we use,

$$\left(\frac{dL^{\mathcal{I}}}{dz}\right)_{z=0} = \left(\frac{dL^{\mathcal{O}}}{dz}\right)_{z=0} = 1$$

to obtain,

$$L^{\mathcal{I}}(z) = k^{-1}\exp[kz], \text{and} \quad L^{\mathcal{O}}(z) = -k^{-1}\exp[-kz]$$

For this problem $\chi(z) = (\nabla_\perp^2)z = 0$. We can now obtain the value of $f(t) \equiv \omega^2$ from the generalized formula 2.1

$$\omega^2(k,m) \equiv T\left\{\frac{\frac{d\chi}{dz} - \lambda(z)}{\rho^{\mathcal{I}}L^{\mathcal{I}}(z) - \rho^{\mathcal{O}}L^{\mathcal{O}}(z)}\right\}_{z=0},$$

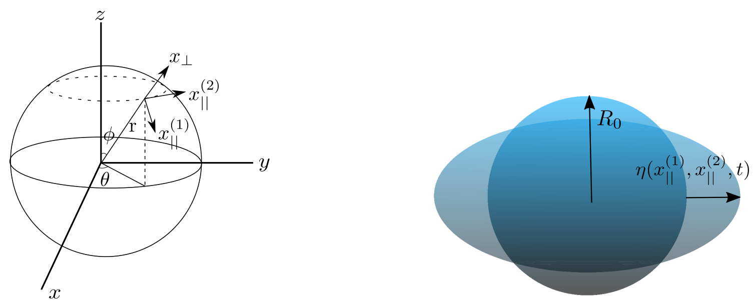
to obtain

$$\boxed{\omega^2(k,m) = \frac{Tk^3}{\rho^{\mathcal{I}} + \rho^{\mathcal{O}}}}$$

which is the dispersion relation for capillary standing waves on a radially unbounded cylindrical pool of infinite depth and is independent of $m$. It is seen that the dispersion relation in a cylindrical radially unbounded pool has the same form as that for a rectangular horizontally unbounded pool.

### 2.3. *Standing capillary waves on a spherical droplet*

We impose perturbations on a spherical droplet as seen in figure 4b. The base state is known to be stable to all linearized pertubations and hence any perturbation leads to oscillations. The dispersion relation for these oscillations was first obtained by Rayleigh (1879).




(a) (b)

Figure 4: a) Unperturbed interface - 3D visualisation. b) Perturbed interface - 3D.

For this problem the horizontal coordinates tangential to the unperturbed interface are $x_{||}^{(1)} = \phi$, $x_{||}^{(2)} = \theta$, the vertical coordinate $x_\perp = r$ and the unperturbed interface has the vertical coordinate $x_\perp^0 = R_0$, $R_0$ being the radius of the unperturbed droplet. The Laplacian operator here is given by (Weisstein 2017*d*)

$$\nabla^2 = \nabla_{||}^2 + \nabla_\perp^2 \equiv \left[\frac{1}{r^2 \sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2}\right] + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$$

In spherical coordinate system, the horizontal part of the Laplacian satisfies $\nabla_{||}^2$ (see Eq. 7.2.11 in Prosperetti (2011) where the opposite definitions for $\theta$ and $\phi$ are chosen)

$$\left[\frac{1}{r^2 \sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2}\right] Y_l^m(\phi,\theta) = -\frac{l(l+1)}{r^2} Y_l^m(\phi,\theta)$$

where $Y_l^m(\phi,\theta)$ is a spherical harmonic. Hence for this problem the eigenvalue $\lambda(r) = -l(l+1)/r^2$ and the equation governing $L(r)$ is,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dL^{\mathcal{I}}}{dr}\right) - \frac{l(l+1)}{r^2}L^{\mathcal{I}}(r) = 0, \quad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dL^{\mathcal{O}}}{dr}\right) - \frac{l(l+1)}{r^2}L^{\mathcal{O}}(r) = 0,$$

which has the solution $L^{\mathcal{I}}(r) = C^{\mathcal{I}} r^l$ and $L^{\mathcal{O}}(r) = C^{\mathcal{O}} r^{-l-1}$ due to boundedness requirements at $r = 0$ and $r = \infty$ respectively. Using $\left(\frac{dL^{\mathcal{I}}}{dr}\right)_{r=R_0} = \left(\frac{dL^{\mathcal{O}}}{dr}\right)_{r=R_0} = 1$, we find $L^{\mathcal{I}}(r) = \frac{R_0}{l}\left(\frac{r}{R_0}\right)^l$ and $L^{\mathcal{O}}(r) = -\frac{R_0}{l+1}\left(\frac{r}{R_0}\right)^{-l-1}$. For this problem $\chi(r) = (\nabla_\perp^2)r = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)r = 2/r$. We can now obtain the value of $f(t) \equiv \omega^2$ from the generalized formula 2.1,

$$\omega^2 \equiv T\left\{\frac{\frac{d\chi}{dr} - \lambda(r)}{\rho^{\mathcal{I}}L^{\mathcal{I}}(r) - \rho^{\mathcal{O}}L^{\mathcal{O}}(r)}\right\}_{r=R_0},$$

leading to

$$\boxed{\omega^2(l,m) = \frac{Tl(l+2)(l-1)(l+1)}{R_0^3\left[\rho^{\mathcal{I}}(l+1)+\rho^{\mathcal{O}}l\right]}}$$

which is the well known dispersion relation for oscillations of an inviscid drop surrounded by another inviscid fluid as obtained by (Rayleigh 1879; Lamb 1932) and is independent of $m$.

## 3. Faraday waves

### 3.1. *Faraday waves on a rectangular pool*

Considering pure capillary waves, we set gravity $g = 0$ and choose $G(z,t) = -h\cos(\Omega t)$ with $\chi = 0$ and $\lambda = -(k^2 + l^2)^{1/2} = -|\mathbf{k}|$ in equation 2.1 corresponding to a rectangular Cartesian geometry (see section 2.1). Thus the Mathieu equation in this case is,

$$\boxed{\ddot{a}(t) + |\mathbf{k}|\left[\frac{T|\mathbf{k}|^2}{\rho^{\mathcal{I}}+\rho^{\mathcal{O}}} + \left(\frac{\rho^{\mathcal{I}}-\rho^{\mathcal{O}}}{\rho^{\mathcal{I}}+\rho^{\mathcal{O}}}\right)h\cos(\Omega t)\right]a(t) = 0}$$

The above equation generalises the Mathieu equation (equation 2.12 in Benjamin & Ursell (1954)) provided in Benjamin & Ursell (1954) taking into account density of both fluids and for the specific case of pure capillary waves on a pool of infinite depth.

### 3.2. *Faraday waves on a cylindrical pool*

Considering pure capillary waves, we set gravity, we set gravity $g = 0$ and choose $G(z,t) = -h\cos(\Omega t)$ with $\chi = 0$ and $\lambda = -k^2$ in equation 2.1, corresponding to cylindrical geometry (see section 2.2). Thus the Mathieu equation in this case is,

$$\boxed{\ddot{a}(t) + k\left[\frac{Tk^2}{\rho^{\mathcal{I}}+\rho^{\mathcal{O}}} + \left(\frac{\rho^{\mathcal{I}}-\rho^{\mathcal{O}}}{\rho^{\mathcal{I}}+\rho^{\mathcal{O}}}\right)h\cos(\Omega t)\right]a(t) = 0}$$

### 3.3. *Faraday waves on a sphere*

For Faraday waves on a sphere, we set $G(r,t) \equiv -h\cos(\Omega t)\frac{r}{R_0}$ with $\chi(r) = 2/r$ and $\lambda = -\frac{l(l+1)}{r^2}$ in equation 2.1, corresponding to spherical geometry (see section 2.3). Thus the Mathieu equation in this case is,

$$\boxed{\ddot{a}(t) + \frac{1}{R_0}\left[\frac{T}{R_0^2}\frac{l(l+2)(l+1)(l-1)}{\rho^{\mathcal{I}}(l+1)+l\rho^{\mathcal{O}}} + \left(\frac{\left(\rho^{\mathcal{I}}-\rho^{\mathcal{O}}\right)l(l+1)}{\rho^{\mathcal{I}}(l+1)+l\rho^{\mathcal{O}}}\right)h\cos(\Omega t)\right]a(t) = 0}$$

which generalises equation 4.5 in Adou & Tuckerman (2016) including densities of both fluids.

### 3.4. *Faraday waves on a cylindrical filament*

For Faraday waves on an axially unbounded cylindrical filament of radius $R_0$, we set $G(r,t) = -h\cos(\Omega t)r/R_0$. Here $\chi(r) = 1/r$ and $\lambda(r) = -\left(m^2/r^2 + k^2\right)$ in equation 2.1, see main manuscript for geometry. Hence the Mathieu equation in this case reads,

$$\boxed{\ddot{a}(t)+\frac{1}{R_0}\left[\frac{T}{R_0^2}\frac{kR_0\left(k^2R_0^2+m^2-1\right)}{\rho^{\mathcal{I}}\frac{I_m(kR_0)}{I_m'(kR_0)}-\rho^{\mathcal{O}}\frac{K_m(kR_0)}{K_m'(kR_0)}}+\left(\frac{\left(\rho^{\mathcal{I}}-\rho^{\mathcal{O}}\right)kR_0}{\rho^{\mathcal{I}}\frac{I_m(kR_0)}{I_m'(kR_0)}-\rho^{\mathcal{O}}\frac{K_m(kR_0)}{K_m'(kR_0)}}\right)h\cos(\Omega t)\right]a(t)=0}$$

## 4. Numerical Floquet analysis

We provide the details of the numerical Floquet analysis used in the paper following Kumar & Tuckerman (1994). We conduct Floquet analysis of the following equation studied in the paper

$$\frac{d^2a}{d\tilde{t}^2}+\left(\mathcal{A}+2\mathcal{B}\cos(\tilde{t})\right)a(\tilde{t})=0, \tag{4.1}$$

Using the Floquet ansatz, we can write

$$a(\tilde{t})/a(0)=\zeta(\tilde{t})\exp\left[\left(\tilde{\mu}+i\tilde{\alpha}\right)\tilde{t}\right]=\sum_{n=-\infty}^{\infty}\zeta_n\exp\left[\left(\tilde{\mu}+i\left(\tilde{\alpha}+n\right)\right)\tilde{t}\,\right] \tag{4.2}$$

where $\zeta(\tilde{t})$ is a periodic function of unit time period and is expanded in a Fourier series. Substituting expression 4.2 in 4.1, we obtain

$$\left\{\left(\tilde{\mu}+i\left(\tilde{\alpha}+n\right)\right)^2+\mathcal{A}\right\}\zeta_n=-\mathcal{B}\left(\zeta_{n-1}+\zeta_{n+1}\right). \quad (n=\ldots,-2,-1,0,1,2\ldots) \tag{4.3}$$

For neutral stability curves, $\tilde{\mu}=0$ leading to

$$\left\{-\left(\tilde{\alpha}+n\right)^2+\mathcal{A}\right\}\zeta_n=-\mathcal{B}\left(\zeta_{n-1}+\zeta_{n+1}\right). \quad (n=\ldots,-2,-1,0,1,2\ldots) \tag{4.4}$$

Define $L_n\equiv-\left(\tilde{\alpha}+n\right)^2+\mathcal{A}$ and hence the above can be written as

$$L_n\zeta_n=-\mathcal{B}\left(\zeta_{n-1}+\zeta_{n+1}\right). \quad (n=\ldots,-2,-1,0,1,2\ldots) \tag{4.5}$$

Negative values of $n$ need not be considered for both harmonic and subharmonic cases due to reality constraints (Kumar & Tuckerman 1994; Kumar 1996). In the harmonic case ($\tilde{\alpha}=0$), the equation for negative $n$ is the complex conjugate of the equation for the corresponding value of positive $n$. Similarly in the subharmonic case ($\tilde{\alpha}=1/2$), the equation for $n=0$ is the complex conjugate of the equation for $n=-1$, the one for $n=1$ is the complex conjugate for $n=-2$ and so on (Kumar & Tuckerman 1994). Consequently, our Fourier series starts with $n=0$ and extends to positive values of $n$ only. Writing $\zeta_n=\zeta_n^r+i\zeta_n^i$, equation 4.5 can be split into real and imaginary parts leading to (Kumar & Tuckerman 1994)

$$L_n\zeta_n^r=-\mathcal{B}\left(\zeta_{n-1}^r+\zeta_{n+1}^r\right), \quad L_n\zeta_n^i=-\mathcal{B}\left(\zeta_{n-1}^i+\zeta_{n+1}^i\right), \quad (n=0,1,2\ldots N) \quad . \tag{4.6}$$

Note that $\mathcal{B}$ being related to the forcing amplitude, is constrained to be real and positive. Thus while writing equations 4.6, we assume that $\mathcal{B}$ is real and hence in our numerical solution, only those eigenvalues are accepted which satisfy this constraint.

### 4.1. *Harmonic case*

For the harmonic case, $\tilde{\alpha} = 0$. Subject to the reality constraint $\zeta_{-1} = \bar{\zeta}_1$, we have the following equations for $N = 20$ from 4.6

$$
\begin{aligned}
L_0\zeta_0^r &= -\mathcal{B}\left(2\zeta_1^r\right)\\
L_0\zeta_0^i &= 0\\
L_1\zeta_1^r &= -\mathcal{B}\left(\zeta_0^r + \zeta_2^r\right)\\
L_1\zeta_1^i &= -\mathcal{B}\left(\zeta_0^i + \zeta_2^i\right)\\
L_2\zeta_2^r &= -\mathcal{B}\left(\zeta_1^r + \zeta_3^r\right)\\
L_2\zeta_2^i &= -\mathcal{B}\left(\zeta_1^i + \zeta_3^i\right)\\
\vdots \quad &\quad \vdots\\
L_{20}\zeta_{20}^r &= -\mathcal{B}\left(\zeta_{19}^r\right)\\
L_{20}\zeta_{20}^i &= -\mathcal{B}\left(\zeta_{19}^i\right)
\end{aligned}
$$

Note that in the last equation $\zeta_{21}^r$ and $\zeta_{21}^i$ have been set to zero because of truncation of Fourier series at $N = 20$. These equations can be written as

$$\mathbf{L}\cdot\boldsymbol{\zeta} = -\mathcal{B}\mathbf{Q}\cdot\boldsymbol{\zeta} \tag{4.7}$$

which written explicitly using matrices are

$$
\begin{pmatrix}
L_0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0\\
0 & L_0 & 0 & 0 & 0 & 0 & \dots & 0 & 0\\
0 & 0 & L_1 & 0 & 0 & 0 & \dots & 0 & 0\\
0 & 0 & 0 & L_1 & 0 & 0 & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & 0 & 0 & \dots & L_{20} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & L_{20}
\end{pmatrix}
\begin{pmatrix}
\zeta_0^r\\ \zeta_0^i\\ \zeta_1^r\\ \zeta_1^i\\ \vdots\\ \vdots\\ \vdots\\ \zeta_{20}^r\\ \zeta_{20}^i
\end{pmatrix}
= -\mathcal{B}
\begin{pmatrix}
0 & 0 & 2 & 0 & 0 & 0 & 0 & \dots & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0\\
1 & 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0\\
0 & 1 & 0 & 0 & 0 & 1 & 0 & \dots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\zeta_0^r\\ \zeta_0^i\\ \zeta_1^r\\ \zeta_1^i\\ \vdots\\ \vdots\\ \vdots\\ \zeta_{20}^r\\ \zeta_{20}^i
\end{pmatrix}
$$

### 4.2. *Sub-harmonic case*

For the sub-harmonic case, $\tilde{\alpha} = 1/2$, subject to the reality constraint $\zeta_{-1} = \bar{\zeta}_0$. We have the following equations for $N = 20$ from 4.6

$$
\begin{aligned}
L_0\zeta_0^r &= -\mathcal{B}\left(\zeta_0^r + \zeta_1^r\right)\\
L_0\zeta_0^i &= -\mathcal{B}\left(-\zeta_0^i + \zeta_1^i\right)\\
L_1\zeta_1^r &= -\mathcal{B}\left(\zeta_0^r + \zeta_2^r\right)\\
L_1\zeta_1^i &= -\mathcal{B}\left(\zeta_0^i + \zeta_2^i\right)\\
L_2\zeta_2^r &= -\mathcal{B}\left(\zeta_1^r + \zeta_3^r\right)\\
L_1\zeta_1^i &= -\mathcal{B}\left(\zeta_1^i + \zeta_3^i\right)\\
\vdots \quad &\quad \vdots\\
L_{20}\zeta_{20}^r &= -\mathcal{B}\left(\zeta_{19}^r\right)\\
L_{20}\zeta_{20}^i &= -\mathcal{B}\left(\zeta_{19}^i\right)
\end{aligned}
$$

Note that in the last equation $\zeta_{21}^r$ and $\zeta_{21}^i$ have been set to zero because of truncation of Fourier series at $N = 20$. These equations can be written as

$$\mathbf{L} \cdot \boldsymbol{\zeta} = -\mathcal{B}\mathbf{Q} \cdot \boldsymbol{\zeta} \tag{4.8}$$

which written explicitly using matrices are

$$\begin{pmatrix} L_0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & L_0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & L_1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & L_1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & L_{20} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & L_{20} \end{pmatrix} \begin{pmatrix} \zeta_0^r \\ \zeta_0^i \\ \zeta_1^r \\ \zeta_1^i \\ \vdots \\ \vdots \\ \vdots \\ \zeta_{20}^r \\ \zeta_{20}^i \end{pmatrix} = -\mathcal{B} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \zeta_0^r \\ \zeta_0^i \\ \zeta_1^r \\ \zeta_1^i \\ \vdots \\ \vdots \\ \vdots \\ \zeta_{20}^r \\ \zeta_{20}^i \end{pmatrix}$$

## 5. Perturbative solution to the Mathieu equation

Using the multiple scales technique, we define $\tilde{t}_0 \equiv \tilde{t}$ , $\tilde{t}_1 \equiv \mathcal{B}\tilde{t}$ , $\tilde{t}_2 \equiv \mathcal{B}^2\tilde{t}$ and solve equation 5.1 upto $\mathcal{O}\left(\mathcal{B}^2\right)$

$$\frac{d^2 a}{d\tilde{t}^2} + \left(\mathcal{A} + 2\mathcal{B}\cos(\tilde{t})\right) a(\tilde{t}) = 0, \tag{5.1}$$

We use the expansion

$$a(\tilde{t}_0, \tilde{t}_1, \tilde{t}_2) = a_0(\tilde{t}_0, \tilde{t}_1, \tilde{t}_2) + a_1(\tilde{t}_0, \tilde{t}_1, \tilde{t}_2)\mathcal{B} + a_2(\tilde{t}_0, \tilde{t}_1, \tilde{t}_2)\mathcal{B}^2 + \mathcal{O}(\mathcal{B}^3) \tag{5.2}$$

Define $D_n \equiv \frac{\partial}{\partial \tilde{t}_n}$ and at $\mathcal{O}(1)$, $\mathcal{O}(\mathcal{B})$ and $\mathcal{O}(\mathcal{B}^2)$ the following equations are obtained respectively,

$$\left(D_0^2 + \mathcal{A}\right) a_0 = 0, \tag{5.3}$$

$$\left(D_0^2 + \mathcal{A}\right) a_1 = -2\left(D_0 D_1 + \cos(\tilde{t}_0)\right) a_0, \tag{5.4}$$

$$\left(D_0^2 + \mathcal{A}\right) a_2 = -\left(D_1^2 + 2D_0 D_2\right) a_0 - 2\left(D_0 D_1 + \cos(\tilde{t}_0)\right) a_1. \tag{5.5}$$

### 5.1. *Stable solutions:* $\mathcal{A} - n^2/4 = \mathcal{O}(1), \quad n = 1, 2, 3 \dots$

The solution of equation 5.3 is,

$$a_0(\tilde{t}_0, \tilde{t}_1, \tilde{t}_2) = C_0(\tilde{t}_1, \tilde{t}_2)\exp(i\tilde{t}_0\sqrt{\mathcal{A}}) + \texttt{c.c} \tag{5.6}$$

where `c.c.` stands for complex conjugate and $C_0$ is a complex function of $\tilde{t}_1$ and $\tilde{t}_2$. Using the expression for $a_0$, equation 5.4 can be written as,

$$\left(D_0^2 + \mathcal{A}\right) a_1 = -2\left[\frac{\partial C_0}{\partial \tilde{t}_1} i\sqrt{\mathcal{A}}\exp\left(i\tilde{t}_0\sqrt{\mathcal{A}}\right) + \frac{C_0}{2}\left\{\exp\left[i\tilde{t}_0\left(\sqrt{\mathcal{A}}+1\right)\right] + \exp\left[i\tilde{t}_0\left(\sqrt{\mathcal{A}}-1\right)\right]\right\}\right] + \texttt{c.c.} \tag{5.7}$$

In addition to the first term on the right hand side of equation 5.7, additional resonant forcing terms can arise if $\sqrt{\mathcal{A}} \pm 1 = \pm\sqrt{\mathcal{A}}$. Hence for $\mathcal{A} = 1/4$ (and in general for $\mathcal{A} = \frac{n^2}{4}$, $n = 1, 2 \dots$), additional resonant forcing terms can appear on the right hand side of 5.7. We will presently asssume that this is not the case i.e. $\mathcal{A} - \frac{n^2}{4} = O(1)$. Under this assumption, the first term on the right hand side of equation 5.7 is the only term which produces a resonant forcing term. In order to eliminate this, we set $\frac{\partial C_0}{\partial \tilde{t}_1} = 0$ and thus $C_0(\tilde{t}_2)$, the solution of equation 5.7 is,

$$a_1(\tilde{t}_1, \tilde{t}_2) = C_0(\tilde{t}_2)\left(\frac{1}{2\sqrt{A}+1}\exp\left[i\tilde{t}_0\left(\sqrt{\mathcal{A}}+1\right)\right] - \frac{1}{2\sqrt{A}-1}\exp\left[i\tilde{t}_0\left(\sqrt{\mathcal{A}}-1\right)\right]\right) + \texttt{c.c} \tag{5.8}$$

In order to determine $C_0(\tilde{t}_2)$, we need to proceed to the next order. With the expressions of $a_0$ and $a_1$, equation 5.5 becomes,

$$
\begin{aligned}
\left(D_0^2+\mathcal{A}\right)a_2 = {} & -2\left[\frac{dC_0}{d\tilde{t}_2} i\sqrt{\mathcal{A}}\exp\left(i\tilde{t}_0\sqrt{\mathcal{A}}\right)\right. \\
& +C_0(\tilde{t}_2)\left\{\frac{\exp(i\tilde{t}_0)+\exp(-i\tilde{t}_0)}{2}\right\}\left\{\frac{1}{2\sqrt{\mathcal{A}}+1}\exp\left[i\tilde{t}_0\left(\sqrt{\mathcal{A}}+1\right)\right]\right. \\
& \left.\left.-\frac{1}{2\sqrt{\mathcal{A}}-1}\exp\left[i\tilde{t}_0\left(\sqrt{\mathcal{A}}-1\right)\right]\right\}\right] + \texttt{c.c.} \\
= {} & -2\left[\frac{dC_0}{d\tilde{t}_2} i\sqrt{\mathcal{A}}\exp\left(i\tilde{t}_0\sqrt{\mathcal{A}}\right)+\frac{C_0}{2}\left\{\frac{1}{2\sqrt{\mathcal{A}}+1}\exp\left[i\tilde{t}_0\left(\sqrt{\mathcal{A}}+2\right)\right]\right.\right. \\
& -\frac{1}{2\sqrt{\mathcal{A}}-1}\exp\left[i\tilde{t}_0\left(\sqrt{\mathcal{A}}\right)\right]+\frac{1}{2\sqrt{\mathcal{A}}+1}\exp\left[i\tilde{t}_0\left(\sqrt{\mathcal{A}}\right)\right] \\
& \left.\left.-\frac{1}{2\sqrt{\mathcal{A}}-1}\exp\left[i\tilde{t}_0\left(\sqrt{\mathcal{A}}-2\right)\right]\right\}\right] + \texttt{c.c.}
\end{aligned}
\tag{5.9}
$$

Like earlier, additional resonant forcing terms arise if $\sqrt{\mathcal{A}}\pm 2 = \pm\sqrt{\mathcal{A}}$ viz. $\mathcal{A}=1$. Assuming that this is not the case, the elimination of the resonant forcing terms on the right hand side of equation 5.9 leads to the following equation for $C_0(\tilde{t}_2)$.

$$
i\sqrt{\mathcal{A}}\frac{dC_0}{d\tilde{t}_2}-\frac{C_0}{2}\frac{1}{2\sqrt{\mathcal{A}}-1}+\frac{C_0}{2}\frac{1}{2\sqrt{\mathcal{A}}+1}=0 \tag{5.10}
$$

Equation 5.10 has the solution

$$
C_0(\tilde{t}_2)=c_0\exp\left[\frac{-i\tilde{t}_2}{\sqrt{\mathcal{A}}\left(4\mathcal{A}-1\right)}\right] \tag{5.11}
$$

where $c_0$ is a complex constant. Equation 5.9 with resonant forcing terms on the right hand side eliminated, is

$$
\begin{aligned}
\left(D_0^2+\mathcal{A}\right)a_2 = {} & -C_0(\tilde{t}_2)\left\{\frac{1}{2\sqrt{\mathcal{A}}+1}\exp\left[i\tilde{t}_0\left(\sqrt{\mathcal{A}}+2\right)\right]-\frac{1}{2\sqrt{\mathcal{A}}-1}\exp\left[i\tilde{t}_0\left(\sqrt{\mathcal{A}}-2\right)\right]\right\} \\
& +\texttt{c.c.}
\end{aligned}
\tag{5.12}
$$

Equation 5.12 implies that

$$
a_2(\tilde{t}_0,\tilde{t}_2)=\frac{C_0(\tilde{t}_2)}{4}\left\{\frac{\exp\left[i\tilde{t}_0\left(\sqrt{\mathcal{A}}+2\right)\right]}{\left(\sqrt{\mathcal{A}}+1\right)\left(2\sqrt{\mathcal{A}}+1\right)}+\frac{\exp\left[i\tilde{t}_0\left(\sqrt{\mathcal{A}}-2\right)\right]}{\left(\sqrt{\mathcal{A}}-1\right)\left(2\sqrt{\mathcal{A}}-1\right)}\right\}+\text{c.c} \tag{5.13}
$$

Hence the solution upto $\mathcal{O}(\mathcal{B}^2)$ is,

$$
\begin{aligned}
\frac{a(t)}{c_0} \approx {} & \exp\left[i\varphi t\right]+\frac{\mathcal{B}}{2\sqrt{\mathcal{A}}+1}\exp\left[it\left(\varphi+1\right)\right]-\frac{\mathcal{B}}{2\sqrt{\mathcal{A}}-1}\exp\left[it\left(\varphi-1\right)\right] \\
& +\frac{\mathcal{B}^2}{4\left(\sqrt{\mathcal{A}}+1\right)\left(2\sqrt{\mathcal{A}}+1\right)}\exp\left[it\left(\varphi+2\right)\right]+\frac{\mathcal{B}^2}{4\left(\sqrt{\mathcal{A}}-1\right)\left(2\sqrt{\mathcal{A}}-1\right)}\exp\left[it\left(\varphi-2\right)\right]+\text{c.c}
\end{aligned}
\tag{5.14}
$$

where $\varphi \equiv \sqrt{\mathcal{A}} - \frac{\mathcal{B}^2}{\sqrt{\mathcal{A}}(4\mathcal{A}-1)}$. Writing $c_0 = \frac{r}{2}\exp(i\Theta)$ we obtain from equation 5.14

$$\begin{aligned}\frac{a(t)}{r} &= \cos\left[\varphi t + \Theta\right] + \frac{\mathcal{B}}{2\sqrt{\mathcal{A}}+1}\cos\left[(\varphi+1)\,t+\Theta\right] - \frac{\mathcal{B}}{2\sqrt{\mathcal{A}}-1}\cos\left[(\varphi-1)\,t+\Theta\right]\\ &+\frac{\mathcal{B}^2}{4\left(\sqrt{\mathcal{A}}+1\right)\left(2\sqrt{\mathcal{A}}+1\right)}\cos\left[(\varphi+2)\,t+\Theta\right]\\ &+\frac{\mathcal{B}^2}{4\left(\sqrt{\mathcal{A}}-1\right)\left(2\sqrt{\mathcal{A}}-1\right)}\cos\left[(\varphi-2)\,t+\Theta\right] + \mathcal{O}(\mathcal{B}^3).\end{aligned} \tag{5.15}$$

### 5.2. *Unstable solution: $\mathcal{A} - 1/4 = \mathcal{O}(\mathcal{B})$*

We obtain a solution upto $\mathcal{O}(\mathcal{B})$. For $\mathcal{A} - 1/4 = \mathcal{O}(\mathcal{B})$, we have (Bender & Orszag 2010),

$$a(\tilde{t}_0,\tilde{t}_1) = a_0(\tilde{t}_0,\tilde{t}_1) + \mathcal{B}a_1(\tilde{t}_0,\tilde{t}_1) + \mathcal{O}(\mathcal{B}^2),\ \ \mathcal{A} = 1/4 + \mathcal{A}_1\mathcal{B} + \mathcal{O}(\mathcal{B}^2), \tag{5.16}$$

with the following equations being obtained at $\mathcal{O}(1)$, $\mathcal{O}(\mathcal{B})$ respectively.

$$\left(D_0^2 + 1/4\right)a_0 = 0, \tag{5.17}$$

$$\left(D_0^2 + 1/4\right)a_1 = -2\left(D_0D_1 + \mathcal{A}_1/2 + \cos(\tilde{t}_0)\right)a_0, \tag{5.18}$$

The solution to equation 5.17 is,

$$a_0(\tilde{t}_0,\tilde{t}_1) = E_0(\tilde{t}_1)\exp(i\tilde{t}_0/2) + \mathtt{c.c} \tag{5.19}$$

where $E_0$ is a complex function. Equation 5.18 is now given by

$$\begin{aligned}\left(D_0^2 + 1/4\right)a_1 &= -2\left[\frac{\partial E_0}{\partial\tilde{t}_1}\frac{i}{2}\exp(i\tilde{t}_0/2) + \frac{\mathcal{A}_1E_0}{2}\exp(i\tilde{t}_0/2) + \frac{E_0}{2}\exp(3i\tilde{t}_0/2) + \frac{\bar{E}_0}{2}\exp(i\tilde{t}_0/2)\right]\\ &+ \mathtt{c.c.}\end{aligned} \tag{5.20}$$

Note that an overbar denotes complex conjugation. Elimination of resonant forcing terms leads to

$$i\frac{\partial E_0}{\partial\tilde{t}_1} + \mathcal{A}_1E_0 + \bar{E}_0 = 0. \tag{5.21}$$

Writing $E_0 = E_0^R + iE_0^I$, we have the following equations for $E_0^R$ and $E_0^I$.

$$\frac{\partial E_0^R}{\partial\tilde{t}_1} = (1-\mathcal{A}_1)E_0^I,\quad \frac{\partial E_0^I}{\partial\tilde{t}_1} = (1+\mathcal{A}_1)E_0^R, \tag{5.22}$$

For $-1 < \mathcal{A}_1 < 1$, the solution to the above set of equations is

$$\begin{aligned}E_0^R(\tilde{t}_1) &= \sqrt{\frac{1-\mathcal{A}_1}{1+\mathcal{A}_1}}\left[K_1\exp\left(\tilde{t}_1\sqrt{1-\mathcal{A}_1^2}\right) - K_2\exp\left(-\tilde{t}_1\sqrt{1-\mathcal{A}_1^2}\right)\right]\\ E_0^I(\tilde{t}_1) &= K_1\exp\left(\tilde{t}_1\sqrt{1-\mathcal{A}_1^2}\right) + K_2\exp\left(-\tilde{t}_1\sqrt{1-\mathcal{A}_1^2}\right)\end{aligned} \tag{5.23}$$

where $K_1$ and $K_2$ are real constants. For $-1 < \mathcal{A}_1 < 1$, we get exponentially growing (and decaying) solutions as $t \to \infty$. The expression for $E_0$ is

$$\begin{aligned}E_0(\tilde{t}_1) = K_1&\left(i + \sqrt{\frac{1-\mathcal{A}_1}{1+\mathcal{A}_1}}\right)\exp\left(\tilde{t}_1\sqrt{1-\mathcal{A}_1^2}\right)\\ &+K_2\left(i - \sqrt{\frac{1-\mathcal{A}_1}{1+\mathcal{A}_1}}\right)\exp\left(-\tilde{t}_1\sqrt{1-\mathcal{A}_1^2}\right)\end{aligned}$$

With this the solution to equation 5.20 is

$$a_1(\tilde{t}_0,\tilde{t}_1) = \frac{E_0(\tilde{t}_1)}{2}\exp\left(3i\tilde{t}_0/2\right) + \mathtt{c.c.} \tag{5.24}$$

Thus $a_0$ can be written as

$$a_0(t) = 2K_1 \exp\left(\mathcal{B}t\sqrt{1-\mathcal{A}_1^2}\right)\left(\sqrt{\frac{1-\mathcal{A}_1}{1+\mathcal{A}_1}}\cos(t/2) - \sin(t/2)\right)$$
$$-2K_2 \exp\left(-\mathcal{B}t\sqrt{1-\mathcal{A}_1^2}\right)\left(\sqrt{\frac{1-\mathcal{A}_1}{1+\mathcal{A}_1}}\cos(t/2) + \sin(t/2)\right) \tag{5.25}$$

Similarly $a_1$ is

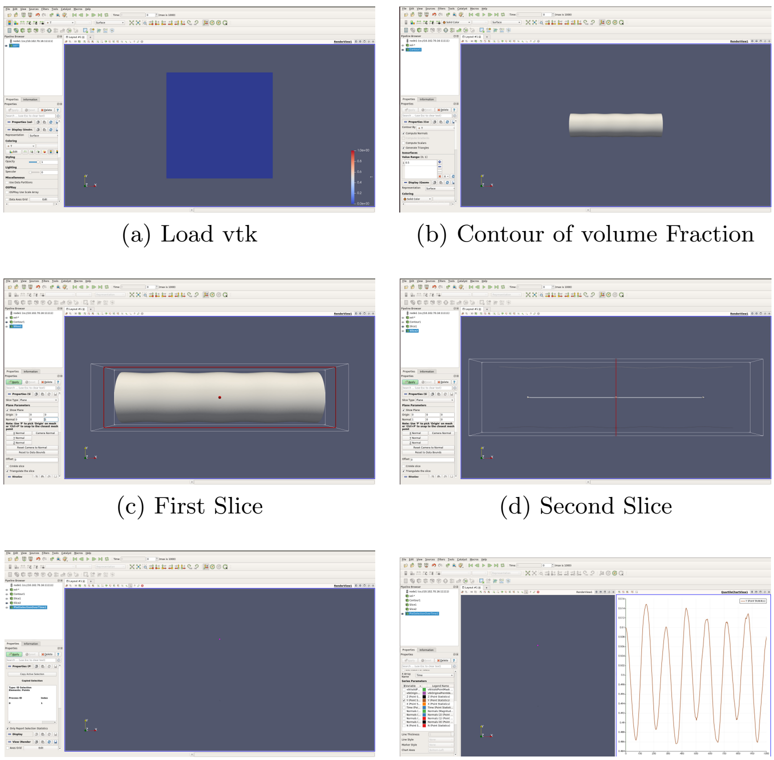
$$a_1(t) = K_1 \exp\left(\mathcal{B}t\sqrt{1-\mathcal{A}_1^2}\right)\left(\sqrt{\frac{1-\mathcal{A}_1}{1+\mathcal{A}_1}}\cos(3t/2) - \sin(3t/2)\right)$$
$$-K_2 \exp\left(-\mathcal{B}t\sqrt{1-\mathcal{A}_1^2}\right)\left(\sqrt{\frac{1-\mathcal{A}_1}{1+\mathcal{A}_1}}\cos(3t/2) + \sin(3t/2)\right) \tag{5.26}$$

and thus $a(t)$ can be written as,

$$a(t) = a_0(t) + \mathcal{B}a_1(t) + \mathcal{O}(\mathcal{B}^2) \tag{5.27}$$

## 6. Data extraction using Paraview

The GFS files generated by `Gerris3D` are converted to Paraview (Ahrens *et al.* 2005) readable format VTK legacy file using GfsOutputSimulation event. These files are renamed with integer time series for paraview compatibility. To extract the amplitude of interface, we apply contour filter on volume fractions (see Fig. 5(a-b)) for level=0.5. Slice filter is applied in succession to get a single point on interface face (see Fig.5(c-d)). The point is selected(see Fig.5(e)) and plot selection overtime(see Fig.5(f)) is applied to get the time signal of the interface.



(a) Load vtk

(b) Contour of volume Fraction

(c) First Slice

(d) Second Slice

(e) Plot selection over time

(f) Time series

Figure 5

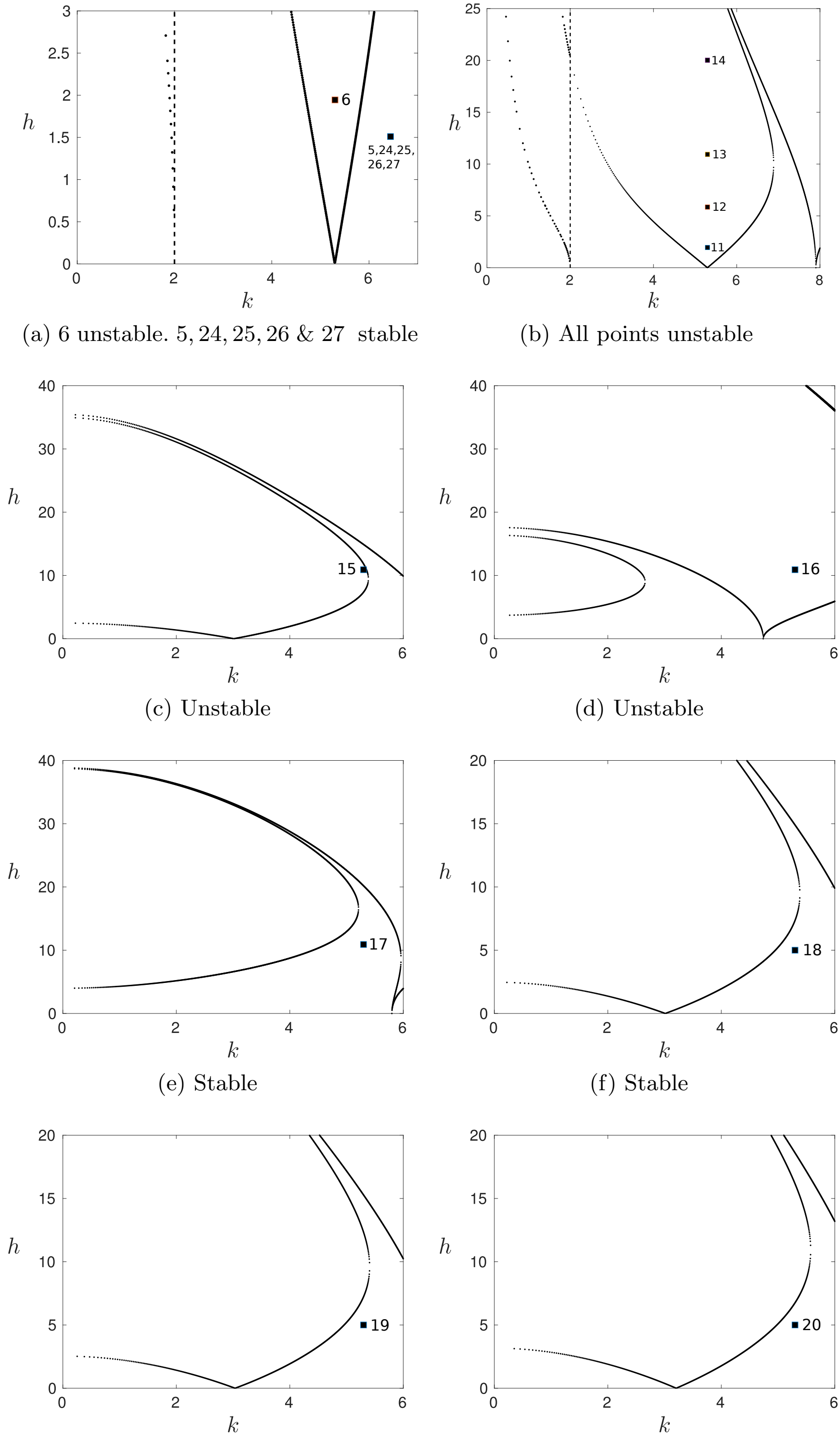
## 7. DNS Parameters

Table 1: **DNS Parameters (Dimensional)**

| **Case** | **Comments** | $a(0)$ | $m$ | $k$ | $\rho^{\mathcal{I}}$ | $\rho^{\mathcal{O}}$ | $R_0$ | $h$ | $\Omega$ | $T$ | $L$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | Free perturbations | 1/600 | 0 | 4 | 1 | 0.01 | 1/6 | 0 | NA | 1 | $4\pi$ |
| 2 | | 1/600 | 0 | 8 | 1 | 0.01 | 1/6 | 0 | NA | 1 | $4\pi$ |
| 3 | | 1/600 | 2 | 4 | 1 | 0.01 | 1/6 | 0 | NA | 1 | $4\pi$ |
| 4 | | 1/600 | 2 | 8 | 1 | 0.01 | 1/6 | 0 | NA | 1 | $4\pi$ |
| 5 | Linear theory validation | 0.01 | 0 | 6.4373 | 1 | 0.01 | 0.5 | 1.5127 | $2\pi$ | 0.1 | 3.9042 |
| 6 | | 0.01 | 0 | 5.2959 | 1 | 0.01 | 0.5 | 1.9431 | $2\pi$ | 0.1 | 4.7457 |
| 7 | RP stabilisation | 1/120 | 0 | 4 | 1 | 0.01 | 1/6 | 80 | $4\pi$ | 1 | $4\pi$ |
| 8 | | 1/120 | 0 | 4 | 1 | 0.01 | 1/6 | 120 | $4\pi$ | 1 | $4\pi$ |
| 9 | | 1/120 | 0 | 2 | 1 | 0.01 | 1/6 | 80 | $4\pi$ | 1 | $4\pi$ |
| 10 | | 1/120 | 0 | 2 | 1 | 0.01 | 1/6 | 210 | $4\pi$ | 1 | $4\pi$ |
| 11 | Variation of $\Gamma$ | 0.01 | 0 | 5.2959 | 1 | 0.01 | 0.5 | 3.8862 | $2\pi$ | 0.1 | 4.7457 |
| 12 | | 0.01 | 0 | 5.2959 | 1 | 0.01 | 0.5 | 5.8293 | $2\pi$ | 0.1 | 4.7457 |
| 13 | | 0.01 | 0 | 5.2959 | 1 | 0.01 | 0.5 | 10.9431 | $2\pi$ | 0.1 | 4.7457 |
| 14 | | 0.01 | 0 | 5.2959 | 1 | 0.01 | 0.5 | 20.0 | $2\pi$ | 0.1 | 4.7457 |
| 15 | Variation of $\epsilon_{az}$ | 0.01 | 2 | 5.2959 | 1 | 0.01 | 0.5 | 10.9431 | $2\pi$ | 0.1 | 4.7457 |
| 16 | | 0.01 | 3 | 5.2959 | 1 | 0.01 | 0.5 | 10.9431 | $2\pi$ | 0.1 | 4.7457 |
| 17 | | 0.01 | 4 | 5.2959 | 1 | 0.01 | 0.5 | 10.9431 | $2\pi$ | 0.1 | 4.7457 |
| 18 | Variation of $\rho_r$ | 0.01 | 2 | 5.2959 | 1 | 0.01 | 0.5 | 5 | $2\pi$ | 0.1 | 4.7457 |
| 19 | | 0.01 | 2 | 5.2959 | 1 | 0.02 | 0.5 | 5 | $2\pi$ | 0.1 | 4.7457 |
| 20 | | 0.01 | 2 | 5.2959 | 1 | 0.1 | 0.5 | 5 | $2\pi$ | 0.1 | 4.7457 |
| 21 | Variation of $\beta$ | 0.01 | 2 | 5.2959 | 1 | 0.01 | 0.5 | 1.5 | $\pi$ | 0.1 | 4.7457 |
| 22 | | 0.01 | 2 | 5.2959 | 1 | 0.01 | 0.5 | 1.5 | $4\pi$ | 0.1 | 4.7457 |
| 23 | | 0.01 | 2 | 5.2959 | 1 | 0.01 | 0.5 | 1.5 | $6\pi$ | 0.1 | 4.7457 |
| 24 | Variation of $\epsilon_{ax}$ | 0.05 | 0 | 6.4373 | 1 | 0.01 | 0.5 | 1.5127 | $2\pi$ | 0.1 | 3.9042 |
| 25 | | 0.1 | 0 | 6.4373 | 1 | 0.01 | 0.5 | 1.5127 | $2\pi$ | 0.1 | 3.9042 |
| 26 | | 0.15 | 0 | 6.4373 | 1 | 0.01 | 0.5 | 1.5127 | $2\pi$ | 0.1 | 3.9042 |
| 27 | | 0.2 | 0 | 6.4373 | 1 | 0.01 | 0.5 | 1.5127 | $2\pi$ | 0.1 | 3.9042 |

Dimensional parameters for DNS.

## 8. Stability Charts




(a) 6 unstable. 5, 24, 25, 26 & 27 stable

(b) All points unstable

(c) Unstable

(d) Unstable

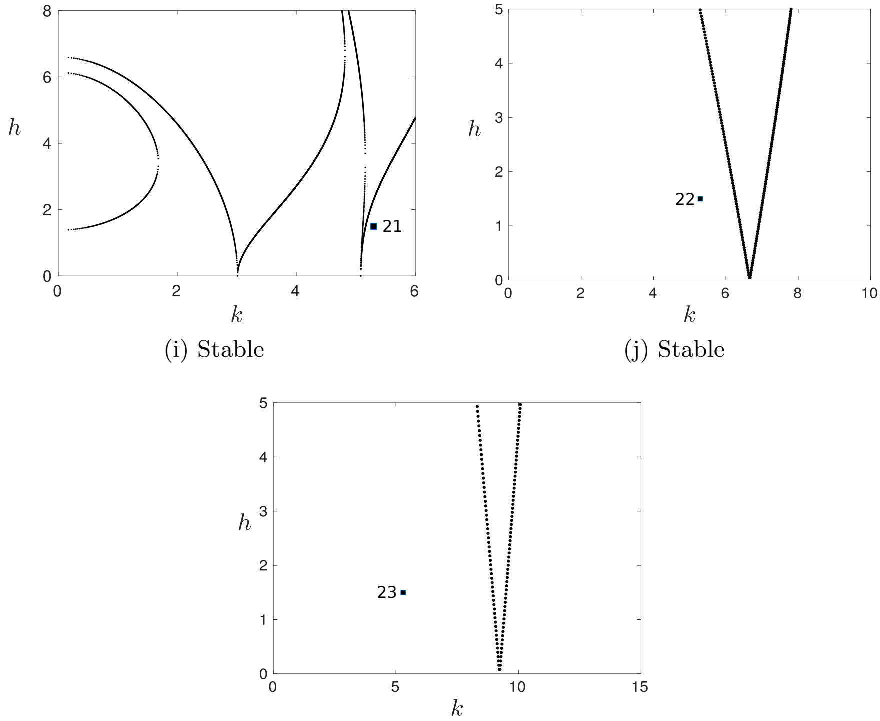
(e) Stable

(f) Stable

(g) Stable

(h) Stable

Figure 6: Stability charts for cases in table 1 . The number for each point corresponds to the case number in the table.




(i) Stable

(j) Stable

(k) Stable

Figure 6: Stability charts for cases in table 1. The number for each point corresponds to the case number in the table.

## 9. Numerical dissipation in DNS

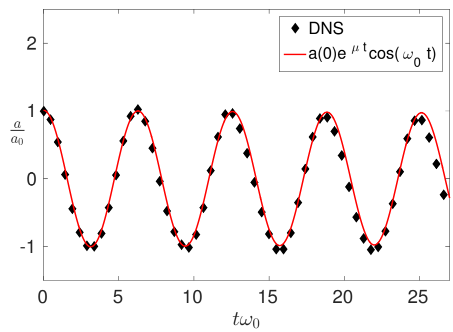



Figure 7: Numerical dissipation. A fit with $a(t) = a(0)e^{\mu t}\cos(\omega_0 t)$ leads to $\mu = -6.0 \times 10^{-2}, \omega_0 = 48.3576$.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