Supporting Information on "Effect of transverse temperature gradient on the migration of a deformable droplet in a Poiseuille flow"

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In this supplementary material we focus on the representation of the different field variables in terms of spherical harmonics. The details of the asymptotic methodology as well as the expressions for the leading order solutions for the velocity and temperature field are provided below.

1. Spherical harmonic representation of field variables

As the governing equation for the temperature field is a Laplace equation (refer to equation (5) of the main text), the general solution for the temperature distribution for either of the phases can be written in the following form

$$\left.\begin{aligned} T_i &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[a_{n,m} r^n \cos(m\varphi) + \hat{a}_{n,m} r^n \sin(m\varphi)\right] P_{n,m}(\cos\theta), \\ T_e &= \zeta r \sin\varphi P_{1,1}(\cos\theta) + \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[b_{-n-1,m} r^{-n-1}\cos(m\varphi) + \hat{b}_{-n-1,m} r^{-n-1}\sin(m\varphi)\right] P_{n,m}(\cos\theta), \end{aligned}\right\} \quad (1)$$

where $P_{n,m}$ is an associate Legendre polynomial of order *m* and degree *n*. The unknown constants present in the above expression $\left(a_{n,m}, \hat{a}_{n,m}, b_{-n-1,m}, \text{and}\, \hat{b}_{-n-1,m}\right)$ can be determined by using the continuity of temperature and heat flux boundary conditions. The surface temperature can be expressed as

$$T_s = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[T_{n,m}\cos(m\varphi) + \hat{T}_{n,m}\sin(m\varphi)\right] P_{n,m}(\cos\theta), \quad (2)$$

where the constant coefficients, $T_{n,m}$ and $\hat{T}_{n,m}$ represent surface spherical harmonics.

The general expression for the velocity and the pressure fields inside the droplet $(\mathbf{u}_i, p_i)$, which obey the Stokes equation as well as the continuity equation, can be written in terms of growing spherical harmonics by using the Lamb's general solution as (Hetsroni & Haber 1970)

$$\left.\begin{aligned} \mathbf{u}_i &= \sum_{n=1}^{\infty}\left[\nabla\times(\mathbf{r}\chi_n) + \nabla\Phi_n + \frac{n+3}{2(n+1)(2n+3)\lambda} r^2\nabla p_n - \frac{n}{(n+1)(2n+3)\lambda}\mathbf{r} p_n\right], \\ p_i &= \sum_{n=0}^{\infty} p_n, \end{aligned}\right\} \quad (3)$$

where p_n, Φ_n and χ_n are growing solid spherical harmonics of degree *n*, the expressions of which are provided below

$$\left.\begin{aligned} p_n &= \lambda r^n \sum_{m=0}^{n} \left[A_{n,m} \cos(m\varphi) + \hat{A}_{n,m} \sin(m\varphi) \right] P_{n,m}(\cos\theta), \\ \Phi_n &= r^n \sum_{m=0}^{n} \left[B_{n,m} \cos(m\varphi) + \hat{B}_{n,m} \sin(m\varphi) \right] P_{n,m}(\cos\theta), \\ \chi_n &= r^n \sum_{m=0}^{n} \left[C_{n,m} \cos(m\varphi) + \hat{C}_{n,m} \sin(m\varphi) \right] P_{n,m}(\cos\theta). \end{aligned}\right\} \tag{4}$$

In a similar manner, using the Lamb's solution the velocity and pressure field outside the droplet can be expressed as (Hetsroni & Haber 1970)

$$\left.\begin{aligned} \mathbf{u}_e &= (\mathbf{V}_\infty - \mathbf{U}) + \sum_{n=1}^{\infty} \left[\nabla \times (\mathbf{r}\chi_{-n-1}) + \nabla\Phi_{-n-1} - \frac{n-2}{2n(2n-1)} r^2 \nabla p_{-n-1} + \frac{n+1}{n(2n-1)} \mathbf{r} p_{-n-1} \right], \\ p_e &= p_\infty + \sum_{n=0}^{\infty} p_{-n-1}, \end{aligned}\right\} \tag{5}$$

where $(\mathbf{V}_\infty, p_\infty)$ are the velocity and pressure fields as $r \to \infty$ and $p_{-n-1}, \Phi_{-n-1}, \chi_{-n-1}$ represent the decaying solid spherical harmonics showcased below

$$\left.\begin{aligned} p_{-n-1} &= r^{-n-1} \sum_{m=0}^{n} \left[A_{-n-1,m} \cos(m\varphi) + \hat{A}_{-n-1,m} \sin(m\varphi) \right] P_{n,m}(\cos\theta), \\ \Phi_{-n-1} &= r^{-n-1} \sum_{m=0}^{n} \left[B_{-n-1,m} \cos(m\varphi) + \hat{B}_{-n-1,m} \sin(m\varphi) \right] P_{n,m}(\cos\theta), \\ \chi_{-n-1} &= r^{-n-1} \sum_{m=0}^{n} \left[C_{-n-1,m} \cos(m\varphi) + \hat{C}_{-n-1,m} \sin(m\varphi) \right] P_{n,m}(\cos\theta). \end{aligned}\right\} \tag{6}$$

The constant coefficients $A_{n,m}$, $B_{n,m}$, $C_{n,m}$, $A_{-n-1,m}$, $B_{-n-1,m}$, $C_{-n-1,m}$, $\hat{A}_{n,m}$, $\hat{B}_{n,m}$, $\hat{C}_{n,m}$, $\hat{A}_{-n-1,m}$, $\hat{B}_{-n-1,m}$ and $\hat{C}_{-n-1,m}$ in the equations (4) and (6) can be obtained by substituting equations (3) and (5) in the boundary conditions at the interface of the droplet (the kinematic boundary conditions, the tangential velocity continuity and the tangential stress balance conditions) and then solving them.

2. Leading Order solution

The leading order solution is obtained for an undeformed droplet $(Ca = 0)$. The expression for the leading order temperature field for both the phases, obtained by solving the leading order boundary conditions for the temperature field, is provided below

$$\left.\begin{aligned} T_e^{(0)} &= r\zeta\left\{1+\frac{1}{r^3}\left(\frac{1-\xi}{2+\xi}\right)\right\}\cos\varphi P_{1,1}\left(\cos\theta\right), \\ T_i^{(0)} &= r\zeta\left(\frac{3}{\xi+2}\right)\cos\varphi P_{1,1}\left(\cos\theta\right). \end{aligned}\right\} \tag{7}$$

The surface temperature thus obtained from above is given by $T_s^{(0)} = 3\zeta/\left(\xi+2\right)\cos\varphi P_{1,1}\left(\cos\theta\right)$ where the only non zero coefficient of the surface harmonic $\cos\varphi P_{1,1}\left(\cos\theta\right)$ is given by $T_{1,1} = 3\zeta/\left(\xi+2\right)$.

The leading order solution for the velocity and pressure fields are obtained by substituting equations (3), (5), (4) and (6) in the boundary conditions for leading order. We then use the force-free condition to calculate the droplet migration velocity. This force-free condition is obtained by equating the net drag force on the droplet to zero which is given by

$$\mathbf{F}_D^{(0)} = 4\pi\nabla\left(r^3 p_{-2}^{(0)}\right), \tag{8}$$

where the decaying solid harmonic p_{-2} is given by

$$p_{-2} = r^{-3}\left[A_{-2,0}^{(0)}P_{2,0}\left(\cos\theta\right) + A_{-2,1}^{(0)}\cos\phi P_{2,1}\left(\cos\theta\right) + \hat{A}_{-2,1}^{(0)}\sin\phi P_{2,1}\left(\cos\theta\right)\right]. \tag{9}$$

The constant coefficients $A_{-2,0}^{(0)}$, $A_{-2,1}^{(0)}$ and $\hat{A}_{-2,1}^{(0)}$ which has been found previously from the boundary conditions are substituted in equation (9) to calculate p_{-2} and hence the migration velocity is calculated from the condition $\mathbf{F}_D^{(0)} = 0$. Thus the expressions of the leading order axial velocity (along *x*-direction), $U_x^{(0)}$ and the cross-stream velocity (along *y*-direction) $U_y^{(0)}$, thus obtained, are provided below

$$\begin{aligned} U_x^{(0)} &= -\frac{4\lambda}{H^2\left(3\lambda+2\right)} + 4\frac{y_d}{H}\left(1-\frac{y_d}{H}\right), \\ U_y^{(0)} &= \frac{2\zeta Ma_T}{\left(\xi+2\right)\left(3\lambda+2\right)}, \\ U_z^{(0)} &= 0. \end{aligned} \tag{10}$$

It can be seen from the above expressions that for the leading order, the interfacial temperature distribution has no effect on the axial migration velocity, whereas the cross-stream migration velocity in the *y*-direction originates solely due to the transversely applied constant temperature gradient.

References

HETSRONI, G. & HABER, S. 1970 The flow in and around a droplet or bubble submerged in an unbound arbitrary velocity field. *Rheol. Acta* **9**(4), 488–496.