Supplementary material for: Three dimensional free-surface flow over arbitrary bottom topography

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16 April 2018

This document contains supplementary material for Buttle *et al.* (2018).

1. Algebraic equations for linearised problem

The linear problem presented in §4 is discretised in the same manner as the nonlinear problem in §3.1 and evaluated on the half mesh points $(x_k^*, y_l^*) = ((x_k + x_{k+1})/2, y_l)$. The resulting vector function, $\boldsymbol{G}$, is:

$$\boldsymbol{G}_{1(k,l)} = \phi^*_{x(k,l)} + \frac{\zeta^*_{(k,l)}}{F_H^2} - 1, \tag{1.1}$$

$$\begin{aligned}\boldsymbol{G}_{2(k,l)} = &\sum_{i=1}^{N}\sum_{j=1}^{M} w(i,j)\left\{\left(\zeta_{x(i,j)} - \zeta^*_{x(k,l)}\right) K_{5(i,j,k,l)}\right.\\ &\left. + \beta_{x(i,j)} K_{7(i,j,k,l)} + (\psi_{(i,j)} - x_i) K_{6(i,j,k,l)}\right\}\\ &+ \zeta^*_{x(k,l)} \iint K_5 \,\mathrm{d}x\,\mathrm{d}y - 2\pi\left(\phi^*_{(k,l)} - x_k^*\right),\end{aligned} \tag{1.2}$$

$$\boldsymbol{G}_{3(l)} = x_1 \phi_{x(1,l)} + n\phi_{(1,l)} - x_1(n+1), \tag{1.3}$$

$$\boldsymbol{G}_{4(l)} = \frac{x_1}{\Delta x}\phi_{x(2,l)} + \left(n - \frac{x_1}{\Delta x}\right)\phi_{x(1,l)} - n, \tag{1.4}$$

$$\boldsymbol{G}_{5(l)} = x_1 \zeta_{x(1,l)} + n\zeta_{(1,l)}, \tag{1.5}$$

$$\boldsymbol{G}_{6(l)} = \frac{x_1}{\Delta x}\zeta_{x(2,l)} + \left(n - \frac{x_1}{\Delta x}\right)\zeta_{x(1,l)}, \tag{1.6}$$

$$\begin{aligned}\boldsymbol{G}_{7(k,l)} = &\sum_{i=1}^{N}\sum_{j=1}^{M} w(i,j)\Big\{\zeta_{x(i,j)} K_{7(i,j,k,l)}\\ &+ \left(-\beta^*_{x(k,l)} + \beta_{x(i,j)}\right) K_{5(i,j,k,l)}\\ &+ (\phi_{(i,j)} - x_i) K_{6(i,j,k,l)}\Big\}\\ &+ \beta^*_{x(k,l)} \iint K_5 \,\mathrm{d}x\,\mathrm{d}y\\ &- 2\pi\left(\psi^*_{(k,l)} - x_k^*\right),\end{aligned} \tag{1.7}$$

$$\boldsymbol{G}_{8(l)} = x_1 \psi_{x(1,l)} + n\psi_{(1,l)} - x_1(n+1), \tag{1.8}$$

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$$\boldsymbol{G}_{9(l)} = \frac{x_1}{\Delta x}\psi_{x(2,l)} + \left(n - \frac{x_1}{\Delta x}\right)\psi_{x(1,l)} - n, \tag{1.9}$$

$$\tag{1.10}$$

for $k = 1, 2, ..., N-1$, $l = 1, 2, ..., M$ and

$$K_{5(i,j,k,l)} = \left((x_i - x_k^*)^2 + (y_j - y_l^*)^2\right)^{-1/2}, \tag{1.11}$$

$$K_{6(i,j,k,l)} = \left((x_i - x_k^*)^2 + (y_j - y_l^*)^2 + 1\right)^{-3/2}, \tag{1.12}$$

$$K_{7(i,j,k,l)} = \left((x_i - x_k^*)^2 + (y_j - y_l^*)^2 + 1\right)^{-1/2}. \tag{1.13}$$

The order of the functions in the vector function is:

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{G}_{5(1)} \\ \boldsymbol{G}_{6(1)} \\ \boldsymbol{G}_{1(1,1)} \\ \boldsymbol{G}_{1(2,1)} \\ \vdots \\ \boldsymbol{G}_{1(N-1,1)} \\ \boldsymbol{G}_{5(2)} \\ \vdots \\ \boldsymbol{G}_{1(N-1,M)} \\ \boldsymbol{G}_{3(1)} \\ \boldsymbol{G}_{4(1)} \\ \boldsymbol{G}_{2(1,1)} \vdots \\ \boldsymbol{G}_{2(N-1,M)} \\ \boldsymbol{G}_{8(1)} \\ \boldsymbol{G}_{9(1)} \\ \boldsymbol{G}_{7(1,1)} \\ \vdots \\ \boldsymbol{G}_{7(N-1,M)} \end{bmatrix}. \tag{1.14}$$

The simplicity of the linear discretised problem allows for the Jacobian to be computed analytically. This is done by differentiating the equations with respect to each of the unknowns.

$$\frac{\mathrm{d}\boldsymbol{G}}{\mathrm{d}\boldsymbol{u}} = \begin{bmatrix} \frac{\partial \boldsymbol{G}_{(1)}}{\partial \boldsymbol{u}_1} & \frac{\partial \boldsymbol{G}_{(1)}}{\partial \boldsymbol{u}_2} & \cdots & \frac{\partial \boldsymbol{G}_{(1)}}{\partial \boldsymbol{u}_{3(N+1)M}} \\ \frac{\partial \boldsymbol{G}_{(2)}}{\partial \boldsymbol{u}_1} & \frac{\partial \boldsymbol{G}_{(2)}}{\partial \boldsymbol{u}_2} & \cdots & \frac{\partial \boldsymbol{G}_{(2)}}{\partial \boldsymbol{u}_{3(N+1)M}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \boldsymbol{G}_{(3(N+1)M)}}{\partial \boldsymbol{u}_1} & \frac{\partial \boldsymbol{G}_{(3(N+1)M)}}{\partial \boldsymbol{u}_2} & \cdots & \frac{\partial \boldsymbol{G}_{(3(N+1)M)}}{\partial \boldsymbol{u}_{3(N+1)M}} \end{bmatrix} \tag{1.15}$$

The derivatives of the function are:

$$\frac{\partial \boldsymbol{G}_{1(k,l)}}{\partial \phi_{(1,m)}} = 0, \tag{1.16}$$

$$\frac{\partial \boldsymbol{G}_{1(k,l)}}{\partial \phi_{x(n,m)}} = \begin{cases} \frac{1}{2} & \text{for } n = k,\ n = k+1,\ m = l, \\ 0 & \text{otherwise.} \end{cases}, \tag{1.17}$$

$$\frac{\partial \boldsymbol{G}_{1(k,l)}}{\partial \zeta_{(1,m)}} = \begin{cases} \frac{1}{F_H^2} & \text{for } m = l, \\ 0 & \text{otherwise.} \end{cases}, \tag{1.18}$$

$$\frac{\partial \boldsymbol{G}_{1(k,l)}}{\partial \zeta_{x(n,m)}} = \begin{cases} -\frac{\mathit{\Delta} x}{2F_H^2} \quad \text{for } n=1,\ m=l, \\ \frac{\mathit{\Delta} x}{F_H^2} \quad \text{for } 1<n<k,\ m=l, \\ \frac{3\mathit{\Delta} x}{4F_H^2} \quad \text{for } n=k,\ m=l, \\ \frac{\mathit{\Delta} x}{4F_H^2} \quad \text{for } n=k+1,\ m=l, \\ 0, \quad \text{otherwise.} \end{cases}, \tag{1.19}$$

$$\frac{\partial \boldsymbol{G}_{1(k,l)}}{\partial \psi_{x(n,m)}} = 0, \tag{1.20}$$

$$\frac{\partial \boldsymbol{G}_{2(k,l)}}{\partial \phi_{(1,m)}} = \begin{cases} -2\pi \quad \text{for } m=l, \\ 0 \quad \text{otherwise.} \end{cases}, \tag{1.21}$$

$$\frac{\partial \boldsymbol{G}_{2(k,l)}}{\partial \phi_{x(n,m)}} = \begin{cases} -\frac{\pi}{2}\mathit{\Delta} x \quad \text{for } n=1,\ k=1,\ m=l, \\ -\pi\mathit{\Delta} x \quad \text{for } n=1,\ k>1,\ m=l, \\ -2\pi\mathit{\Delta} x \quad \text{for } 1<n<k,\ m=l, \\ -\frac{3\pi\mathit{\Delta} x}{2} \quad \text{for } n=k,\ m=l, \\ -\frac{\pi\mathit{\Delta} x}{2} \quad \text{for } n=k+1,\ m=l, \\ 0 \quad \text{otherwise.} \end{cases}, \tag{1.22}$$

$$\frac{\partial \boldsymbol{G}_{2(k,l)}}{\partial \zeta_{(1,m)}} = 0, \tag{1.23}$$

$$\frac{\partial \boldsymbol{G}_{2(k,l)}}{\partial \zeta_{x(n,m)}} = \begin{cases} \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{M} w(i,j)K_{5(i,j,k,l)} \\ \qquad -\frac{1}{2}\iint K_5 \\ \qquad - w(n,m)K_{5(n,m,k,l)} \\ \qquad\qquad\qquad \text{for } n=k,\ m=l, \\ -w(n,m)K_{5(n,m,k,l)} \quad \text{otherwise.} \end{cases}, \tag{1.24}$$

$$\frac{\partial \boldsymbol{G}_{2(k,l)}}{\partial \psi_{(1,m)}} = 0, \tag{1.25}$$

$$\frac{\partial \boldsymbol{G}_{2(k,l)}}{\partial \psi_{x(n,m)}} = \begin{cases} \sum_{i=2}^{N}\left\{\frac{\mathit{\Delta} x}{2}w(i,m)K_{6(i,m,k,l)}\right\} \quad \text{for } n=1, \\ \frac{\mathit{\Delta} x}{2}w(n,m)K_{6(n,m,k,l)} \\ \qquad + \mathit{\Delta} x\sum_{i=n+1}^{N}\left\{w(i,m)K_{6(i,m,k,l)}\right\} \\ \qquad\qquad\qquad \text{for } 1<n<N, \\ \frac{\mathit{\Delta} x}{2}w(n,m)K_{6(n,m,k,l)} \quad \text{for } n=N. \end{cases}, \tag{1.26}$$

$$\frac{\partial \boldsymbol{G}_{3(l)}}{\partial \phi_{(1,m)}} = n \quad \text{for } m=l, \tag{1.27}$$

$$\frac{\partial \boldsymbol{G}_{3(l)}}{\partial \phi_{x(1,m)}} = \begin{cases} x_1 \quad \text{for } m=l, \\ 0 \quad \text{otherwise.} \end{cases}, \tag{1.28}$$

$$\frac{\partial \boldsymbol{G}_{3(l)}}{\partial \zeta_{(1,m)}} = 0, \tag{1.29}$$

$$\frac{\partial \boldsymbol{G}_{3(l)}}{\partial \zeta_{x(n,m)}} = 0, \tag{1.30}$$

$$\frac{\partial \boldsymbol{G}_{3(l)}}{\partial \psi_{x(k,l)}} = 0, \tag{1.31}$$

$$\frac{\partial \boldsymbol{G}_{4(l)}}{\partial \phi_{(1,l)}} = 0, \tag{1.32}$$

$$\frac{\partial \boldsymbol{G}_{4(l)}}{\partial \phi_{x(n,m)}} = \begin{cases} n - \frac{x_1}{\Delta x} & \text{for } n = 1,\ m = l, \\ \frac{x_1}{\Delta x} & \text{for } n = 2,\ m = l, \\ 0 & \text{otherwise.} \end{cases}, \tag{1.33}$$

$$\frac{\partial \boldsymbol{G}_{4(l)}}{\partial \zeta_{(1,m)}} = 0, \tag{1.34}$$

$$\frac{\partial \boldsymbol{G}_{4(l)}}{\partial \zeta_{x(n,m)}} = 0, \tag{1.35}$$

$$\frac{\partial \boldsymbol{G}_{4(l)}}{\partial \psi_{x(n,m)}} = 0, \tag{1.36}$$

$$\frac{\partial \boldsymbol{G}_{5(l)}}{\partial \phi_{(1,m)}} = 0, \tag{1.37}$$

$$\frac{\partial \boldsymbol{G}_{5(l)}}{\partial \phi_{x(n,m)}} = 0, \tag{1.38}$$

$$\frac{\partial \boldsymbol{G}_{5(l)}}{\partial \zeta_{(1,m)}} = \begin{cases} n & \text{for } m = l, \\ 0 & \text{otherwise.} \end{cases}, \tag{1.39}$$

$$\frac{\partial \boldsymbol{G}_{5(l)}}{\partial \zeta_{x(n,m)}} = \begin{cases} x_1 & \text{for } n = 1,\ m = l, \\ 0 & \text{otherwise.} \end{cases}, \tag{1.40}$$

$$\frac{\partial \boldsymbol{G}_{5(l)}}{\partial \psi_{x(k,l)}} = 0, \tag{1.41}$$

$$\frac{\partial \boldsymbol{G}_{6(l)}}{\partial \phi_{(1,m)}} = 0, \tag{1.42}$$

$$\frac{\partial \boldsymbol{G}_{6(l)}}{\partial \phi_{x(n,m)}} = 0, \tag{1.43}$$

$$\frac{\partial \boldsymbol{G}_{6(l)}}{\partial \zeta_{(1,m)}} = 0, \tag{1.44}$$

$$\frac{\partial \boldsymbol{G}_{6(l)}}{\partial \zeta_{x(n,m)}} = \begin{cases} n - \frac{x_1}{\Delta x} & \text{for } n = 1,\ m = l, \\ \frac{x_1}{\Delta x} & \text{for } n = 2,\ m = l, \\ 0 & \text{otherwise.} \end{cases}, \tag{1.45}$$

$$\frac{\partial \boldsymbol{G}_{6(l)}}{\partial \psi_{x(n,m)}} = 0, \tag{1.46}$$

$$\frac{\partial \boldsymbol{G}_{7(k,l)}}{\partial \phi_{(1,m)}} = 0, \tag{1.47}$$

$$\frac{\partial \boldsymbol{G}_{7(k,l)}}{\partial \phi_{x(n,m)}} = \begin{cases} \sum_{i=2}^{N} \left\{ \frac{\Delta x}{2} w(i,m) K_{6(i,m,k,l} \right\} \\ \qquad\qquad\qquad \text{for } n = 1, \\ \frac{\Delta x}{2} w(n,m) K_{6(n,m,k,l)} \\ \quad + \Delta x \sum_{i=n+1}^{N} \left\{ w(i,m) K_{6(i,m,k,l)} \right\} \\ \qquad\qquad\qquad \text{for } 1 < n < N, \\ \frac{\Delta x}{2} w(n,m) K_{6(n,m,k,l)} \quad \text{for } n = N. \end{cases}, \tag{1.48}$$

$$\frac{\partial \boldsymbol{G}_{7(k,l)}}{\partial \zeta_{(1,m)}} = 0, \tag{1.49}$$

$$\frac{\partial \boldsymbol{G}_{7(k,l)}}{\partial \zeta_{x(n,m)}} = \begin{cases} -w(n,m) K_5 \quad \text{for } m = l, \\ 0 \quad \text{otherwise.} \end{cases}, \tag{1.50}$$

$$\frac{\partial \boldsymbol{G}_{7(k,l)}}{\partial \psi_{x(n,m)}} = \begin{cases} -\pi \Delta x \quad \text{for } n = 1,\ m = l, \\ -2\pi \Delta x \quad \text{for } 1 < n < k,\ m = l, \\ -\frac{3\pi \Delta x}{2} \quad \text{for } n = k,\ m = l, \\ -\frac{\pi \Delta x}{2} \quad \text{for } n = k+1,\ m = l, \\ 0 \quad \text{otherwise.} \end{cases}, \tag{1.51}$$

$$\frac{\partial \boldsymbol{G}_{8(l)}}{\partial \phi_{(1,m)}} = 0, \tag{1.52}$$

$$\frac{\partial \boldsymbol{G}_{8(l)}}{\partial \phi_{x(n,m)}} = 0, \tag{1.53}$$

$$\frac{\partial \boldsymbol{G}_{8(l)}}{\partial \zeta_{(1,m)}} = 0, \tag{1.54}$$

$$\frac{\partial \boldsymbol{G}_{8(l)}}{\partial \zeta_{x(n,m)}} = 0, \tag{1.55}$$

$$\frac{\partial \boldsymbol{G}_{8(l)}}{\partial \psi_{(1,m)}} = n \quad \text{for } m = l, \tag{1.56}$$

$$\frac{\partial \boldsymbol{G}_{8(l)}}{\partial \psi_{x(1,m)}} = \begin{cases} x_1 \quad \text{for } m = l, \\ 0 \quad \text{otherwise.} \end{cases}, \tag{1.57}$$

$$\frac{\partial \boldsymbol{G}_{9(l)}}{\partial \psi_{(1,l)}} = 0, \tag{1.58}$$

$$\frac{\partial \boldsymbol{G}_{9(l)}}{\partial \psi_{x(n,m)}} = \begin{cases} n - \frac{x_1}{\Delta x} \quad \text{for } n = 1,\ m = l, \\ \frac{x_1}{\Delta x} \quad \text{for } n = 2,\ m = l, \\ 0 \quad \text{otherwise.} \end{cases} \tag{1.59}$$

REFERENCES

Buttle, N. R., Pethiyagoda, R., Moroney, T. J. & McCue, S. W. 2018 Three dimensional free-surface flow over arbitrary bottom topography. *Journal of Fluid Mechanics*, doi:10.1017/jfm.2018.254.