

Rm	γ	l_{max}	n_{max}	r_t	\mathbf{U} restart ?	remark
10	-9.295	16	8	10^{-4}	no	
20	-8.681	16	8	10^{-4}	no	
30	-7.838	16	8	10^{-4}	no	
40	-6.415	16	8	10^{-3}	no	
50	-4.496	16	8	10^{-3}	no	
60	-2.004	16	8	10^{-3}	no	
65	-0.479	16	8	10^{-3}	yes	
70	1.312	16	8	10^{-3}	yes	
75	3.461	16	8	10^{-3}	yes	
64.1	-0.178	16	8	3.3×10^{-5}	yes	
64.2	-0.143	16	8	3.4×10^{-5}	yes	
64.3	-0.107	16	8	3.5×10^{-5}	yes	
64.4	-0.071	16	8	3.5×10^{-5}	yes	
64.5	-0.036	16	8	3.5×10^{-5}	yes	
64.6	0.000	16	8	3.7×10^{-5}	yes	
64.7	0.036	16	8	3.6×10^{-5}	yes	
64.8	0.072	16	8	4.0×10^{-5}	yes	
64.9	0.108	16	8	4.0×10^{-5}	yes	
65.0	0.144	16	8	4.0×10^{-5}	yes	
65.2	0.150	16	8	4.4×10^{-4}	yes	
65.4	0.219	16	8	4.5×10^{-4}	yes	
65.6	0.071	16	8	3.5×10^{-4}	yes	
65.8	0.137	16	8	3.5×10^{-4}	yes	
66.0	0.169	16	8	3.8×10^{-4}	yes	
64.1	-0.131	24	12	10^{-4}	yes	
64.2	-0.095	24	12	10^{-4}	yes	
64.3	-0.058	24	12	10^{-4}	yes	
64.4	-0.022	24	12	10^{-4}	yes	
64.5	0.015	24	12	10^{-4}	yes	
64.6	0.052	24	12	10^{-4}	no	
64.7	0.089	24	12	10^{-4}	yes	
64.8	0.126	24	12	10^{-4}	yes	
65.0	0.200	24	12	10^{-4}	yes	
64.1	-0.128	24	12	5.1×10^{-5}	yes	
64.2	-0.092	24	12	5.1×10^{-5}	yes	
64.3	-0.055	24	12	2.6×10^{-5}	yes	
64.4	-0.018	24	12	2.6×10^{-5}	yes	
64.43	-0.007	24	12	8×10^{-6}	yes	
64.44	-0.003	24	12	8×10^{-6}	yes	
64.45	0.001	24	12	8×10^{-6}	yes	R1
64.45	0.003	24	24	1.2×10^{-4}	no	
64.45	0.001	24	12	8×10^{-6}	no	R2
64.45	0.002	24	12	5×10^{-5}	yes	$T = 2.5$

Table 1: The optimisation results, listed in four groups with gradually refining ranges of Rm . The growth rates listed here are also plotted in Figure 2 in the main text. We use time step $\Delta t \leq 10^{-4}$ and $\alpha_1, \alpha_2 \in [0.04, 0.1]$ in the final phase of the relaxation scheme. γ is the asymptotic growth rate of magnetic field, l_{max} is the maximum spherical harmonic degree, n_{max} is the maximum degree in radial basis functions, r_t is the residue and “ \mathbf{U} restart ?” indicates either a restart from previously stored fields (yes) or not (no, meaning a random start). Without specification the total time window $T = 2$. The initial starting points for models R1 and R2 are independent of each other.

Rm	γ	r_t	U restart ?	Rm	γ	r_t	U restart ?
30	-8.138	10^{-3}	no	55	-3.278	1.9×10^{-3}	no
30	-7.844	10^{-3}	no	55	-3.275	1.5×10^{-3}	no
30	-8.139	10^{-3}	no	55	-3.241	1.5×10^{-3}	no
30	-8.140	10^{-3}	no	55	-3.253	1.7×10^{-3}	no
30	-8.139	10^{-3}	no	55	-3.422	2.3×10^{-3}	no
30	-7.961	3.4×10^{-3}	no	55	-3.375	1.8×10^{-3}	no
30	-7.860	1.8×10^{-3}	no	55	-3.406	2.4×10^{-3}	no
30	-8.139	10^{-3}	no	55	-3.222	1.4×10^{-3}	no
35	-7.170	10^{-3}	no	57	-2.734	2.1×10^{-3}	no
35	-7.197	2.0×10^{-3}	no	57	-4.875	3.5×10^{-3}	no
35	-7.430	5.4×10^{-3}	no	57	-4.713	10^{-3}	no
35	-7.280	3.5×10^{-3}	no	57	-3.053	2.8×10^{-3}	no
35	-7.198	1.8×10^{-3}	no	57	-2.783	1.7×10^{-3}	no
35	-7.178	10^{-3}	no	57	-2.840	2.5×10^{-3}	no
35	-7.175	10^{-3}	no	57	-2.555	1.5×10^{-3}	no
35	-7.175	10^{-3}	no	57	-2.830	2.3×10^{-3}	no
40	-6.588	4.5×10^{-3}	no	59	-2.915	6.4×10^{-3}	no
40	-6.490	1.1×10^{-3}	no	59	-2.106	2.3×10^{-3}	no
40	-6.438	1.8×10^{-3}	no	59	-2.371	2.9×10^{-3}	no
40	-6.424	1.2×10^{-3}	no	59	-2.228	2.5×10^{-3}	no
40	-6.438	1.9×10^{-3}	no	59	-2.329	1.6×10^{-3}	no
40	-6.518	3.0×10^{-3}	no	59	-2.062	2.7×10^{-3}	no
40	-6.475	2.3×10^{-3}	no	59	-2.273	2.2×10^{-3}	no
40	-6.919	3.1×10^{-3}	no	59	-2.802	2.8×10^{-3}	no
45	-5.578	1.6×10^{-3}	no	60	-1.680	1.6×10^{-3}	no
45	-5.673	3.3×10^{-3}	no	60	-1.749	1.9×10^{-3}	no
45	-5.607	2.1×10^{-3}	no	60	-2.002	2.8×10^{-3}	no
45	-5.542	10^{-3}	no	60	-2.063	3.5×10^{-3}	no
45	-5.566	1.6×10^{-3}	no	60	-1.659	1.5×10^{-3}	no
45	-5.623	2.3×10^{-3}	no	60	-1.972	2.8×10^{-3}	no
45	-5.704	3.1×10^{-3}	no	60	-1.634	1.5×10^{-3}	no
45	-5.617	2.3×10^{-3}	no	60	-2.553	6.7×10^{-3}	no
50	-4.529	1.1×10^{-3}	no	62	-1.621	4.1×10^{-3}	no
50	-4.541	1.7×10^{-3}	no	62	-1.104	1.8×10^{-3}	no
50	-4.416	1.4×10^{-3}	no	62	-1.344	3.0×10^{-3}	no
50	-4.581	1.9×10^{-3}	no	62	-1.276	1.8×10^{-3}	no
50	-4.507	1.2×10^{-3}	no	62	-1.249	2.0×10^{-3}	no
50	-4.535	1.4×10^{-3}	no	62	-1.278	2.1×10^{-3}	no
50	-4.767	3.5×10^{-3}	no	62	-1.248	2.6×10^{-3}	no
50	-4.488	10^{-3}	no	62	-1.044	2.0×10^{-3}	no

Table 2: Additional optimisation runs, all start from random initial fields. We do not see local optimal states other than the optimum we have identified before. The scatter in the growth rates are due to less stringent numerical schemes. Here we have used a lower resolution $(l_{max}, n_{max}) = (16, 8)$, higher tolerance $r_t \geq 10^{-3}$ and a shorter time window $T = 1.5$ in exchange of more independent runs. The time step $\Delta t = 10^{-4}$ and the relaxation parameters $\alpha_1, \alpha_2 \in [0.04, 0.15]$.